Local and global aspects of almost global stability

Pablo Monzón and Rafael Potrie

Abstract— **In this work we introduce several known and new results on almost global stability. We focus on how local properties of equilibrium points of dynamical systems are related to the existence of density functions and to the almost global stability property. We present some examples that illustrate those results.**

I. INTRODUCTION

Consider the autonomous differential equation

$$\dot{x} = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 field (enough to ensure the existence and uniqueness of solutions to the initial value problem) and $f(0) = 0$. In this context, a *density function* is a function $\rho \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ satisfying

$$\nabla \cdot (\rho f) > 0 \quad m\text{-a.e.}$$

where m is the Lebesgue measure. It can be proved than the existence of a function satisfying this hypothesis globally, plus being positive (adding some integrability conditions) results on *almost global stability* of the system, meaning that almost all trajectories of the system will converge to the origin; by almost all, we refer to a set whose complement is insignificant in the Lebesgue sense. This result can be found in (Rantzer, 2001).

Theorem 1 (Rantzer)**.** *Given $\dot{x} = f(x)$ satisfying $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, and if there exists $\rho \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ non negative which satisfies $\rho(x)f(x)/|x|$ is integrable in $\{x : |x| \geq \epsilon\}$ $\forall \epsilon > 0$ and*

$$\nabla \cdot (\rho f) > 0 \quad m\text{-c.t.p.}$$

Then, for almost every $x_0 \in \mathbb{R}^n$ (in Lebesgue sense) the solution $x(t)$ of the differential equation with initial condition x_0 exists and it is well defined in $[0, +\infty)$ and also $x(t) \to 0$ when $t \to +\infty$.

The previous Theorem has triggered a new research direction on nonlinear systems, about converse results, topological restrictions and control applications (Rantzer, 2003; Monzón, 2003; Angeli, 2003; Prajna and Rantzer and Parrilo, 2004; Angeli, 2004; Monzón, 2004). The paper is organized as follows. In Section II, we introduce the monotone measures and a particular result for two dimensional systems that we want to generalize to higher dimensions; in Section III we review some properties of density and Lyapunov functions; after that we present some motivating examples that show how some hypothesis may be handled; the next Section deals with the relevance of local stability for necessary conditions for the existence of density functions. Finally, we present some relationships between local properties of equilibrium points and density functions and some concluding remarks.

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Pablo Monzón is with Faculty of Engineering, Universidad de la Republica, Uruguay, monzon@fing.edu.uy

Rafael Potrie is with Faculty of Engineering, Universidad de la Republica, Uruguay, rpotrie@fing.edu.uy

II. MONOTONE MEASURES

The proof of this Theorem is based on the following Lemma (similar to the Liouville's Theorem):

Lema 2. *Let $f \in C^1(D, \mathbb{R}^n)$, with $D \subset \mathbb{R}^n$ open and let $\rho \in C^1(D, \mathbb{R})$ integrable. For $x_0 \in \mathbb{R}^n$ let $f^t(x_0)$ be the solution of $\dot{x} = f(x)$ in the time t such that $f^0(x_0) = x_0$. Given a measurable set Z and assuming that $f^\tau(Z) \subset D$ $\forall \tau \in [0, t]$. Then*

$$\int_{f^t(Z)} \rho(x)dx - \int_Z \rho(x)dx = \int_0^t \int_{f^\tau(Z)} [\nabla \cdot (\rho f)](x) dx d\tau$$

This Lemma, besides its use in the already mentioned theorem, allows us to think of the density ρ as a measure which grows on the trajectories defined by $\mu(E) = \int_E \rho$. We now give a definition, of a generalization of the notion of densities:

Definition 1. Given a $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ field, a Borel measure μ is said *monotone* iff

- $m \ll \mu$ (meaning that $\forall A$ such that $\mu(A) = 0 \Rightarrow m(A) = 0$)
- $\forall Y$ such that if $0 < \mu(Y) < \infty$ then $\mu(Y) < \mu(f^t(Y))$ $\forall t > 0$.
- $\mu(B^c(0, \epsilon)) < \infty$ $\forall \epsilon > 0$.

With arguments similar to the ones used by Rantzer in (Rantzer, 2001), it can be proved that the existence of a monotone measure allows us to ensure almost global stability. Converse results about existence of monotone measures have been introduced in (Monzón, 2004). It is also proved, in (Monzón, 2005), that in dimension 2 the existence of a measure with similar characteristics (less restrictive in the sense of monotonicity but the measure must be bounded on compact sets) added to almost global stability implies also local asymptotical stability.

Theorem 3. *Given the system $\dot{x} = f(x)$ ($f \in C^2(\mathbb{R}^2, \mathbb{R}^2)$) with a discrete set of fixed points. If there exists a measure $\mu \ll m$ bounded (meaning that for any bounded set Y $\mu(Y) < \infty$) satisfying that for every bounded set X such that $0 < \mu(X)$ there exists $t \neq 0$ such that $\mu(f^t(X)) \neq \mu(X)$ then almost global stability at the origin implies local asymptotical stability.*

This result is important because to the effects of control theory, almost global stability is not good enough, but is very well complemented by local stability. This result is not easily generalized to higher dimensions since it uses Poincaré-Bendixson theory (Khalil, 1996) which stands in properties intrinsic to the plane. We attempt here to give some steps towards the generalization.

The following property relates the existence of a measure with the existence of a density. In this property there is no result on the regularity of the density, we only find an $L^1(m)$ density whose induced measure grows with the flow.

Remark 1. We recall that by the Radon-Nikodym theorem, given two $\sigma-$finite measures η,ν we can decompose η as $\rho+\lambda$ such that $\rho \ll \nu$ (see Definition 1 that implies that $\rho(E)=\int_e h d\nu$ for some h) and $\lambda \perp \nu$ (meaning that we can decompose the space in two sets L and L^c such that $\nu(L)=0$ and $\lambda(L^c)=0$.

Property 4. *Let f be a C^1 field and μ a monotone measure then if $\mu=\nu+\lambda$ is μ's decomposition with $\nu \ll m$ and $\lambda \perp m$, so ν is monotone.*

Proof: First we can observe that the differentiability of the field ensures us that $m(E)=0 \Rightarrow m(f^t(E))=0$ $\forall t$ since it implies Lipschitz condition which can be used to send a arbitrarily small cover of E by f^t to again a arbitrarily small cover of $f^t(E)$. There are two sets L and R such that $\mathbb{R}^n=L\cup R$, $\nu(L)=0$ and $\lambda(R)=0$. Then, using what we just proved

$$\nu(f^t(L))=0 \quad \forall t$$

Now, given Y such that $0<\nu(Y)<\infty$ we have that $\forall t$:

$$0<\nu(Y)=\nu(Y\cap R)=\nu(Y\cap R\cap f^{-t}(R))=$$

$$\mu(Y\cap R\cap f^{-t}(R))<$$

$$<\mu(f^t(Y)\cap f^t(R)\cap R)=\nu(f^t(Y)\cap f^t(R))=\nu(f^t(Y))$$

■

III. DENSITIES AND LYAPUNOV FUNCTIONS

In this Section we review some properties and relationships between densities and Lyapunov functions. Some of them appeared in (Rantzer, 2001).

Property 5. *Given a C^1 density ρ , it verifies $\lim_{x\to 0}\rho(x)=+\infty$ in a non integrable way.*

Proof: If we consider the measure given by ρ, using Lemma 2 it's easy to see that it is monotone. Now, if we consider the region of attraction of $x=0$, we can see that it is invariant by the field, then, since the measure given by ρ is monotone, it must measure 0 or infinity, and since ρ can´t be 0 almost everywhere we deduce that $\int_{\mathcal{R}}\rho=\infty$ (where $\mathcal{R}$ is the region of attraction) then, since ρ is integrable on the complement of every neighborhood of the origin we deduce $\lim_{x\to 0}\rho(x)=+\infty$. ■

Proposition 6. *Given $V(x)>0$ $\forall x\neq 0$ satisfying*

$$\alpha\nabla V\cdot f<V\nabla\cdot f \quad m-ctp$$

for some $\alpha>0$. Then for $\rho(x)=V(x)^{-\alpha}$ we have $\nabla\cdot(\rho f)>0$ $m-ae$.

Proposition 7. *If for every $x\neq 0$, ρ satisfies*

$$\nabla\cdot(\rho f)>0 \quad \nabla\cdot f\leq 0 \quad \rho>0$$

Then $V(x)=\rho(x)^{-1}$ satisfies $\nabla V\cdot f<0$.

The proof of this properties is very simple and are based on the formula

$$\nabla\cdot(\rho f)=\nabla\rho f+\rho\nabla\cdot f$$

We present here the proof of Proposition 7 since it is possible to generalize it for the case when the hypothesis are satisfied locally (Lyapunov functions only need to be defined locally).

Proof:

$$\nabla\cdot(\rho f)=\nabla\rho f+\rho\nabla\cdot f>0 \quad m-a.e.$$

using positivity of ρ and that in a neighborhood of the origin $\nabla\cdot f\leq 0$ we have that $V=\rho^{-1}>0$ implies $\nabla\rho f\geq 0$ so

$$\dot V=\nabla V f=-\rho^{-2}(\nabla\rho f)\leq 0$$

■

IV. SOME MOTIVATING EXAMPLES

We attempt to make a step towards the generalization of the result in Theorem 3. Also, we intend to add some extra conditions on the fields to ensure local asymptotical stability, or maybe, less ambitiously, local stability. Here we present some examples of almost globally stable systems but not locally asymptotically stable.

Example 1. This simple example shows how an almost globally stable system, admitting a C^1 density may not be locally asymptotically stable. A not so trivial example can be found in (Prajna and Rantzer and Parrilo, 2004).

$$\begin{cases}\dot x_1 &= -x_1^3\\ \dot x_2 &= -x_1^2x_2\end{cases}$$

then for $\rho=(x_1^2+x_2^2)^{-3}$ we have

$$\nabla\cdot(\rho f)=6(x_1^2+x_2^2)^{-4}(x_1^4+x_1^2x_2^2)-4(x_1^2+x_2^2)^{-3}x_1^2=$$
$$=2(x_1^2+x_2^2)^{-3}x_1^2>0 \quad m-a.e.$$

We see that in this example the origin is not an isolated fixed point for the flow f^t. That will be one of the expected hypothesis for the system to be locally asymptotically stable.

Example 2. We show here that the field

$$\begin{cases}\dot x_1 &= x_1^2-x_2^2\\ \dot x_2 &= 2x_1x_2\end{cases}$$

does not accept $\rho=\|x\|^{-\alpha}$ as a density function. We observe that

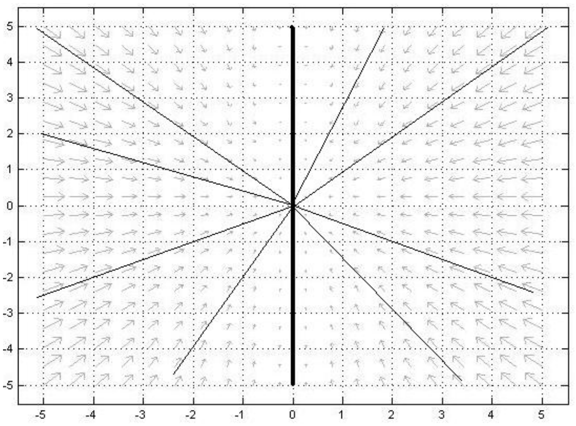

Fig. 1. *Trajectories for the system of Example 1. The dark line represents the fixed points.*

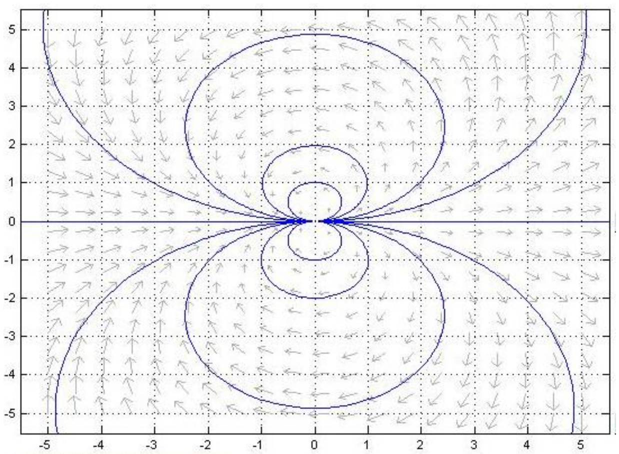

Fig. 2. *Trajectories for the system of Example 2*

$$\nabla \cdot (\rho f) = \|x\|^{-\alpha} x_1 (4 - 2\alpha)$$

so ρ can not be a density function for this system. We do not know if the system admits a density function of other kind.

V. NECESSARY CONDITION FOR ALMOST GLOBAL STABILITY

We present in this section some results of great utility which have appeared in (Monzón, 2003).

Theorem 8. *Given a class C^2 vector field f in $\mathbb{R}^n$ such that the origin is an asymptotically stable equilibrium and such that the region of attraction is the whole space. Then, there exists a diffeomorphism of class C^2 conjugating the system $\dot{x} = f(x)$ with $\dot{y} = -y$ assuming the field is complete.*

Remark 2. For the special case of $n = 4$ it is necessary to add the hypothesis of the equilibrium being hyperbolic (see (Monzón, 2003))

Remark 3. The hypothesis of being globally asymptotically stable can be removed by a weaker one of being locally asymptotically stable and in that case, the conjugacy will be defined from the region of attraction of the origin and $\mathbb{R}^n$.

In the proof of this theorem, the diffeomorphism is calculated explicitly from a diffeomorphism (H) between a level manifold ($\mathcal{E}$) of the Lyapunov function given by Massera's Theorem (Massera, 1949) and a $n-1$ sphere in $\mathbb{R}^n$ in the following way: given $x \in \mathcal{R}$ (the region of attraction) there exists $t_x \in \mathbb{R}$ such that $f^{t_x}(x) \in \mathcal{E}$, in fact, t_x is unique, so we can send $x \mapsto g^{-t_x}(H(f^{t_x}(x))) = e^{t_x} H(f^{t_x}(x))$ or reversely $x \mapsto g^{t_x}(H(f^{t_x}(x))) = e^{-t_x} H(f^{t_x}(x))$. We shall call h_1 to the first map, and h_2 to the second, they satisfy:

$$h_1 \circ f^t(x) = g^t \circ h_1(x) \quad , \quad h_2 \circ f^t(x) = g^{-t} \circ h_2(x)$$

This diffeomorphism allow us to define a C^1 density for $f(x)$ from a given density for the field $g(y) = -y$. However, we must notice that this density will only be defined for the region of attraction $\mathcal{R}$; we are interested in extending it to $\mathcal{R}^c$ in a way that the resulting density may be of class C^1. The density is defined from a density ρ for g in this way

$$\overline{\rho}(x) = \rho(h_1(x)) \left| \frac{\partial h_1}{\partial x}(x) \right|$$

Theorem 9. *The density is well defined*

Proof: First of all , is easy to notice that it is non negative (taking into account that h_1 is orientation preserving) and is C^1 in $\mathcal{R} \backslash \{0\}$ since h_1 is C^2.

To see that $\nabla \cdot (\rho f) > 0$ *c.t.p.* we consider a measurable set $\mathcal{Z}$ not containing the origin in its closure so:

$$\int_{\mathcal{Z}} \overline{\rho}(x) dx = \int_{\mathcal{Z}} \rho(h_1(x)) \left| \frac{\partial h_1}{\partial x}(x) \right| dx = \int_{h_1(\mathcal{Z})} \rho(y) dy$$

Since $h_1 \circ f^t(x) = g^t \circ h_1(x)$ we have $\forall x \in \mathcal{R}$

$$\left| \frac{\partial h_1}{\partial x}(f^t(x)) \right| \left| \frac{\partial f^t}{\partial x}(x) \right| = \left| \frac{\partial g^t}{\partial x}(h_1(x)) \right| \left| \frac{\partial h_1}{\partial x}(x) \right|$$

Then

$$\nabla \cdot (\overline{\rho} f) = \frac{\partial}{\partial t} \left\{ \overline{\rho}(f^t(x)) \cdot \left| \frac{\partial f^t}{\partial x}(x) \right| \right\} \Big|_{t=0} =$$

$$\frac{\partial}{\partial t} \left\{ \rho(h_1(f^t(x))) \cdot \left| \frac{\partial h_1}{\partial x}(f^t(x)) \right| \cdot \left| \frac{\partial f^t}{\partial x}(x) \right| \right\} \Big|_{t=0} =$$

$$= \frac{\partial}{\partial t} \left\{ \rho(g^t(h_1(x))) \cdot \left| \frac{\partial g^t}{\partial x}(h_1(x)) \right| \left| \frac{\partial h_1}{\partial x}(x) \right| \right\} \Big|_{t=0} =$$

$$= \left| \frac{\partial h_1}{\partial x}(x) \right| \nabla \cdot (\rho g)(h_1(x)) > 0 \quad c.t.p.$$

■

Remark 4. This procedure is valid in the other way, i.e. given a density function for $\dot{x} = f(x)$ we can build one for $\dot{y} = -y$ in an analogous way.

Using the previous results, it can be proved (Monzón, 2003) that global asymptotical stability implies the existence of a C^1 density function. We attempt to give a more general result

keeping the hypothesis of local asymptotical stability and changing global asymptotical stability for almost global stability. The result is almost the same but the derivative of the density we find may not be continuous in a region of 0 Lebesgue measure, specifically in the complement of the region of attraction of the origin. However, the density we find is particular, in the sense that it is zero (and has zero derivative although it may not be continuous) in this region. After, we present an example from (Rantzer, 2001) where an almost globally stable system (and not globally asymptotically stable) admits a density which is strictly positive everywhere.

Theorem 10. *Given a complete differential equation $\dot{x} = f(x)$ with $f \in C^1$ such that the origin is a almost globally stable and locally asymptotically stable fixed point for the flow f^t, there exists a density ρ differentiable and with continuous derivative up to a set of zero Lebesgue measure. Also, this density can be constructed such that it is zero in the complement of the basin of attraction.*

Proof: We have defined a candidate for density in this way

$$\overline{\rho}(x) = \rho(h_1(x)) \left| \frac{\partial h_1}{\partial x}(x) \right|$$

Where ρ is a density for the field $\dot{y} = -y$. We want to see that defined this way can be extended to the complement of the basin of attraction $\mathcal{R}^c$.

We know that given $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ tal que $x_n \to z \in \mathcal{R}^c$ we have that $t_{x_n} \to +\infty$ so $h_1(x_n) \to \infty$ that means that if we make $\rho(y) \left| \frac{\partial h_1}{\partial x}(h_1^{-1}(y)) \right| \to 0$ as $y \to \infty$ we'll achieve continuity of $\overline{\rho}$ in $\mathcal{R}^c$. Also we can see that:

$$\frac{\partial \overline{\rho}}{\partial x}(x) = \frac{\partial \rho}{\partial y}(h_1(x)) \frac{\partial h_1}{\partial x}(x) \left| \frac{\partial h_1}{\partial x}(x) \right| + \rho(h_1(x)) \nabla \left| \frac{\partial h_1}{\partial x}(x) \right|$$

And this equation holds $\forall x \in \mathcal{R}^c$. So, if we bound ρ and $\frac{\partial \rho}{\partial y}$ adequately we can make $\overline{\rho}$ of class C^1.

This may not be possible, since we only have control on ρ and we have to handle two inequalities, one to make the derivative continuous and another one to ensure that ρ is a density for the system $\dot{y} = -y$.

Let

$$j(r) = r^2 \sup_{\|y\| \le r} \left\{ \left| \frac{\partial h_1}{\partial x}(h_1^{-1}(y)) \right|, \right.$$
$$\left. \left\| \nabla \left| \frac{\partial h_1}{\partial x}(h_1^{-1}(y)) \right| \right\|, \frac{1}{d(h_1^{-1}(y), \mathcal{R}^c)} \right\}$$

j is well defined, because $D_r = \{y : |y| \le r\}$ is compact, and the functions defined continuous (the one that could be doubted is the distance but $\mathcal{R}^c$ is closed and $h_1^{-1}(D_r) \cap \mathcal{R}^c = \emptyset$ then $\inf\{d(h_1^{-1}(y), \mathcal{R}^c) > 0$).

Now we define $\beta : \mathbb{R}^n \to \mathbb{R}$ C^∞ increasing such that outside some neighborhood of 0 satisfies $\beta(y) > j(\|y\|)$ (and be constant in $\|y\| =$ constant) and such that $\beta(0) = 0$. Then we have that β is a Lyapunov function for $\dot{y} = -y$.

Also, we can consider β to be convex in each direction, that is, having in mind that β is defined radially that $\beta(tx + (1-t)y) \le t\beta(x) + (1-t)\beta(y)$. We prove that if $\alpha > n$, $\beta^{-\alpha}$ will be a density for $\dot{y} = -y$. It's enough to see (because of Proposition 6) that

$$-\alpha \nabla \beta(y) y < \nabla \cdot (-y) \beta(y) \Rightarrow \alpha \nabla \beta(y) y > n \beta(y)$$

But since the function is radial (its gradient is parallel to the function $g(y) = -y$) we can verify the property in one direction. So we see that

$$n\beta(x_1, 0, \ldots, 0) < \alpha x_1 \frac{\partial \beta}{\partial x_1}(x_1, 0, \ldots, 0)$$

And it is easy to see that for a real valued convex function w such that $w(0) = 0$ we have that $w(x) \le x w'(x)$ so if $\alpha > n$ we can have what we wanted.

Now, given $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ such that $x_n \to z \in \mathcal{R}^c$ we have

$$\overline{\rho}(x_n) = \rho(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right| =$$
$$\beta^{-\alpha}(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right| \to 0$$

So $\overline{\rho}(x) = 0$ $\forall x \in \mathcal{R}^c$. To see that this function is differentiable we consider $z \in \mathcal{R}^c$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ (there is no problem considering the sequence in $\mathcal{R}$ since it is zero in its complement) such that $x_n \to z$ and we have

$$\frac{\|\overline{\rho}(x_n) - \overline{\rho}(z)\|}{\|x_n - z\|} = \frac{\|\overline{\rho}(x_n)\|}{\|x_n - z\|} =$$
$$\frac{\beta^{-\alpha}(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right|}{\|x_n - z\|} \to 0$$

Since if $\alpha > 2$

$$\beta^{-1}(h_1(x_n)) \left| \frac{\partial h_1}{\partial x}(x_n) \right| \to 0$$

$$\frac{\beta^{-1}(h_1(x_n))}{\|x_n - z\|} < \frac{1}{\|h_1(x_n)\|^2} \underbrace{\frac{d(h_1^{-1}(h_1(x_n)), \mathcal{R}^c)}{\|x_n - z\|}}_{<1} \to 0$$

So we have that $\overline{\rho}$ is differentiable in all $\mathbb{R}^n$, the continuity of its derivative in $\mathcal{R}$ is immediate, but to have continuity in all $\mathbb{R}^n$ we need that for some α

$$\frac{\partial \rho}{\partial y}(y) = (-\alpha \beta^{-\alpha-1}(y)) \nabla \beta(y)$$

goes to zero as fast as ρ when $y \to \infty$, since in this case

$$\frac{\partial \overline{\rho}}{\partial x}(x_n) = \frac{\partial \rho}{\partial y}(h_1(x_n)) \frac{\partial h_1}{\partial x}(x_n) \left| \frac{\partial h_1}{\partial x}(x) \right|$$
$$+ \rho(h_1(x_n)) \nabla \left| \frac{\partial h_1}{\partial x}(x_n) \right| \to 0$$

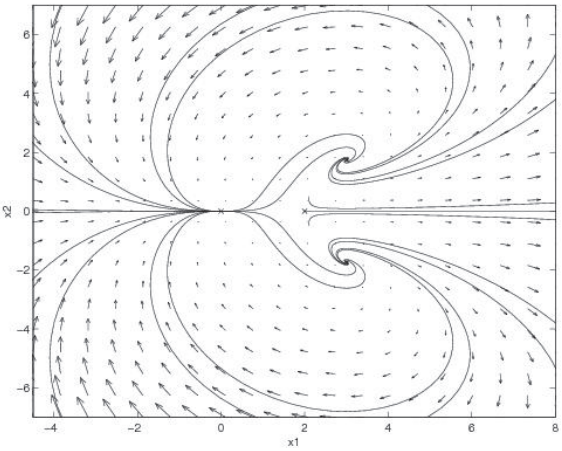

Fig. 3. *Phase portrait of the system of Example 3*

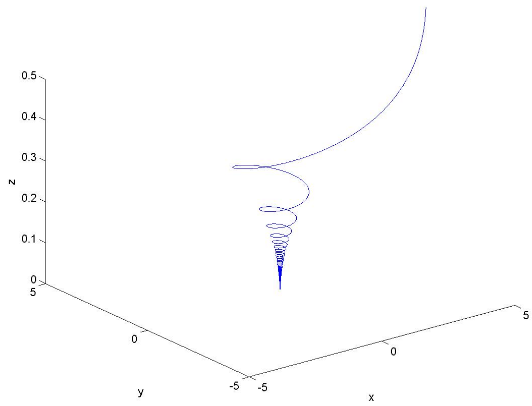

Fig. 4. *A trajectory of the system of Example 4*

Unfortunately this is not true in general since a convex function can be constructed such that its derivative is larger than any power of the function in a sequence going to infinity.

However, if β can be constructed satisfying that property then the density will be C^1. ∎

We present an example from (Rantzer, 2001) showing how for a system satisfying the hypothesis of the previous Theorem it can be found a density function which is strictly positive. It shows that it can exist a density which is non zero in the complement of the basin of attraction (see figure 3).

Example 3. Consider the system

$$\begin{cases} \dot{x}_1 &= -2x_1 + x_1^2 - x_2^2 \\ \dot{x}_2 &= -6x_2 + 2x_1x_2 \end{cases}$$

for which $\rho = \|x\|^{-4}$ is a density function.

VI. CLASSIFICATION OF FIXED POINTS

In this Section we try to classify the possible behaviors of the origin as a fixed point depending on the eigenvalues of the derivative of the field in the origin. We can conclude that if there is at least one with positive real part then almost global stability is not possible. On the other hand the existence of one eigenvalue with negative real part added to the existence of a density function implies local asymptotical stability. This will lead us to a generalization of Theorem 3.

Proposition 11. *Let $\dot{x} = f(x)$ with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\frac{\partial f}{\partial x}(0)$ has at least one eigenvalue λ with $Re(\lambda) > 0$ then the set of points such that $\lim_{t\to\infty} f^t(x) = 0$ has zero Lebesgue measure.*

Proof: This proof is based on the existence of local invariant manifolds (Hirsch and Pugh and Shub, 1977). In this case we are interested on the unstable manifold (given by the expanding eigenvalues of the derivative of f) and the center stable manifold ($W^{cs}_{loc}(0)$). This manifolds exist only locally and has zero Lebesgue measure since its dimension is strictly smaller than the dimension of the space. Also we know that if a point in $\mathbb{R}^n$ is to converge to the origin as $t \to \infty$ then for some $n \in \mathbb{N}$ $f^n(x) \in W^{cs}_{loc}$, so , we have that the region of attraction of the origin is contained in $\bigcup_{n\in\mathbb{N}} f^{-n}(W^{cs}_{loc})$ which is a countable union of sets with zero measure so it has zero Lebesgue measure. ∎

Proposition 12. *Let $\dot{x} = f(x)$ with $f \in C^1\mathbb{R}^n, \mathbb{R}^n$ such that $\frac{\partial f}{\partial x}(0)$ has all eigenvalues with $Re(\lambda) \leq 0$ and at least one with negative real part then the existence of a differentiable density function implies local (asymptotical) stability.*

Proof: The fact that the eigenvalues are as said implies that $\nabla \cdot f < 0$. Proposition 7 implies the existence of a Lyapunov function. ∎

We now present an example where we show that in the remaining case, i.e. when all eigenvalues have null real part, the result is not valid even in the case that the origin is an isolated fixed point (in Example 1, it was already shown that when the origin was not an isolated equilibrium a density could be found).

Example 4.

$$\begin{cases} \dot{x}_1 &= x_2 - 2x_1x_3^2 \\ \dot{x}_2 &= -x_1 - 2x_2x_3^2 \\ \dot{x}_3 &= -x_3^3 \end{cases}$$

The trajectories of this field are periodic orbits in the plane $x_3 = 0$ and spirals converging to the origin when the initial condition in x_3 is not 0 (see figure 4).

We can see that $\nabla \cdot f = -7x_3^2$ so, using $\rho = (x_1^2 + x_2^2 + x_3^2)^{-4}$ we have that $\nabla \cdot (\rho f) = x_3^2(x_1^2 + x_2^2 + x_3^2)^{-4}$ that is indeed positive almost everywhere. Note that $\nabla \cdot (\rho f)$ equals 0 in the plane $x_3 = 0$, as could be expected since $f|_{x_3=0}$ is not almost globally stable (for the system $f|_{x_3=0}$, the set of points that not converge to zero have full measure if we consider the Lebesgue measure in the plane). Although the origin is not asymptotically stable, is a locally stable equilibrium.

VII. CONCLUSIONS AND FUTURE WORKS

In this work we have put together several aspects of almost global stability, monotone measures, density and Lyapunov functions and local and global properties of equilibrium points. We show how local stability plus almost global stability of the origin can be combined in order to construct a density function. We also gave a first step into the generalization to higher dimensions of a planar result about the relationships between almost global stability and local asymptotical stability. We proved that the existence of a positive eigenvalue of the Jacobian matrix at the origin denies almost global stability. On the other hand, negative divergence of the field at the origin together with a density function implies local asymptotical stability. In future works we will analyze the remaining case of zero divergence, trying to establish conditions for local stability, as is motivated by Example 4.

VIII. ACKNOWLEDGMENTS

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