Lecture 4

Thm (HP) $f: M \to M$ p.h. $\pi_1(M)$ solvable

then:
- $\to$ $M = \mathbb{T}^3$ and f leaf conj to Anosov
- $\to$ $M = \mathbb{T}^3$ or Nil and f leaf conj to skew prod.
- $\to$ $M =$ Sol and f leaf conj to suspensi[on] of Anosov
- $\to$ $\exists T$ torus tangent to E^{cs} or E^{cu}.

(mod finite cover + iterate)

First 3 lectures: (Brin-Burago-Ivanov/Parwani

"One can restrict to those manifolds"

$f : \mathbb{T}^3 \to \mathbb{T}^3$ p.h. $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$

$f_* : \pi_1(\mathbb{T}^3) \to \pi_1(\mathbb{T}^3)$ is Anosov

$$|\lambda_1| \leq |\lambda_2| < 1 < |\lambda_3|$$

<u>Thm</u>: f is dynamically coherent.

<u>Recall</u>: $\exists$ f-invariant branching foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ $\pitchfork$ to E^{cs} and E^{cu}.

<u>Goal</u>: Show that they don't branch.

1) Classify ~~Reebless~~ foliations
w.o. tori

<u>Ideas</u> : Plante, Roussarie, Gabai, ..

$\mathcal{F}_1, \mathcal{F}_2$ br. foliations are ALMOST PARALL

$\exists R$
- $\forall L \in \tilde{\mathcal{F}}_1 \ \exists L' \in \tilde{\mathcal{F}}_2$ s.t. $d_H(L, L') < R$
- $\forall L \in \tilde{\mathcal{F}}_2 \ \exists L' \in \tilde{\mathcal{F}}_1$ s.t. $d_H(L, L') < R$

Branching foliations are almost parallel to true foliations. (BI)

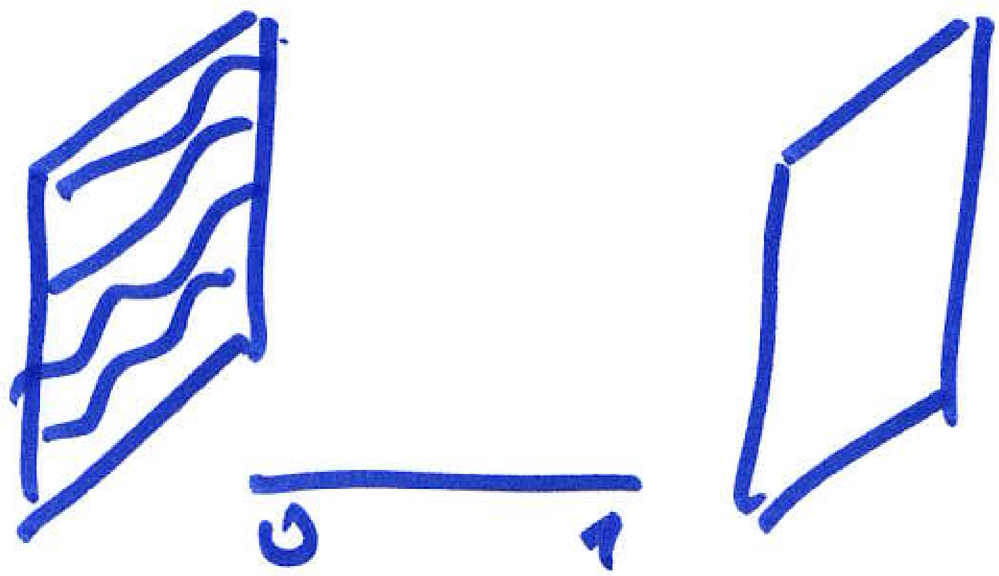
$B \in SL(2,\mathbb{Z})$

$$M = \mathbb{T}^2 \times [0,1] \Big/ (x,1) \sim (Bx, 0)$$

$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ — Nil

$B = A$ hyp. $\longrightarrow$ Solv.

Thm (Roussarie/Gabai) $\mathcal{F}$ a foliation w.o. torus leaves in M and T is incompressible torus $\Longrightarrow$ T is isotopic to a torus transverse to $\mathcal{F}$.



__Consequence__: In $\mathbb{T}^3$ every foliation w.o. torus leaves is A.P. to a linear foliation.

$$\mathcal{F}^{cs} \overset{AP}{\sim} \mathcal{L}^{cs}$$

$$\mathcal{F}^{cu} \overset{AP}{\sim} \mathcal{L}^{cu}$$

2) Show that L^{cs} and L^{cu} are f_*-invariant.

$d_{C^0}(\tilde{f}, f_*) < K \qquad \tilde{f}(\mathcal{L}) = \mathcal{L}$

$d(f_* L^\sigma, L^\sigma)$ bounded

$f_* L^\sigma = L^\sigma \qquad \sigma = cs, cu$

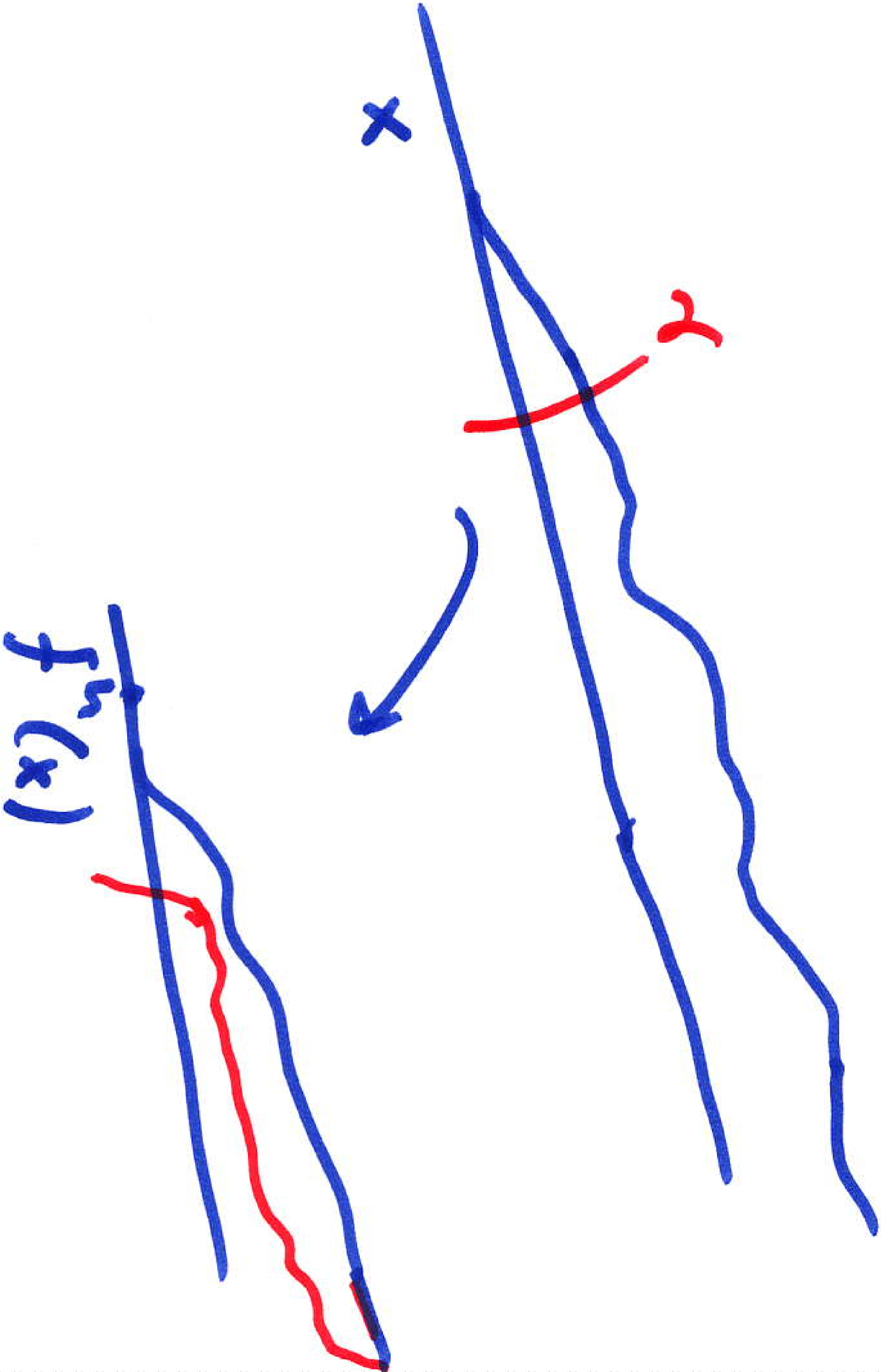
3) GPS $\Rightarrow$ dyn. coherence

$\mathcal{F}^{cs}$ and $\mathcal{W}^u$ have GPS

if $\forall L \in \tilde{\mathcal{F}}^{cs}$ and $\forall \gamma \in \tilde{\mathcal{W}}^u$

we have $\# L \cap \gamma = 1$.

<u>We know</u> $\# L \cap \gamma \leq 1$.


x
γ
f^n(x)

<u>Comments</u>: → We have still not used that f_* is Anosov

→ We DO use that $M = \mathbb{T}^3$
(At least that leafs of $\mathcal{L}^{cs}$ are parallel to each other)

→ $\exists$ unique f-inv. foliation $\mathcal{L}^{cs}$ and $\mathcal{L}^{cu}$ resist perturbations

4) Prove GPS when f_* is Anosov

<u>Proposition</u>: $\mathcal{L}^{cu}$ and $\mathcal{L}^{cs}$ are totally irrational.

<u>Pf</u>: f_* is Anosov.

[Novikov / Hector Hirsch ($\dim M = 3$)

A foliation by simply connected leafs has GPS with every transverse foliation.]

<u>Consequence</u>: We got GPS

$\Longrightarrow$ dyn. coherence.

$$\mathcal{F}^{cs} \rightsquigarrow \mathcal{W}^{cs} — L^{cs}$$

$$\mathcal{F}^{cu} \rightsquigarrow \mathcal{W}^{cu} — L^{cu}$$

<u>Lema</u> $L^{cs} \neq L^{cu}$

<u>Consequence</u>: $3 \neq$ real eigenvalues

$\tilde{f}$ has $\tilde{W}^{cs}, \tilde{W}^{cu}$ — L^{cs}, L^{cu}

f_* $\quad |\lambda_1| < |\lambda_2| < 1 < |\lambda_3|$

$$\begin{array}{ccc} | & | & | \\ E^1_* & E^2_* & E^3_* \end{array}$$

Goal: $L^{cs} = E^1_* \oplus E^2_* \quad L^{cu} = E^2_* \oplus E^3_*$

L^{cs} and L^{cu} resist perturbations

$\exists H : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $d(H, id) < K$

$$H \circ \tilde{f} = f_* \circ H$$

$$H(\tilde{W}^u(x)) = E^3_* + H(x)$$

$$H(\tilde{W}^s(x)) \subseteq E^1_* \oplus E^2_* + H(x)$$

Implies: $L^{cu} \neq E^1_* \oplus E^2_*$

$L^{cs} = E^1_* \oplus E^2_*$.

<u>Problem</u>: $L^{cu} \stackrel{?}{=} E^1_* \oplus E^3_*$