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ON THE SHORTEST SPANNING SUBTREE OF A GRAPH AND THE TRAVELING SALESMAN PROBLEM

JOSEPH B. KRUSKAL, JR.

Several years ago a typewritten translation (of obscure origin) of [1] raised some interest. This paper is devoted to the following theorem: If a (finite) connected graph has a positive real number attached to each edge (the *length* of the edge), and if these lengths are all distinct, then among the spanning[1] trees (German: Gerüst) of the graph there is only one, the sum of whose edges is a minimum; that is, the shortest spanning tree of the graph is unique. (Actually in [1] this theorem is stated and proved in terms of the "matrix of lengths" of the graph, that is, the matrix $\|a_{ij}\|$ where a_{ij} is the length of the edge connecting vertices i and j. Of course, it is assumed that $a_{ij}=a_{ji}$ and that $a_{ii}=0$ for all i and j.)

The proof in [1] is based on a not unreasonable method of constructing a spanning subtree of minimum length. It is in this construction that the interest largely lies, for it is a solution to a problem (Problem 1 below) which on the surface is closely related to one version (Problem 2 below) of the well-known traveling salesman problem.

PROBLEM 1. Give a practical method for constructing a spanning subtree of minimum length.

PROBLEM 2. Give a practical method for constructing an unbranched spanning subtree of minimum length.

The construction given in [1] is unnecessarily elaborate. In the present paper I give several simpler constructions which solve Problem 1, and I show how one of these constructions may be used to prove the theorem of [1]. Probably it is true that any construction

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[1] A subgraph spans a graph if it contains all the vertices of the graph.

which solves Problem 1 may be used to prove this theorem.

First I would like to point out that there is no loss of generality in assuming that the given connected graph G is complete, that is, that every pair of vertices is connected by an edge. For if any edge of G is "missing," an edge of great length may be inserted, and this does not alter the graph in any way which is relevant to the present purposes. Also, it is possible and intuitively appealing to think of missing edges as edges of infinite length.

CONSTRUCTION A. Perform the following step as many times as possible: Among the edges of G not yet chosen, choose the shortest edge which does not form any loops with those edges already chosen. Clearly the set of edges eventually chosen must form a spanning tree of G, and in fact it forms a shortest spanning tree.

CONSTRUCTION B. Let V be an arbitrary but fixed (nonempty) subset of the vertices of G. Then perform the following step as many times as possible: Among the edges of G which are not yet chosen but which are connected either to a vertex of V or to an edge already chosen, pick the shortest edge which does not form any loops with the edges already chosen. Clearly the set of edges eventually chosen forms a spanning tree of G, and in fact it forms a shortest spanning tree. In case V is the set of all vertices of G, then Construction B reduces to Construction A.

CONSTRUCTION A′. This method is in some sense dual to A. Perform the following step as many times as possible: Among the edges not yet chosen, choose the longest edge whose removal will not disconnect them. Clearly the set of edges *not* eventually chosen forms a spanning tree of G, and in fact it forms a shortest spanning tree. It is not clear to me whether Construction B in general has a dual analogous to this.

Before showing how Construction A may be used to prove the theorem of [1], I find it convenient to combine into a theorem a number of elementary facts of graph theory. The reader should have no trouble convincing himself that these are true. For aesthetic reasons, I state considerably more than I need.

PRELIMINARY THEOREM. *If G is a connected graph with n vertices, and T is a subgraph of G, then the following conditions are all equivalent:*

(a) *T is a spanning tree of G;*

(b) *T is a maximal[2] forest[3] in G;*

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[2] A graph is "maximal" if it is not contained in any larger graph of the same sort; it is "minimal" if it does not contain any smaller graph of the same sort.

[3] A "forest" is a graph which does not have any loops.

(c) *T is a minimal[2] connected spanning graph of G;*
(d) *T is a forest with $n-1$ edges;*
(e) *T is a connected spanning graph with $n-1$ edges.*

The theorem to be proved states that if the edges of G all have distinct lengths, then T is unique, where T is any shortest spanning tree of G. Clearly T may be redefined as any shortest forest with $n-1$ edges.

In Construction A, let the edges chosen be called $a_1, \cdots, a_{n-1}$ in the order chosen. Let A_i be the forest consisting of edges a_1 through a_i. It will be proved that $T=A_{n-1}$. From the hypothesis that the edges of G have distinct lengths, it is easily seen that Construction A proceeds in a unique manner. Thus the A_i are unique, and hence also T.

It remains to prove that $T=A_{n-1}$. If $T\neq A_{n-1}$, let a_i be the first edge of A_{n-1} which is not in T. Then $a_1, \cdots, a_{i-1}$ are in T. $T\cup a_i$ must have exactly one loop, which must contain a_i. This loop must also contain some edge e which is not in A_{n-1}. Then $T\cup a_i-e$ is a forest with $n-1$ edges.

As $A_{i-1}\cup e$ is contained in the last named forest, it is a forest, so from Construction A,

$$\text{length } (e) > \text{length } (a_i).$$

But then $T\cup a_i-e$ is shorter than T. This contradicts the definition of T, and hence proves indirectly that $T=A_{n-1}$.

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