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90m:01058 01A70

Massera, José Luís (UR-UREP)

Mischa in Montevideo.

Analysis and partial differential equations, xxi–xxiii, *Lecture Notes in Pure and Appl. Math.*, 122, *Dekker*, *New York*, 1990.

Reminiscences of Cotlar are given, particularly from the early 1930's in Montevideo and his contacts with Uruguayan mathematicians.

{For the entire collection see MR 90j:00006.}

84k:01059 01A70

Nachbin, Leopoldo (1-RCT)

Massera, one of the greatest mathematicians of Latin America of all times.

Ciênc. Cultura **35** (1983), *no.* 7, 917–919.

Homage to Massera and his work, and a call for Massera's release from prison. {He was released in early 1984.}

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Massera, H. L. [Massera, José Luis]; Šeffer, H. H. [Schäffer, Juan Jorge]

★**Линейные дифференциальные уравнения и функциональные пространства. (Russian) [Linear differential equations and function spaces]**

Translated from the English by A. M. Zverkin and G. A. Kamenskiĭ

Izdat. "*Mir*", *Moscow* 1970 456 *pp.*

The English original has been reviewed [Academic Press, New York, 1966; MR 35#3197].

37 5466 34.40

Lewowicz, Jorge; Massera, José L.

Asymptotic directions of the solutions of linear differential equations. (English. Spanish summary)

Univ. Repúb. Fac. Ingen. Agrimens. Montevideo Publ. Inst. Mat. Estadíst **4** 1967 107–113 (1967)

The authors consider the linear system of differential equations $x_i{}' = f_i(t)x_i + \sum_{j=1}^n g_{ij}(t)x_j$ $(i = 1, \cdots, n)$. Theorems related to the dichotomies of solutions are given under the assumptions that the g_{ij} are absolutely integrable and the f_i satisfy (i) $f_1(t) - f_j(t) \geq 0$ and (ii) $\int^\infty (f_1(t) - f_j(t))\,dt = +\infty$, $j \neq 1$. *J. S. W. Wong*

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Massera, José Luis; Schäffer, Juan Jorge

★**Linear differential equations and function spaces.**

Pure and Applied Mathematics, Vol. 21

Academic Press, *New York-London* 1966 xx+404 *pp.*

The book contains in a more systematic and detailed form the main results in the theory of linear differential equations which the authors have obtained in the last years. The theory is developed for equations in a Banach space and considerable emphasis is placed on the methods of functional analysis and on the use of function spaces. The book is addressed primarily to readers interested in the theory of differential equations, but no specialized knowledge in this field is required. A certain acquaintance of the reader with Banach space theory ("soft" and "hard") is assumed, but not an intimate knowledge of operator theory.

Let (1) $\dot{x}+A(t)x=0$ be a homogeneous linear differential equation and (2) $\dot{x}+A(t)x=f(t)$ the corresponding non-homogeneous equation. Here x represents a function with values in a Banach space X, $\dot{x}=dx/dt$, the real variable t ranging over $R_+=[0,\infty)$. Let $\tilde{X}$ be the algebra of endomorphisms (i.e., bounded operators) of X; then $A(t)$ is a mapping of R_+ into $\tilde{X}$ and $f(t)$ is a mapping of R_+ into X. There are two main classes of properties under investigation, termed "admissibility" and "dichotomy" and described as follows. (I) Admissibility: Let $\mathbf{B},\mathbf{D}$ be two function spaces whose elements are mappings from R_+ into X. The pair $(\mathbf{B},\mathbf{D})$ is called admissible for equation (2) if for each $f\in\mathbf{B}$ there exists a solution of (2), say x, such that $x\in\mathbf{D}$. It is supposed throughout the book that $\mathbf{B},\mathbf{D}$ are Banach spaces such that convergence in norm implies convergence in the mean on each compact subinterval of R_+. A basic result is the following proposition, which can be proved by use of the open mapping theorem. If the pair $(\mathbf{B},\mathbf{D})$ is admissible with respect to (2), then there exists a solution $x\in\mathbf{D}$ for which $|x|_{\mathbf{D}}\le K|f|_{\mathbf{B}}$, where K is independent of $f\in\mathbf{B}$. (II) Dichotomy: One says that a (closed) subspace Y of X induces a dichotomy of the solutions of (1) if there exist positive constants γ,N,N',γ_0 such that the following properties hold. (Di) For any solution $y(t)$ of (1) with $y(0)\in Y$, one has $\|y(t)\|\le N\|y(t_0)\|$ for $t\ge t_0\ge 0$. (Dii) For any solution $z(t)$ of (1) such that $z(0)$ has an angular distance $\ge\gamma$ from every nonzero element of Y one has $\|z(t)\|\ge N'^{-1}\|z(t_0)\|$ for $t\ge t_0\ge 0$. (Diii) For any pair of nonzero solutions $y(t),z(t)$ of (1) with the properties described in (Di) and (Dii), the angular distance between $y(t)$ and $z(t)$ is $\ge\gamma_0$. The angular distance between nonzero elements of X is defined by

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$\gamma[x,y] = \| \|y\|^{-1}y - \|x\|^{-1}x\|$. If one replaces (Di) and (Dii) by $\|y(t)\| \leq N\|y(t_0)\| \times \exp\{-\nu(t-t_0)\}$ and $\|z(t)\| \geq N'^{-1}\|z(t_0)\| \exp\{\nu'(t-t_0)\}$, where ν and ν' are positive constants, then one speaks of an exponential dichotomy. One of the main aims of the authors is to establish various relationships between admissibility and dichotomy. The typical form of such a relationship is: If the pair $(\mathbf{B},\mathbf{D})$ is admissible with respect to (2), then (1) possesses a certain dichotomy property. Converse propositions are also valid.

The book is divided into three Parts. Part I contains preliminary material: Chapter I treats some special problems concerning the geometry of Banach spaces. Chapter II is devoted to the theory of function spaces. There are results concerning the Lebesgue spaces $\mathbf{L}^p(R_+,X)$, but the main attention is paid to new classes of function spaces. In Chapter III, some general facts about linear differential equations are collected. Part II, the "core" of the book, contains Chapter IV on dichotomies, Chapter V on admissibility, and Chapter VI on the relations between these properties; Chapter VII is concerned with the dependence of dichotomies and admissibility on the operator-valued function $A(t)$, and Chapter VIII treats similar problems about equations on the real axis. Part III contains some complements to the general theory and includes also certain special cases. The definition of dichotomy properties suggests that these concepts are closely related to those of conditional stability and conditional exponential stability. Chapter IX gives a sketch of the possibility of adapting Ljapunov's direct method to the characterization of dichotomies. In Chapter X a theory of almost periodic equations is given and Chapter XI provides a treatment of periodic equations. Finally, Chapter XII is concerned with higher-order differential equations. Namely, instead of (1) and (2), the equations (1′) $w^{(m+1)} + \sum_0^m A_k(t)w^{(k)} = 0$, (2′) $w^{(m+1)} + \sum_0^m A_k(t)w^{(k)} = h(t)$ are studied. The results of this last chapter are essentially due to P. Hartman [Math. Ann. 147 (1962), 378–421; MR 26#1549]. As mentioned above, the results included in this book have already been published by the authors [Ann. of Math. (2) 67 (1958), 517–573; MR 20#3466; ibid. (2) 69 (1959), 88–104; MR 21#756; ibid. (2) 69 (1959), 535–574; MR 21#3638; Math. Ann. 139 (1960), 287–342; MR 22#8181] and the second author [ibid. 137 (1959), 209–262; MR 21#287; ibid. 138 (1959), 141–144; MR 21#6529; ibid. 140 (1960), 308–321; MR 22#8182; ibid. 145 (1961/62), 354–400; MR 25#4193; ibid. 149 (1962/63), 1–32; MR 26#6499; ibid. 150 (1963), 111–118; MR 27#5961; ibid. 151 (1963), 57–100; MR 27#5960].

It should be noted that the theory expounded in this book has

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wide applications to nonlinear problems. For reasons of space, the authors omitted a treatment of such problems. Some contributions along these lines may be found in the following papers: the authors [first reference cited above], the first author [Ann. Mat. Pura Appl. (4) 51 (1960), 95–105; MR 22#12292], the reviewer [An. Sti. Univ. "Al. I. Cuza" Iasi Sect. I (N.S.) 5 (1959), 33–36; MR 22#5766; ibid. (N.S.) 6 (1960), 257–260; MR 23#A1906], P. Hartman and N. Onuchic [Pacific J. Math. 13 (1963), 1193–1207; MR 28#293], P. Hartman [Ordinary differential equations, Wiley, New York, 1964; MR 30#1270].

The book is well-written, self-contained and constitutes undoubtedly an important tool in this field of investigation. The reviewer believes that the topics and methods of this book will find interesting developments in other branches of modern research such as control theory and functional equations. *C. Corduneanu*

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Massera, J. L.

Sur une équation intégrale provenant d'un problème de mécanique des fluides. (French. Spanish summary)

Univ. Repúb. Fac. Ingen. Agrimens. Montevideo Publ. Inst. Mat. Estadíst **3** 1962 171–187 (1962)

The author considers the integral equation

$$(*) \qquad \int_{x-1}^{x} [f(x)-f(t)]^{-1/2}\,dt = A$$

for constant A and $x \geq 0$. The function f equals a given function f_0 on the interval $[-1, 0]$. If f_0 is bounded and measurable and its supremum m satisfies the inequality $\int_{-1}^{0}[m-f(t)]^{-1/2}\,dt \geq A$, it is shown that there is a continuous monotone f which satisfies $(*)$ except possibly in a set of first category. Sufficient conditions are given on f_0 for f to satisfy $(*)$ for all $x \geq 0$. The questions of uniqueness and of continuous dependence on initial data of solutions are considered.

T. W. Mullikin

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Massera, J. L.

Some observations on the order of growth of vector and matrix solutions of a system of linear differential equations. (Spanish. English summary)

Univ. Repúb. Fac. Ingen. Agrimens. Montevideo Publ. Inst. Mat. Estadíst **4** 1964 1–11 (1964)

Consider the vector differential equation (1) $\dot{x} + Ax = 0$, and the corresponding operator equation (2) $\dot{U} + AU = 0$ with $U(0) = I$, defined in a Banach space X, where the dot means d/dt, and $t \in J = [0, +\infty)$. Let F be the family of solutions $x(t)$ of (1) with $\|x(0)\| = 1$. The author considers conditions of the following type: (3) $\lim_{t\to\infty} \|x(t)\|/\|U(t)\| = 1$ for some $x \in F$; (4) $\|x(t)\|/\|U(t)\| \geq k$ for some $x \in F$, all $t \in J$ and some $k > 0$; (5) $\lim \|x(t)\|/\|U(t)\| = 0$ for all $x \in F$; (6) $\limsup \|x(t)\|/\|U(t)\| = 1$ for some $x \in F$. Among other results, it is shown that if $\dim X < \infty$, then (6) is true; if, moreover, (1) is reducible, then (4) is also true. By counter-examples it is shown that (3), (4) are not always satisfied. The importance of the operator A being real is also considered. *E. O. Roxin*

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Massera, J. L.

The meaning of stability. (English. Spanish summary)

Univ. Repúb. Fac. Ingen. Agrimens. Montevideo Publ. Inst. Mat. Estadíst **4** 1964 23–47 (1964)

This is a reprinting of the article reviewed above [#4335].

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Massera, J. L.

The meaning of stability. (English. Spanish summary)

Bol. Fac. Ingen. Agrimens. Montevideo **8** 1964 405–429

Some new types of stability are introduced which neither compare phase-space positions at precisely the same time, as does Lyapunov stability, nor yet totally ignore the time dependence, as does orbital stability. The author feels that these new stabilities might well be more useful than either Lyapunov or orbital stability. A precise definition of one of the stabilities is as follows: Let T be a topology on clock-space, the set of all functions on $J = [0, \infty)$ into J. The identity function in clock-space is called the perfect clock. A motion $x(\cdot, \alpha_0)$ is T-stable if given ε and a T-neighborhood U of the perfect clock, there is a δ such that for all α, $d(\alpha, \alpha_0) < \delta$, there exists a clock $s(\cdot) \in U$ with

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$d(x(t,\alpha), x(s(t),\alpha_0)) < \varepsilon$, $t \in J$. Versions of asymptotic stability are defined and the effect of various clock-space topologies is discussed. In particular, the chaotic topology (two open sets) corresponds to orbital stability, and the discrete topology to Lyapunov stability. A number of theorems give relations between the different stabilities. Six examples further distinguish between the stabilities. The paper concludes with a treatment of the ε^+-periodic motions introduced previously by the author [Univ. Repub. Fac. Ingen. Agrimens. Montevideo Publ. Inst. Mat. Estadist. 2, 135–145 (1954); MR 15, 957]. *C. S. Coleman*

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Massera, J. L.

Some remarks on the order of growth of vector and matrix solutions of a linear system of differential equations. (Spanish. English summary)

Bol. Fac. Ingen. Agrimens. Montevideo **8** 1962 239–249

The author considers the equation (1) $\dot{x} + Ax = 0$, where x and t vary in a Banach space X and in $J = [0,\infty)$, respectively, and where $A = A(t)$ is a function from J to the space $\tilde{X}$ of endomorphisms of X (generalized matrix functions). Together with (1), the corresponding operator equation (2) $\dot{U} + AU = 0$ is considered, where $U\colon J \to \tilde{X}$. The solution of (2) with $U(0) = I$ is denoted by $U(t)$. F is the family of solutions of (1) with $\|x(0)\| = 1$. The purpose of the paper is to estimate the growth of the norm of $x(t) \in F$ relative to that of $U(t)$. Main theorems: (1) $\dim X < \infty$ implies $\limsup_{t\to\infty} \|x(t)\|/\|U(t)\| = 1$ for some $x \in F$; (2) The condition (6) that $\|U(t)\| = O(\|x(t)\|)$ for some $x \in F$ is invariant under t-similarity (in the sense of Conti [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 19 (1955), 247–250; MR 18, 483]); (3) If $\dim X < \infty$, and the system is reducible, i.e., A is t-similar to a constant matrix, then (6) holds; (4) If $\dim X \leq 2$, and A is "essentially real", meaning that it is t-similar to a real matrix, then the condition (6) holds. Various examples are given to show that results cannot be strengthened. *P. Seibert*

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Massera, J. L.

Sur une équation intégrale provenant d'un problème de mécanique des fluides. (French. Spanish summary)

Bol. Fac. Ingen. Agrimens. Montevideo **8** 1962 27–43

The author investigates the integral equation

$$\int_{x-1}^{x} [f(x)-f(t)]^{-1/2}\,dt = A,$$

where A is a constant. The integral equation is to be satisfied for $x \geq 0$, and $f(x)$ must coincide with a given, bounded, measurable function $f_0(x)$ on the interval $-1 \leq x < 0$.

A continuous function $\overline{f}$ is constructed to satisfy the integral equation except possibly on a set of the first category; this function $\overline{f}$ becomes a strict solution for all $x \geq 0$ if f_0 satisfies the additional condition

$$f_0(y) - f_0(x) \geq p_0(y-x) \quad \text{for some } p_0 > 0, \qquad -1 \leq x \leq y < 0.$$

The author derives a number of properties of $\overline{f}$ and lists some related unsolved questions. *I. Stakgold*

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Massera, José L.

On the existence of bounded periodic solutions of quasilinear systems of differential equations. (Spanish)

Rev. Un. Mat. Argentina **20** 1962 303–304

Consider the equations (1) $\dot{x} + A(t)x = h(x,t)$, (2) $\dot{x} + A(t)x = 0$, where $t \in J = [0,\infty]$, x is an element of a reflexive Banach space X, and $A(t)$ an element of the space of endomorphisms of X, this for each t of J. Moreover, $A(t)$ is Bochner-integrable in each finite subinterval of J. One assumes finally that for a subspace X_0 and its complement X_i the solutions of (2) admit an exponential dichotomy. The author states (without proof) the following generalization of a result of Demidovic and Corduneanu. Let $\varphi(r)$ be defined and continuous in $[0,\infty]$ with $\varphi(f) = O(r)$ for $r \to \infty$. Let also $h(x,t)$ be weakly continuous, $X \times J \to X$, and $\|h\| \leq \varphi(\|x\|)$. Then, for every φ_0 in X_0 there exists at least one bounded solution $x(t,\zeta)$ of (1) such that the projection $x(0,\zeta)$ in X_i is ζ.

Similar result: Under the same hypotheses, if $A(t)$ and $h(x,t)$ have period 1 in t, (1) has at least one solution with the same period.

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[References: the author and Schaffer, Ann. of Math. (2) 67 (1958), 517–573; MR 20#3466; ibid. 69 (1959), 535–574; MR 21#3638; Math. Ann. 139 (1960), 287–342; MR 22#8181; Schaffer, ibid. 140 (1960), 308–321; MR 22#8182.] *S. Lefschetz*

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Massera, José L.

On the existence of Lyapunov functions.

Fac. Ingen. Montevideo Publ. Inst. Mat. Estadíst **3** 1959/1960 111–124

In a previous paper [Ann. of Math. (2) 64 (1956), 182–206; MR 18, 42] the author proved some theorems about the existence of a Lyapunov function $V(x,t)$ when the trivial solution of the system $\dot{x} = f(x,t)$, $f(0,t) = 0$, has a certain type of stability. Subsequently, a more general result was proved independently by Kurzweil [Czechoslovak Math. J. 6 (81) (1956), 217–259, 455–484; MR 19, 33], but the proof is much more complicated. In the present paper the author corrects some details of his proof. *M. M. Peixoto*

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Massera, J. L.

Function spaces with translations and their application to linear differential equations. 1961

Proc. Internat. Sympos. Linear Spaces (*Jerusalem*, 1960) *pp.* 327–334 *Jerusalem Academic PRess*, *Jerusalem*; *Pergamon*, *Oxford*

This is a summary of work by the author and the reviewer on the applications of functional analysis to linear differential equations. The use of function spaces with translations, studied by the reviewer [Math. Ann. 137 (1959), 209–262; MR 21#287; ibid. 138 (1959), 141–144; MR 21#6529] is stressed. The main results mentioned in this paper are contained in a paper of Massera and the reviewer [ibid. 139 (1960), 287–342; MR 22#8181]. *J. J. Schäffer*

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Massera, J. L.

Converse theorems of Lyapunov's second method.

Bol. Soc. Mat. Mexicana (2) **5** 1960 158–163

This paper is a very good summary of the problems involved and the results that have been obtained concerning the converse theorems of Ljapunov's second method. *J. K. Hale*

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Massera, José L.

Differential equations and functional analysis. (Spanish)

Rev. Un. Mat. Argentina **19** 1960 179–186 (1960)

Expository summary of the author's joint papers with J. J. Schaffer of similar title, especially the most recent one [Math. Ann. 139 (1960), 287–342; MR 22#8181].

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Massera, J. L.; Schäffer, J. J.

Linear differential equations and functional analysis.

Bol. Soc. Mat. Mexicana (2) **5** 1960 42–48

This is a report (without proofs) on partially completed and published research by the authors [Ann. of Math. (2) 67 (1958), 517–573; 69 (1959), 88–104, 535–574; Math. Ann. 139 (1960), 287–342; MR 20#3466; 21 #756, 3638; 22 #8181] on the application of functional analysis to linear (and some non-linear) ordinary differential equations. The equations considered are essentially of the form $\dot{x}(t)+A(t)x(t)=f(t)$ where x and f are elements of Banach function spaces D and B, the central problem being to establish, given f in B, the existence of a solution in D. Among the related papers are those of Krein [Uspehi Mat. Nauk 3 (1948), no. 3 (25), 166–169; MR 10, 128] and Kucer [Dokl. Akad. Nauk SSSR 69 (1949), 603–606; MR 11, 360]. *H. H. Schaefer*

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Massera, J. L.

Sur l'existence de solutions bornées et périodiques des systèmes quasi-linéaires d'équations différentielles. (French)

Ann. Mat. Pura Appl. (4) **51** 1960 95–105

Let $t\to A(t)$ be a mapping of $J=[0,\infty)$ into the Banach space of continuous linear operators of a real or complex reflexive Banach space X into itself which is uniformly Bochner integrable in every bounded subinterval of J, and suppose that the set $X_0\subset X$ of initial points of bounded solutions of (1) $x'+A(t)x=0$ is a closed subspace and has a closed complement. Let h be a weakly continuous mapping of $X\times J$ into X such that $\|h(x,t)\|\le\varphi(\|x\|)$ where φ is continuous in J and $\varphi(r)=o(r)$ as $t\to\infty$. The author proves that if there exists an exponential dichotomy for the solutions of (1) then (2) $x'+A(t)x=h(x,t)$ has for each $x_0\in X_0$ at least one bounded solution u such that $P_0u(0)=x_0$, where P_0 is the projection associated with

X_0. Moreover, if $A(t+1) = A(t)$, $h(x, t+1) = h(x, t)$ for every $t \in J$, $x \in X$, then (2) has at least one periodic solution with period 1. This extends earlier joint results of the author and Schaffer [Ann. of Math. (2) 67 (1958), 517–573; 69 (1959), 88–104; MR 20#3466; 21 #756] where h is supposed to be Lipschitzian in $X \times J$ for a sufficiently small constant. It also generalizes a theorem of Demidovic [Mat. Sb. (N.S.) 40 (82) (1956), 73–94; MR 18, 738] for finite-dimensional X in which A is assumed to be constant and to have eigenvalues with non-zero real part (which is equivalent to exponential dichotomy). The present result on the existence of a bounded solution of (2) is very similar to a theorem of Corduneanu for the case $X = R^n$ [An. Sti. Univ. "Al. I. Cuza" Iasi. Sect. I (N.S.) 5 (1959), 33–36; MR 22#5766]. *H. A. Antosiewicz*

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Massera, J. L.; Schäffer, J. J.

Linear differential equations and functional analysis. IV.

Math. Ann. **139** 1960 287–342 (1960)

[For part III see Ann. of Math. (2) 69 (1959), 535–574; MR 21#3638.] Let X be a Banach space, $J = [0, \infty)$, and suppose $t \to A(t)$ is a mapping of J into the Banach space of continuous linear operators of X into itself which is uniformly Bochner integrable on every bounded subinterval of J. A closed linear subspace Y of X is said to induce a dichotomy [resp. exponential dichotomy] of the solutions of (I) $x' + A(t)x = 0$ if there exist positive constants $\gamma < 1$, γ_0, N_i [and ν_i], $i = 0, 1$, such that: $x(0) \in Y$ implies, for $t \ge t_0 \ge 0$,

(1) $\|x(t)\| \le N_0 \|x(t_0)\|$

$$\text{[resp. } \|x(t)\| \le N_0 \exp(-\nu_0(t - t_0)) \|x(t_0)\| \text{]};$$

$x(0) \in' Y$ and $\gamma[Y, x(0)] \ge \gamma$ imply, for $t \ge t_0 \ge 0$,

(2) $\|x(t)\| \ge N_1 \|x(t_0)\|$

$$\text{[resp. } \|x(t)\| \ge N_1 \exp(\nu_1(t - t_0)) \|x(t_0)\| \text{]};$$

$x_0(0) \in Y$, $x_1(0) \in' Y$, $x_i(0) \ne 0$, and $\gamma[Y, x_1(0)] \ge \gamma$ imply $\gamma[x_0(t), x_1(t)] \ge \gamma_0$ for $t \ge 0$. Here $\gamma[y, x] = \big\| y/\|y\| - x/\|x\| \big\|$ and $\gamma[Y, x] = \inf\{\gamma[y, x] : y \in Y - \{0\}\}$ for $x \in' Y$. If $Y = \{0\}$ it is only required that $x(0) \in' Y$ imply (2).

In part I [ibid. 67 (1958), 517–573; MR 20#3466] the authors proved, among others, the following result concerning the linear manifold $X_0 \subset X$ of initial points $x(0)$ of bounded solutions of (I). If X_0 is closed and has a closed complement, then X_0 induces a dichotomy if

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and only if (II) $x'+A(t)x=f(t)$ has at least one bounded solution for every $f\in L^1$; and X_0 induces an exponential dichotomy if and only if (II) has at least one bounded solution for every $f\in L^p$, $1<p\le\infty$, provided $(*)$ $\sup_{t\in J}\int_t^{t+1}\|A(s)\|\,ds<\infty$.

In the present part IV the authors attack the question as to when a closed linear subspace of X induces an ordinary or exponential dichotomy in a much more general setting and from a more unified point of view. Their new results represent not only considerable extensions and refinements of their earlier ones; also they concern new types of stability properties, less local in nature than (1), (2), in that they are formulated in terms of different norms and mean values over certain "slices" of the solutions of (I). Only a bare outline of the principal results can be given here.

Let B, D be Banach spaces of mappings of J into X such that convergence in either implies convergence in the mean on every bounded subinterval of J. The authors call the pair (B,D) admissible if, for every $f\in B$, (II) has at least one solution which belongs to D. The pair (B_1,D_1) is stronger than (B_2,D_2) if B_2, D_1 are stronger than B_1, D_2, respectively. A pair (B,D) is a $\mathcal{T}$-pair if, roughly speaking, B and D are translation invariant [see Schaffer, Math. Ann. 137 (1959), 209–262; 138 (1959), 141–144; MR 21#287, 6529].

In what follows, suppose that (B,D) is an admissible $\mathcal{T}$-pair (whence D contains non-trivial solutions). Let X_{0D} be the linear manifold of initial points $x(0)$ of solutions of (I) which belong to D, and suppose that X_{0D} is closed (but its complement need not be closed) and $X_{0D}\neq\{0\}$. Then for every solution of (I): $x(0)\in X_{0D}$ implies, for $t\ge t_0\ge0$,

$$\int_t^{t+\Delta}\|x(s)\|\,ds\le M_0(\Delta)\int_{t_0}^{t_0+\Delta}\|x(s)\|\,ds,$$

$$\|\chi_{[t,t+\Delta]}x\|_D\le M_0(\Delta)\|\chi_{[t_0,t_0+\Delta]}x\|_D; \tag{3}$$

and $x(0)\in' X_{0D}$, $\gamma[X_{0D},x(0)]\ge\gamma$, imply, for $t\ge t_0\ge0$,

$$\int_t^{t+\Delta}\|x(s)\|\,ds\ge M_1(\Delta,\gamma)\int_{t_0}^{t_0+\Delta}\|x(s)\|\,ds,$$

$$\|\chi_{[t,t+\Delta]}x\|_D\ge M_1(\Delta,\gamma)\|\chi_{[t_0,t_0+\Delta]}x\|_D. \tag{4}$$

Here χ_I is the characteristic function of I and $(-1)^iM_i$ are non-increasing with $\Delta>0$. If, in addition, (B,D) is not weaker than $(L^1,L_0{}^\infty)$, then the inequalities (3), (4) hold with their right-hand sides multiplied by $e^{-\nu_0(t-t_0)},e^{\nu_1(t-t_0)}$, respectively. Moreover, if D is also stronger than L^∞, then every solution of (I) which belongs to D

satisfies, for $t \geq t_0 \geq 0$,

$$\|x(t)\| \leq R(t_0)\|x(t_0)\|e^{-\nu(t-t_0)},$$

where $R(t_0)$ can not be taken independent of t_0 in general. Using these results, the authors then prove the following theorems on dichotomies.

If (B,D) is stronger than $(\Theta_\tau L^1, L^\infty)$ for some $\tau \geq 0$, where $\Theta_\tau L^1$ denotes the subspace $\{\chi_{(\tau,\infty)}u\colon u \in L^1\}$, then X_{0D} induces a dichotomy. Conversely, if a closed linear subspace induces a dichotomy then (L^1, L^∞) is admissible. If (B,D) is not stronger than $(\Theta_\tau L^1, L^\infty)$ then X_{0D} need not induce a dichotomy unless some other condition such as $(*)$ is imposed, in which case X_{0D} always induces a dichotomy.

If (B,D) is not weaker than $(L^1, {L_0}^\infty)$ and if there exists a subspace Y which induces a dichotomy, then X_{0D} induces an exponential dichotomy. This is a necessary condition as well provided $(*)$ holds.

As in the previous parts, the authors again give numerous examples to illustrate the interplay between the various hypotheses and to show that in almost all cases the results obtained are best possible. They also point out that, given a $\mathcal{T}$-pair (B,D), there does not in general exist an equation (II) with respect to which it is admissible.

H. A. Antosiewicz

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Massera, J. L.

A criterion for the existence of almost periodic solutions of certain systems of almost periodic differential equations. (Spanish. English summary) 1958

Bol. Fac. Ingen. Agrimens. Montevideo 6 (1957/58) 345–349 *Also published as Fac. Ingen. Montevideo. Publ. Inst. Mat. Estadist.* 3

The author considers the "triangular" system of equations

$$\dot{x}_m + a_{m1}(t)x_1 + \cdots + a_{mm}(t)x_m = f_m(t) \quad (m = 1, \cdots, n)$$

with real almost periodic $a_{pq}(t)$, and he poses the problem, apparently novel, under what conditions on the $a_{pq}(t)$ for every set of real almost periodic $f_m(t)$ there will be at least one almost periodic set of solutions x_m. He finds a necessary and sufficient condition, namely, that the mean values of the diagonal coefficients shall be 0; and the almost periodic solutions are then even unique. A related result if the $f_m(t)$ are replaced by more general expressions $f_m(x_1, \cdots, x_n, t)$ which vary slowly with the $x_1, \cdots, x_n$. *S. Bochner*

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Massera, J. L.; Schäffer, J. J.

Linear differential equations and functional analysis. III. Lyapunov's second method in the case of conditional stability.

Ann. of Math. (2) **69** 1959 535–574

In this third part of their work on linear differential equations

$$\dot{x} + A(t)x = 0 \tag{1}$$

in a Banach space X [for parts I, II see same Ann. (2) 67 (1958), 517–573; 69 (1959), 88–104; MR 20#3466; 21 #756], the authors take up the (very considerable) problem of extending Lyapunov's second method to the case of conditional, i.e., non-uniform, simple and asymptotic stability. They are able to prove both "direct" and "converse" theorems and thereby establish the equivalence between the existence of suitably generalized Lyapunov functions with certain properties and the asymptotic behavior of the bounded and unbounded solutions of (1). These theorems they combine with previous ones of part I to obtain a complete characterization, in terms of Lyapunov functions, of the existence of at least one bounded solution of

$$\dot{x} + A(t)x = f(t) \tag{2}$$

with f from a certain function space. The importance of these results, as indeed of all results in Lyapunov's method, lies in the fact that they provide testable criteria in the sense that all properties of the Lyapunov functions involve solely the (homogeneous) differential equation and require therefore no knowledge of the solutions at all.

Only a bare outline of the wealth of theorems proved in this paper can be given here. Recall that, throughout, A is a continuous endomorphism of X, defined on $J = [0, \infty)$ and uniformly Bochner integrable on every finite subinterval of J; M denotes the space of such endomorphisms (or functions) for which $\int_t^{t+1} \|A(t)\|\, dt$ is bounded on J.

The authors first define the concepts of a Lyapunov function $V(x, t)$ on $X \times J$ and of a total derivative $V'(x, t)$ in such a way that the following crucial implication is true under sufficiently general assumptions: If, for a continuous function $a(x, t)$ on $X \times J$, $V' \geq a$ holds "almost everywhere" (in a sense made precise in the paper),

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then for any $t \geq t_0 \geq 0$

$$V[x(t), t] - V[x(t_0), t_0] \geq \int_{t_0}^{t} a[x(t), t]\, dt,$$

where $x(t)$ is a solution of (1). The terms positive definite, infinitely small upper bound are defined as usual. V' is called positive almost definite if V' is positive definite save on a set H such that either its projection on J is a Lebesgue null set or, if $\dim X < \infty$, H itself is such a set; and if V has certain regularity properties. The first two main theorems can then be stated, in part, as follows.

(I) If $A \in M$ and if there exists a V with infinitely small upper bound and one of its total derivatives V' is negative almost definite, then (i) the set X_0 of initial values of bounded solutions is closed and

$$\|x(t)\| \leq N\|x(t_0)\| e^{-\nu(t-t_0)}, \quad t \geq t_0 \geq 0, \tag{3}$$

N, ν being positive constants; (ii) if ${X_1}'$ is a subspace such that either $x \in {X_1}'$ implies $V(x, 0) \leq 0$ $[V(-x, 0) \leq 0]$ or ${X_1}' \cap X_0 = \varnothing$ and $\dim {X_1}' < \infty$, then $x(0) \in {X_1}'$ implies

$$\|x(t)\| \geq N'\|x(t_0)\| e^{\nu'(t-t_0)}, \quad t \geq t_0 \geq 0, \tag{4}$$

N', ν' being positive constants; (iii) if $x_0(0) \in X_0$, $x_1(0) \in {X_1}'$ then (5) $\alpha[x_0(t), x_1(t)] \geq \alpha_0$ for $t \geq 0$, where α is the angular distance defined in part I. The most striking feature is that V is not required to be positive definite as it is in the classical theory. There $A \in M$ need not hold; here (I) is false if $A \in' M$.

(II) If $X_0, X_1 = \mathbf{C}X_0$ are closed, and if $x_0(t_0) \in X_0$ implies (3), $x_1(0) \in X_1$ implies (4), and $x_i(0) \in X_i$ imply (5); then there exist non-negative functions V_0, V_1 such that every total derivative of $V_0 - V_1$ is negative almost definite.

Combined with results from part I these theorems yield the following.

(III) If the hypotheses of (I) hold and if either codim $X_0 < \infty$ or $X_1 = \mathbf{C}X_0$ is closed and $x \in X_1$ implies $V(x, 0) \leq 0$ $[V(-x, 0) \leq 0]$, then (2) has at least one bounded solution for every $f \in M$.

(IV) If $A \in M$ and $X_0, X_1 = \mathbf{C}X_0$ are closed and if the assertion of (III) holds, then the assertion of (II) holds.

The authors point out that these results are closely related to work of Krasovskii [Mat. Sb. (N.S.) 40 (82) (1956), 57–64; MR 19, 34].

In the case of simple conditional stability the authors prove theorems of analogous type. Roughly speaking, they are able to infer from the existence of non-negative, positively homogeneous Lyapunov functions that every bounded solution satisfies $\|x(t)\| \leq N_0\|x(t_0)\|$,

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$t \geq t_0 \geq 0$, and for the other solutions $\|x(t)\| \geq N'\|x(t_0)\|$. Conversely, if this dichotomy holds, there exist such functions. By virtue of theorems of part I there follows a characterization of the bounded solutions of (2) similar to that in (III) and (IV).

Throughout the paper the authors give numerous examples to illustrate the importance of the various hypotheses in the theorems. The methods of proof are essentially classical. *H. A. Antosiewicz*

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Massera, J. L.; Schäffer, J. J.

Linear differential equations and functional analysis. II. Equations with periodic coefficients.

Ann. of Math. (2) **69** 1959 88–104

[For part I, see same Ann. 67 (1958), 517–573; MR 20#3466.] Given a vector-operator equation of the form $x' = A(t)x + f(t)$, where x is an element in a Banach space, a problem of some interest is that of determining properties of the homogeneous equation from those of the solutions of the inhomogeneous equation when $f(t)$ ranges over a suitable set of vector functions. Continuing this investigation begun in their first paper, the authors consider the case in which $A(t)$ is periodic. By means of a generalized Floquet representation which they obtain, they are able to derive a number of results concerning the existence of periodic solutions of the homogeneous equation.

R. Bellman

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Massera, J. L.; Schäffer, J. J.

Linear differential equations and functional analysis. I.

Ann. of Math. (2) **67** 1958 517–573

In 1930, Perron initiated the problem of determining properties of the solution of the nonlinear differential equation $x' = A(t)x + h(x,t)$ from a knowledge of uniform properties shared by all solutions of equations of the form $x' = A(t)x + h(t)$, where $h(t)$ runs over some function class. Although he used classical analytic techniques, it was realized by Krein [Uspehi Mat. Nauk (N.S.) 3 (1948), 166–169; MR 10, 128], Kucer [Dokl. Akad. Nauk SSSR (N.S.) 69 (1949), 603–606; MR 11, 360] and Bellman [Ann. of Math. (2) 49 (1948), 512–522; MR 10, 2] that these problems could be treated by Banach space techniques. In this very detailed paper, the authors tackle, by means of these techniques, but using a more powerful method, the more difficult problem of deducing properties of some of the solutions of

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the nonlinear differential equation from a knowledge of some of the solutions of the set of linear equations.

In addition to deriving a large number of interesting results, the authors present a number of examples showing in what way some of the results are best possible. *R. Bellman*

19,1047b 31.0X

Massera, J. L.; Schäffer, J. J.

Correction to the article "On the level curves of a convex surface". (Spanish. English summary)

Bol. Fac. Ingen. Agrimens. Montevideo 6 = *Fac. Ingen. Agrimens. Montevideo. Publ. Inst. Mat. Estadist.* **3** (1957), 65–67.

In the paper mentioned in the title [Bol. Fac. Ingen. Montevideo 4 (1953), 665–668; MR 15, 737], the authors had stated a theorem giving necessary and sufficient conditions for the existence of a change of the independent variable t transforming a family of non-decreasing functions $f(t)$ simultaneously into convex functions. In the present note, it is announced that the proof was not correct and that an assumption, viz., the existence of left (or right) limits of the derivatives of the functions $f(t)$, has to be added. Correct versions of the theorem and of the part of the proof in question are given. *W. Fenchel*

18,900e 34.0X

Massera, José L.

On the stability of spaces of infinite dimension. (Spanish)

Rev. Un. Mat. Argentina **17** (1955), 135–147 (1956).

The author considers a differential equation $x' = f(x,t)$ in which x and f have values in a Banach space. It is proved that, if $x \equiv 0$ is a uniformly asymptotically stable solution, then there exists a (generalized) Lyapunov function $V(x,t)$ for the differential equation. As a preparation, the author also proves theorems on dependence of solutions on initial conditions and on parameters. *W. Kaplan*

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18,227g 50.0X

Massera, J. L.

On the fundamental notions of projective geometry. (Spanish)

Bol. Fac. Ingen. Agrimens. Montevideo **5** (1956), 408–458 = *Publ. Didact. Inst. Mat. Estadist.* 1 (1956), 1–56.

Les disciplines mathematiques modernes se presentent comme un ensemble de propositions logiquement deduites de certaines propositions initiales admises (postulats). Cette presentation assure la rigueur, mais l'auteur ne pense pas que l'on doive regarder la science mathematique comme un simple jeu logique a partir d'un systeme arbitraire de postulats et de definitions. Les postulats et les principales definitions d'une veritable theorie mathematique proviennent de l'examen de la realite materielle du sujet ou sont naturellement suggeres par le developpement interne meme de la theorie. L'auteur expose ses idees sur le sujet, en prenant comme exemple la geometrie projective. Il montre comment, partant de la notion intuitive d'espace, et precisant les significations de certaines locutions courantes concernant la geometrie euclidienne, on arrive au systeme des dix postulats a partir desquels on peut deduire tous les theoremes de la geometrie projective. Un appendice contient un certain nombre de considerations historiques, et nous fait connaitre les opinions de quelques-uns des geometres qui ont le plus contribue a eriger la geometrie projective en corps de doctrine, et plus particulierement de Desargues, Descartes, Poncelet et Chasles. *P. Vincensini*

18,211a 34.0X

Massera, José L.

Qualitative study of the equation $u''^2 = u + u'$. (Spanish)

Bol. Fac. Ingen. Agrimens. Montevideo **5** (1956), 339–347 = *Fac. Ingen. Agrimens. Montevideo. Publ. Didact. Inst. Mat. Estadist.* 5 (1956), 1–10.

The problem of determining whether or not the equation $u' + u = (u'')^2$ has a solution which approaches zero like e^{-t} as $t \to \infty$, for suitably chosen values of $u(0)$ and $u'(0)$ was proposed by the reviewer [Bull. Amer. Math. Soc. 61 (1955), 192]. In this paper, the author furnishes a complete solution to the problem, with an affirmative answer, by means of a detailed study of the family of solutions of the above differential equation. *R. Bellman*

{For errata and/or addenda to the original MR item see MR 18,1118 Errata and Addenda in the paper version}

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18,42d 34.0X

Massera, José L.

Contributions to stability theory.

Ann. of Math. (2) **64** (1956), 182–206.

This is a thorough study of necessary and sufficient conditions, of the type considered in Lyapunov's second second method, for the stability (simple, asymptotic, asymptotic in the large, total, in the first approximation) and the instability of the solution $x = 0$ of equations $\dot{x} = f(t,x)$ with f continuous on some semi-cylinder $L \times S$, $L = [0,\infty)$, $S = S(0,p)$, and $f(t,0) \equiv 0$ on L. The results, most of which hold in general Banach spaces, represent important advances in stability theory.

Generalizing his earlier work [Ann. of Math. (2) 50 (1949), 705–721; MR 11, 721], on which Malkin [Prikl. Mat. Meh. 18 (1954), 129–138; MR 15, 873] and Barbasin and Krasovskii [ibid. 18 (1954), 345–350; MR 15, 957] based their criteria for the uniform-asymptotic stability (u.a.s) and the u.a.s. in the large of $x = 0$, respectively, the author proves the following much stronger theorems: If $x \in E^n$ and f is locally Lipschitzian on $L \times S$, the u.a.s. (the u.a.s. in the large) of $x = 0$ implies the existence of a positive definite Lyapunov function $V(t,x)$ with $V(t,x) \to 0$ as $x \to 0$ uniformly in t, (and $V(t,x) \to \infty$ as $\|x\| \to \infty$ uniformly in t), having continuous Frechet differentials of all orders, such that $\delta[V(t,x);(1,f(t,x)]$ is negative definite. Among the other results, too numerous to be stated in detail here, are generalizations of a theorem of Lyapunov [cf., e.g., Probleme general de la stabilite du mouvement, Princeton, 1947, pp. 276–278; MR 9, 34] on linear equations with constant coefficients, of a theorem of Cetaev [Dokl. Akad. Nauk SSSR (N.S.), 1 (1934), 529–531; Uc. Zap. Kazan. Gos. Univ. 98 (1938), no. 9, 43–58] and Persidskii [cf., e.g., Uspehi Mat. Nauk (N.S.) 1 (1946), no. 5–6(15–16), 250–255; MR 10, 456] on instability, of a theorem of Malkin [Dokl. Akad. Nauk SSSR (N.S.) 76 (1951), 783–784; MR 12, 827] on stability in the first approximation, and of theorems of Malkin [Prikl. Mat. Meh. 8 (1944), 241–245; MR 7, 298; 11, 439] and Gorsin [Izv. Akad. Nauk Kazah. SSR. 56, Ser. Mat. Meh. 2 (1948), 46–73; MR 14, 48] on total stability.

H. A. Antosiewicz

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16,925c 36.0X

Massera, J. L.

Sur un théorème de G. Sansone sur l'équation di Liénard. (French)

Boll. Un. Mat. Ital. (3) **9,** (1954). 367–369

G. Sansone has proved [Univ. e Politec. Torino Rend. Sem. Mat. 10, 155–171 (1951); MR 13, 746] that the differential equation $d^2i/dt^2 + \omega f(i)di/dt + \omega^2 i = 0$ possesses one and only one periodic solution, provided $f(i)$ satisfies certain hypotheses one of which is that $|f(i)| < 2$. The author shows that this particular hypothesis may be omitted.

W. Wasow

15,965b 46.0X

Massera, Jose L.

Conditional stability of homeomorphisms. (Spanish. English summary)

Bol. Fac. Ingen. Montevideo **4,** (1952). 455–486

Let the Banach space $Z = X \times Y$ be the topological product of Banach spaces X and Y. Let T be a homeomorphism of a neighborhood of the origin 0 of Z onto a neighborhood of 0 leaving 0 fixed. A set E is called a set of stability if, given any neighborhood W of 0, there exists a neighborhood W' of 0 such that, for each $n \geq 0$, $T^n(E \cap W') \subset W$; E is called a set of instability if it is a set of stability for T^{-1}. The object is to connect the existence and properties of sets of stability with regularity properties of T, which vary from Frechet differentiability to analyticity. In most theorems either X or Y is finite-dimensional or T^{-1} is differentiable at 0. Under detailed hypotheses detailed results are obtained, for example, on asymptotic stability, on maximality (in various senses) of sets of stability, and on transformations $x \to y$ (on sets $\|x\| \leq a$ into Y) that are invariant under T and have graphs that are sets of stability.

F. A. Ficken

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15,957g 36.0X

Massera, Jose L.

Total stability and approximately periodic vibrations. (Spanish. English summary)

Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística **2,** (1954). 135–145

The system $(*)$ $\dot{x} = F(x,t)$ is studied with x an n-component vector, F smooth enough to ensure existence and uniqueness of solutions with given initial data, and $F(0,t) \equiv 0$. The solution $x = 0$ is totally stable if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|F_1 - F\| \le \delta$ for $\|x\| \le \varepsilon$ and $t \ge 0$, and $\dot{x}_1(t) = F_1(x_1(t), t)$ with $\|x_1(0)\| \le \delta$, then $\sup_{t\ge 0} \|x_1(t)\| \le \varepsilon$. As the author notes, this definition agrees with that of stability under constantly acting perturbations studied by I. Malkin [Akad. Nauk SSSR. Prikl. Mat. Meh. 8, 241–245 (1944) = Amer. Math. Soc. Translation no. 8 (1950); these Rev. 7, 298; 11, 439]. Malkin's sufficient condition for total stability is also established here. Moreover, if F is T-periodic in t and $(*)$ has a solution x that is T'-periodic in t with T' and T incommensurable, then in order that x be totally stable it is sufficient, under certain other conditions, that the form $\sum y_i y_j \partial f_i(x(t), t)/\partial x_j$ be negative definite for $t \ge 0$. Other theorems involve orbital stability, and approximate periodicity in the following sense: $x(t)$ is $\varepsilon^+ - T$-periodic if given $\varepsilon > 0$ there exists a sequence t_n, with $t_0 = 0$ and $0 < T - \varepsilon < t_{n+1} - t_n < T + \varepsilon$, such that $|x(t_n + \tau) - x(\tau)| < \varepsilon$ for $0 \le \tau \le t_{n+1} - t_n$. *F. A. Ficken*

15,737e 52.0X

Massera, J. L.; Schäffer, J. J.

On the level curves of a convex surface. (Spanish)

Bol. Fac. Ingen. Montevideo **4,** (1953). 665–668

In connection with investigations into the problem of whether or not a nested one-parameter family of convex curves is the set of level curves of a convex function, the authors consider the following problem: given a family F of non-decreasing functions $f(t)$ on the interval $0 \le t \le 1$, does there exist a change of variable $t = T(s)$, monotone, $T(0) = 0$, $T(1) = 1$, such that $f(T(s))$ is convex for all f in F? Toward solving this problem, they define the negative variation $V^-(F, a, b)$ of a family F over an interval $[a, b]$ in the same manner as the negative variation of a function is defined, except that the supremum is taken simultaneously over all f in F as well as over all subdivisions of $[a, b]$. Here L is defined to be the family of all $f'(t)$, f in F. The following theorem is proved: If F contains the function $f(t) \equiv t$, a necessary and

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sufficient condition for the function $T(s)$ as described above to exist is that (a) all the functions of F are absolutely continuous, (b) if $[t', t'']$ is an interval completely interior to $[0,1]$, $V^-(L, t', t'') < \infty$ (V^- is taken to be positive), (c) $\int_0^1 \exp[-V^{-1}(L, \tau, t)]\,dt < \infty$. *J. W. Green*

{For errata and/or addenda to the original MR item see MR 18,1118 Errata and Addenda in the paper version}

13,944e 36.0X

Massera, José L.

Remarks on the periodic solutions of differential equations. (Spanish)

Bol. Fac. Ingen. Montevideo **4,** (1950). (*Año* 14), 37–45 = 43–53
Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 2, 43–53 (1950).

Systems $\dot{x} = f(x, t)$, $x = (x_1, \cdots, x_n)$, $f = (f_1, \cdots, f_n)$ are considered where the right-hand side of the equation is periodic in t, with period T. The author points out that if $n \geq 2$, such a system may well admit periodic solutions whose period T' is incommensurable with that of the system. Such a solution is a simple closed curve, tangent to the given field, passing through no singular point of the system (i.e. $f \neq 0$), and along which f does not depend on t. These conditions are essentially sufficient. Applications of this result are made to systems of a more special type. *F. Bohnenblust*

13,768j 52.0X

Massera, J. L.; Schäffer, J. J.

Minimum figures covering points of a lattice. (Spanish. English summary)

Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística **2,** (1951). 55–74

Let L be the lattice of all parts with integral coordinates in the cartesian plane. A set M is called covering if any set obtained from M by a motion of the plane contains at least one point of L. Santalo put the problem to find the greatest lower bound of the measures of all covering sets. Here various covering sets are constructed, in particular a convex set C of area $4/3$. It is shown that any convex covering set has at least area $2 - 2^{-1/2}$ and it is conjectured that C yields the minimum for convex sets. A sufficient condition for a convex set to be covering is derived, which is also necessary when the diameter of the set is less than $5^{1/2}$. A necessary and sufficient condition is given for

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a convex covering set not to contain a proper convex subset which is still covering. *H. Busemann*

12,705g 36.0X

Massera, José L.

The existence of periodic solutions of systems of differential equations.

Duke Math. J. **17,** (1950). 457–475

Let differential equations (1) $dx_i/dt = X_i(x_1, \cdots, x_m, t)$, $i = 1, \cdots, m$, be given, where the X_i are smooth for all $x = (x_1, \cdots, x_m)$ and t and have period 1 in t. The author establishes several results, partly negative, concerning the existence of solutions $x = x(t)$ of (1) having period 1 or multiples thereof (subharmonics). Thus, for $m = 2$, a solution of period 1 is assured if all solutions $x(t)$ can be continued for arbitrarily large t and if at least one solution is bounded for t sufficiently large. Again for $m = 2$ it is shown that there exist systems (1) having a prescribed number $N(q)$ of solutions of period q, $q = 1, 2, \cdots$, and no other periodic solutions; here $N(q)$ is 0 or ∞ or a multiple of q. A similar statement holds for $m > 2$, even with analytic X_i and the requirement that all solutions $x(t)$ be defined for $-\infty < t < \infty$; in particular one can have $N(q) \equiv 0$ and still have at least one bounded solution. *W. Kaplan*

11,721f 36.0X

Massera, J. L.

On Liapounoff's conditions of stability.

Ann. of Math. (2) **50,** (1949). 705–721

A stable solution $x(t) = (x_1(t), \cdots, x_n(t))$, say $x = 0$ for simplicity, of a system $\dot{x}(t) = X(x, t)$ of ordinary differential equations is called by Liapounoff asymptotically stable if there exists a $\delta > 0$ such that every solution whose initial conditions at $t = 0$ differ from 0 by less than δ tends to 0 as t tends to infinity. The present author introduces the stronger notion of equi-asymptotic stability by requiring the existence of a positive δ such that the $\lim x(t) = 0$ as $t \to \infty$ be uniform for all initial conditions $|x(0)| \leq \delta$. The relationship between these two notions of asymptotic stability is investigated. In general they are different, but coincide for systems which are either of order 1, linear, periodic, or autonomous. Several known theorems on asymptotic stability are strengthened, both by weakening the assumptions and by showing that equi-asymptotic stability is implied. These results lead to necessary and sufficient conditions for asymptotic stability when the system is either linear or periodic. *F. Bohnenblust*

10,709e 36.0X

Massera, J. L.

The number of subharmonic solutions of non-linear differential equations of the second order.

Ann. of Math. (2) **50,** (1949). 118–126

The system $\dot{x} = F(x, y, t)$, $\dot{y} = G(x, y, t)$ of differential equations is considered. The functions F and G are periodic with period 1 and such that for any initial conditions (x_0, y_0, t_0) the solution exists for all $t \geq t_0$ and is unique. A circle and an integer N are assumed to exist such that (1) every solution enters the circle and (2) if at t_1 the solution is in the circle then it remains in this circle for all $t \geq t_1 + N$. The number $N(q)$ of subharmonic solutions of least period q is investigated under the assumption that the periodic solutions of rational periods are simple. In a previous paper, N. Levinson [Ann. of Math. (2) 45, 723–737 (1944); these Rev. 6, 173] proved that $N(q)$ is an even multiple of q for $q > 1$. Masser pointed out that this proof applies only for q odd and that the result must be modified for q even [cf. Levinson, Ann. of Math. (2) 49, 738 (1948); these Rev. 10, 457]. The present paper contains the details of this correction and also a study of examples showing that the conditions which are obtained for $N(q)$ are necessary and sufficient. *F. Bohnenblust*

9,511c 36.0X

Levi, Beppo; Massera, José L.

Study in the large of a differential equation of the second order. (Spanish)

Math. Notae **7,** (1947). 91–155

The nonlinear differential equation $yy'' = x$ is studied in great detail and essentially all qualitative features of the solutions are obtained. The solutions with the singular initial values $y = y_0 \neq 0$, $x = 0$ are obtained in power series form; for $y_0 = 0$, the curve $3y^2 - 4x^3 = 0$ is shown to be the unique solution. The differential equation is shown to be invariant under a group of transformations: $x = \lambda^2 X$, $y = \lambda^3 Y$, and a corresponding reduction to a first order equation is made. The latter equation is then studied thoroughly and the results obtained permit a completion of the analysis of the given second order equation.

W. Kaplan

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8,142a 27.2X

Massera, José L.

On Green's formula. (Spanish)

Publ. Inst. Mat. Univ. Nac. Litoral **6,** (1946). 169–178

The elementary proof of Green's lemma in the plane presented in this paper follows the pattern used by the reviewer [Amer. J. Math. 63, 563–574 (1941); these Rev. 3, 75], with the author giving an alternate proof of a central result on the approximation of a simple closed rectifiable plane curve by simple closed polygons interior to this curve. In addition, the usual conditions on the partial derivatives are relaxed somewhat, with the general result stating the equality of a line integral and an iterated integral. *W. T. Reid*

8,28c 39.0X

Massera, J. L.; Petracca, A.

On the functional equation $f(f(x)) = 1/x$. (Spanish)

Revista Unión Mat. Argentina **11,** (1946). 206–211

To solve the functional equation (1) of the title, the author observes that the transformation $y = 1/x$ can be obtained from successive inversions in the unit circle and the real axis; by a linear transformation the x-plane can be mapped on an x'-plane with the circle and axis being carried into the x'-coordinate axes; successive inversions in the coordinate axes of the x'-plane result in the transformation $y' = -x'$. Consequently, the solution of (1) is equivalent to the solution of (2) $f(f(x)) = -x$. It is then shown that the general solution of (2) is given implicitly by $F(x^2, f^2) = 0$, where F is an arbitrary symmetric function; thus (1) is also solved. *E. F. Beckenbach*

7,488a 65.0X

Massera, Jose L.

The method of Graeffe for solving algebraic equations. (Spanish)

Bol. Fac. Ingen. Montevideo **3,** (1945). (*Año* 10), 1–20

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6,203i 27.2X

Massera, José L.

On differentiable functions. (Spanish. French summary)

Bol. Fac. Ingen. Montevideo **2,** (1944). (*Año* 9), 647–668

This paper is concerned with functions which have a finite or infinite derivative at each point of a closed interval. In the first part, a variety of elementary theorems concerning derivatives are proved by using some simple properties of the difference quotient $p(a, b) = (f(a) - f(b))/(a - b)$. The second part of the paper is concerned chiefly with "Darboux functions": functions such that, for any two points x_1, x_2 in the interval of definition of $f(x)$, $f(x)$ takes on all values between $f(x_1)$ and $f(x_2)$ for values of x between x_1 and x_2. The most important theorems seem to be the following. (A) If $f(x)$ is a Darboux function in the closed interval $[a, b]$ and has a derivative at every interior point, $f'(x)$ has the properties of Rolle, Lagrange, and Darboux. (B) If the function $f(x)$ has a derivative at every point of $[a, b]$ and no discontinuities of the first kind, $f'(x)$ has the properties mentioned in (A). *J. V. Wehausen*

6,203h 27.2X

Massera, José L.

On differentiable functions. (Spanish. French summary)

Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadistica **1,** (1944). (*Año* 9), 71–93

{For the text of this review, see MR 6,203i.} *J. V. Wehausen*

6,203g 27.2X

Massera, José L.

An example of a Jordan curve whose projections on three orthogonal planes fill areas. (Spanish. French summary)

Bol. Fac. Ingen. Montevideo **2,** (1944). (*Año* 9), 669–672

An example is given of a "continuous curve" in three-dimensional Euclidean space which has zero three-dimensional measure but whose projections on three given orthogonal planes fill a hexagon in each plane. *J. V. Wehausen*

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6,203f 27.2X

Massera, José L.

An example of a Jordan curve whose projections on three orthogonal planes fill areas. (Spanish. French summary)

Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadistica **1,** (1944). 95–98

{For the text of this review, see MR 6,203g.} *J. V. Wehausen*

6,53d 65.0X

Massera, José L.

Formulae for finite differences with application to the approximate solution of differential equations of first order. (Spanish)

Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadistica **1,** (1943). 1–69

These papers are identical with a paper by the same author previously reviewed [Publ. Inst. Mat. Univ. Nac. Litoral 4, 99–166 (1943); these Rev. 4, 283].

6,53c 65.0X

Massera, José L.

Formulae for finite differences with applications to the approximate integration of differential equations of first order. (Spanish)

Bol. Fac. Ingen. Montevideo **2,** (1943). (*Año* 8), 439–507

{For the text of this review, see MR 6,53d.}

4,283g 65.0X

Massera, José L.

Formulae for finite differences with applications to the approximate integration of differential equations of first order. (Spanish)

Publ. Inst. Mat. Univ. Nac. Litoral **4,** (1943). 99–166

This paper undertakes a study and comparison of various methods for the numerical solution of differential equations of the first order. As a basis for the study, the author presents a systematic collection of formulas, including the Newton-Cotes open and closed quadrature formulas and various other analagous formulas useful for numerical integration. A detailed study of the error terms for these formulas is given, with particular application to the solution of differential equations. His conclusion is that the best methods for the numeri-

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cal solution of differential equations are those employing symmetric quadrature formulas such as Simpson's rule and similar formulas of higher degree. Numerical examples are supplied for illustration.

W. E. Milne

1 265 661 01A70

Massera, J. L.

The contribution of Professor Julio Rey Pastor to the development of Uruguayan mathematics. (Spanish)

Rev. Un. Mat. Argentina **35** (1989), 1 (1991).

1 108 448 01A70

Massera, J. L.

Address. (Spanish)

Conference in Honor of Mischa Cotlar (Buenos Aires, 1988).

Rev. Un. Mat. Argentina **34** (1988), 6–9 (1990).