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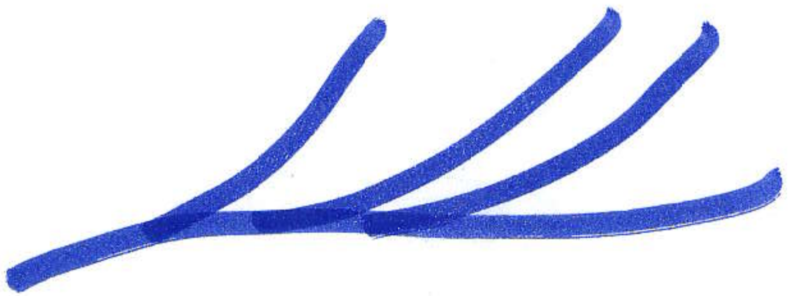
Lecture 3

**Thm** [BI]

$f: M \circlearrowleft$ p.h., $\dim(M)=3$

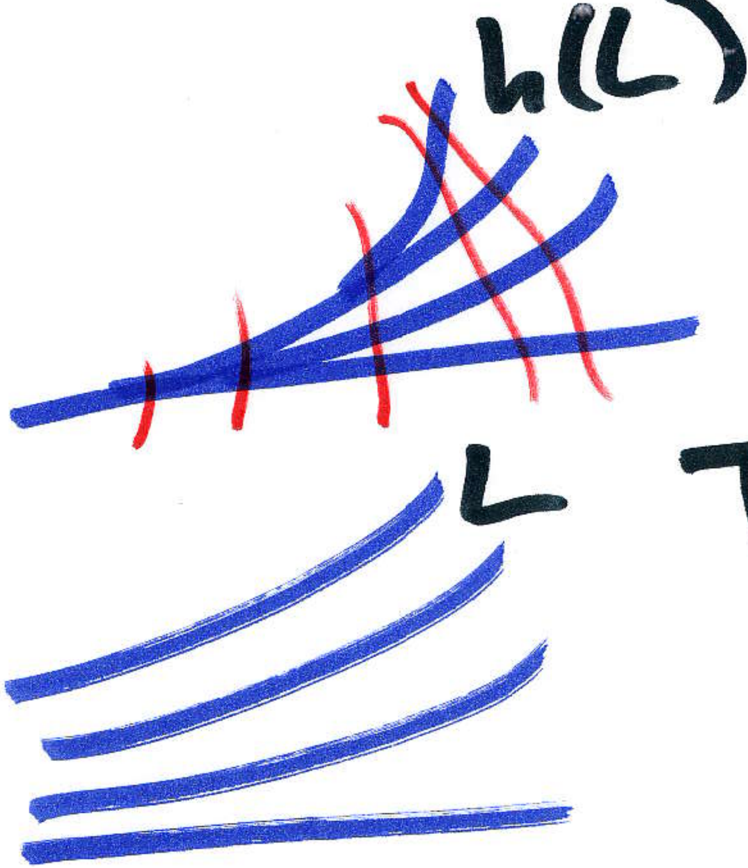
$\exists$ branching foln $\widetilde{\mathcal{F}}^{cs}$

tangent to $E^{cs} = E^c \oplus E^s$



$TM = E^u \oplus E^c \oplus E^s$

$W^u$




smooth v. field $\pitchfork \mathcal{F}$

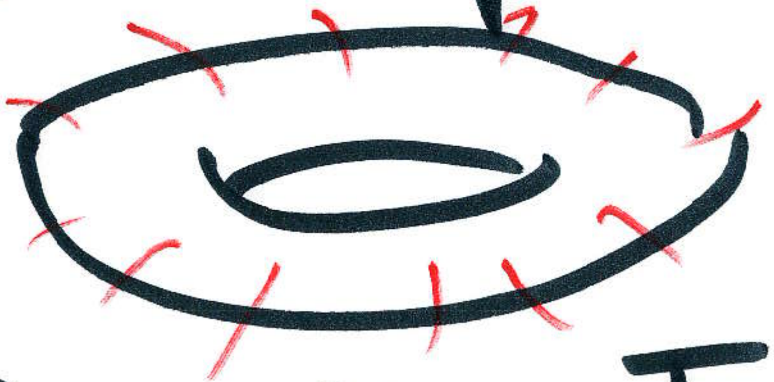
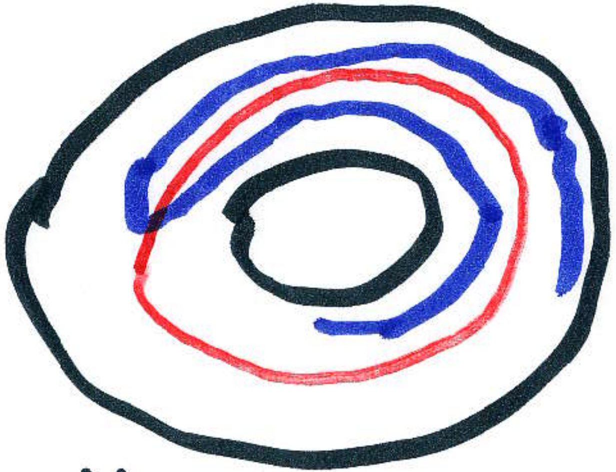
flow to separate leaves

<u>Thm</u> $\forall \varepsilon > 0 \ \exists$ foln $\mathcal{F}_\varepsilon$ and an onto map $h: M \circlearrowleft$

- $\|h - id\| < \varepsilon$
- $T\mathcal{F}_\varepsilon$ is $\varepsilon$-close $E^{cs}$
- $L \in \mathcal{F}_\varepsilon \Rightarrow h(L) \in \mathcal{F}^{cs}$

and $h|_L$ is homeo.

Reeb components

Any foln $\pitchfork$ to $W^u$ is Reebless.

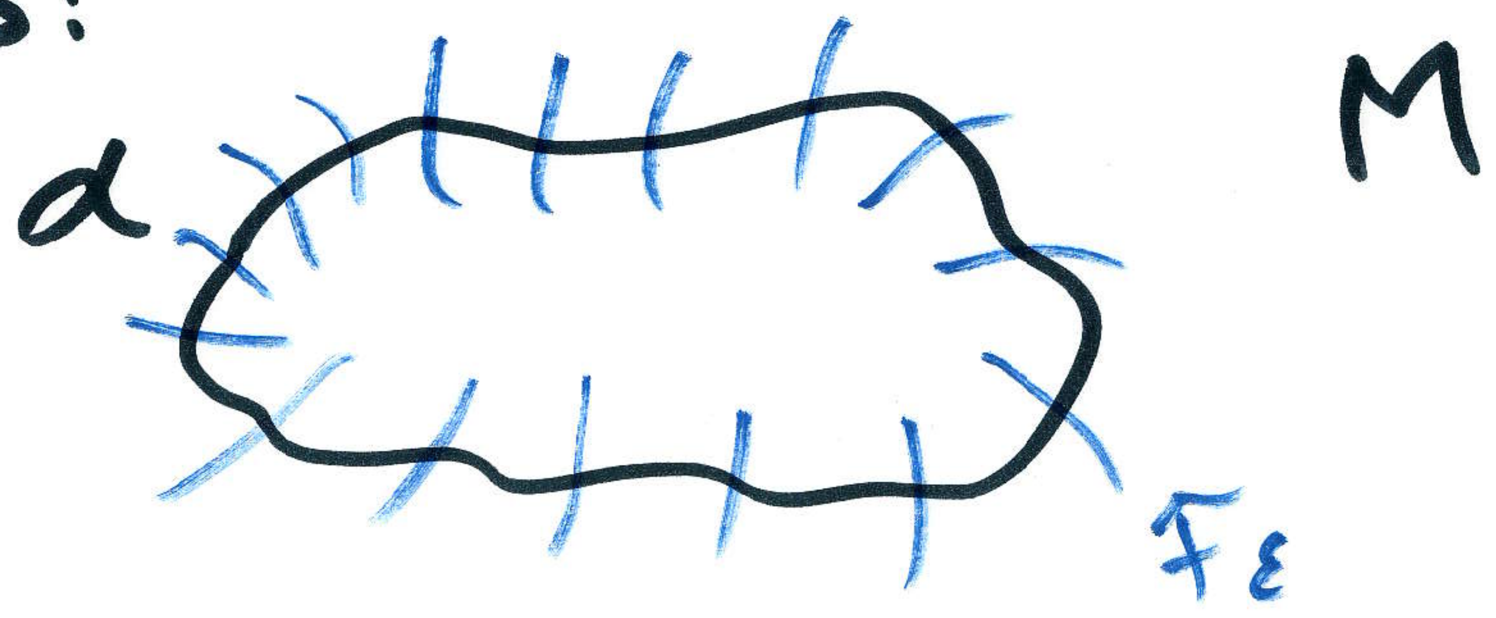
In part, $\mathcal{F}_\varepsilon$ is Reebless.

- each leaf $L \in \mathcal{F}_\varepsilon^\varepsilon$ is $\pi_1$-injective

  $i : L \longrightarrow M$

  $i_* : \pi_1(L) \longrightarrow \pi_1(M)$ is an injection

- no transverse contractible cycles:



$L \in \hat{\mathcal{F}}_\varepsilon$ then $h(L) \in \hat{\mathcal{F}}^{cs}$

an the embeddings are homotopic

$$\begin{tikzcd}
\pi_1(L) \arrow[r, hook] \arrow[d, phantom, "\wr"] & \pi_1(M) \\
\pi_1(h(L)) \arrow[ur] &
\end{tikzcd}$$

injective.

No transverse contractible cycles.

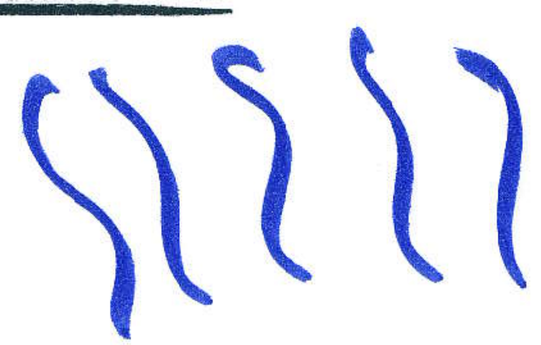
Suppose $\alpha \pitchfork \int E^{cs}$

$$\measuredangle(T\alpha, E^{cs}) > \varepsilon > 0.$$

$\mathcal{F}_\varepsilon$ $\quad \measuredangle(T\mathcal{F}_\varepsilon, E^{cs}) < \varepsilon \qquad E^{cs}$

$$\measuredangle(T\alpha, T\mathcal{F}_\varepsilon) > 0.$$

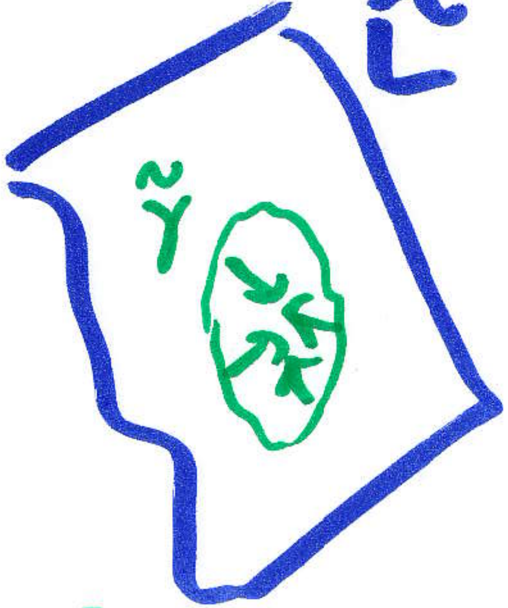
**Prop** If $f: M \circlearrowleft$ p.h. dim $= 3$ then the universal cover $\tilde{M}$ is homeo to $\mathbb{R}^3$.

**proof** $\tilde{M}$  $\tilde{\mathcal{F}}^{cs}$

$\downarrow$

$M$  $\mathcal{F}^{cs}$

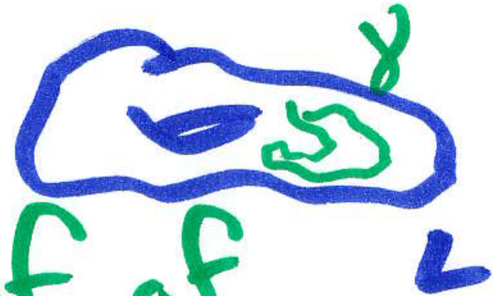
$$\begin{array}{c} \tilde{M} \\ \big\downarrow \\ M \end{array}$$




$\tilde{L}$

$\cancel{S^2}$ or $\mathbb{R}^2$

$T\tilde{L} = E^c \oplus E^s$




Every leaf of $\tilde{\mathcal{F}}^{cs}$ is a plane.

$\tilde{\mathcal{F}}^{cs}$

$\tilde{L}$ is simp conn

$[\gamma] = 0 \in \pi_1(M)$

$\Rightarrow [\gamma] = 0 \in \pi_1(L)$

$\big\downarrow \; \mathcal{F}^{cs}$

$[\tilde{\gamma}] = 0 \in \pi_1(\tilde{L})$

$\widetilde{\mathcal{F}}^{cs} \xleftarrow[\tilde{h}]{\text{planes}} \widetilde{\mathcal{F}}_{\varepsilon}$ is a true foliation by planes.

(Palmeira)

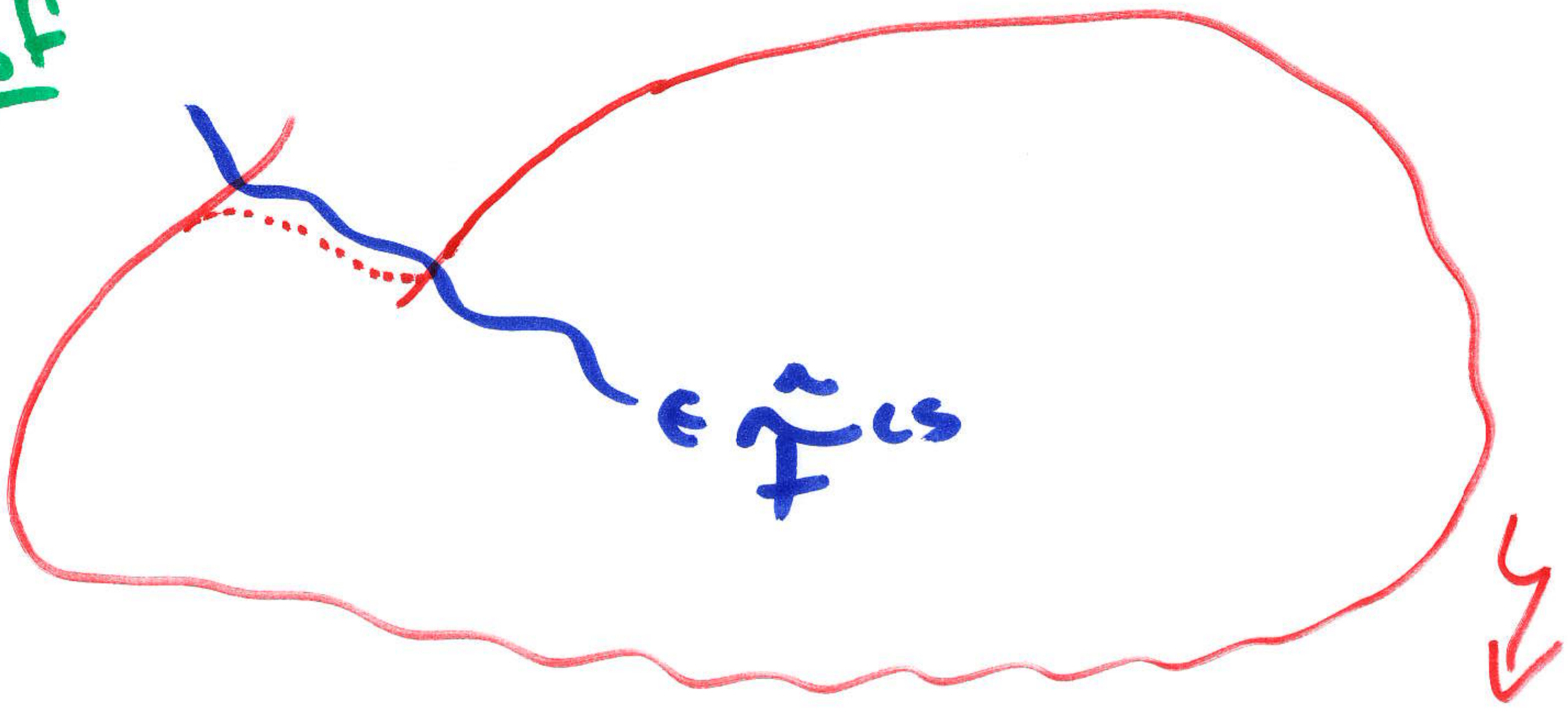
$\Rightarrow \widetilde{M}$ is homeo to $\mathbb{R}^3$.

$\mathcal{F}^{cs} \xleftarrow[h]{} \mathcal{F}_{\varepsilon}$

**Prop** On $\tilde{M}$, each leaf of $\hat{\mathcal{F}}^{cs}$ intersects a leaf $\hat{\mathcal{W}}^{u}$ at most once.

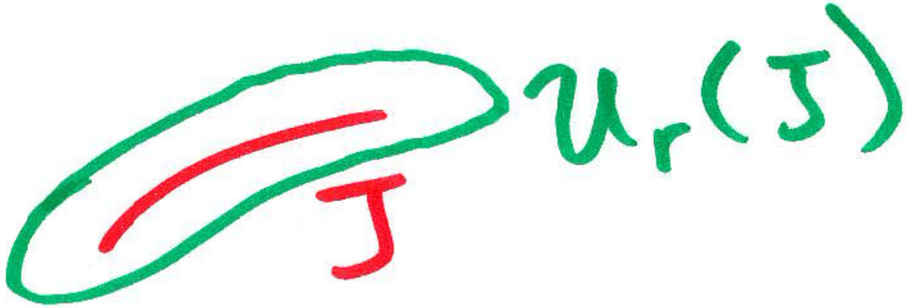
circle $\pitchfork E^{cs}$

**proof**



Define $\mathcal{U}_r(J) = \{x \in \tilde{M} : \operatorname{dist}(x, J) < r\}$

$J \subset \tilde{M}$

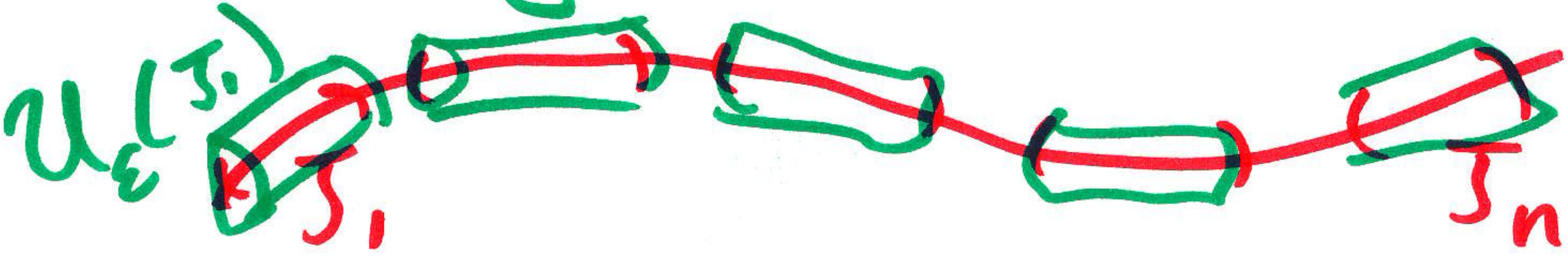



**Prop** There is $C > 0$ s.t.

$$\operatorname{volume} \mathcal{U}_1(J) > C \cdot \operatorname{length} J$$

for any unstable curve $J \subset \tilde{M}$.

**proof** Suppose $J$ is a curve of length $> n$

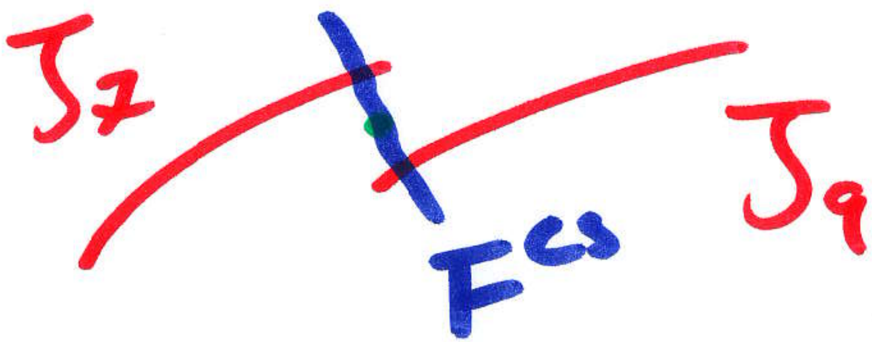



Suppose

$x \in \mathcal{U}_\varepsilon(J_7) \cap \mathcal{U}_\varepsilon(J_9) \neq \emptyset$

each $J_k$ has length $= 1$.

$\mathcal{U}_\varepsilon(J_k) > C$.




$\lightning$

$M = \mathbb{T}^3$ $\quad \tilde{M} = \mathbb{R}^3$ w/ std volume.

Suppose $f : \mathbb{T}^3 \circlearrowleft$ p.h.

Lift $\tilde{f} : \mathbb{R}^3 \circlearrowleft$ there is

a group aut $f_* : \pi_1(\mathbb{T}^3) \circlearrowleft$

s.t.

$\gamma \in \pi_1(\mathbb{T}^3)$ $\quad f_*(\gamma) \cdot \tilde{f} = \tilde{f} \circ \gamma.$

$\gamma : \mathbb{R}^3 \circlearrowleft$

$f_* \curvearrowright \pi_1(\mathbb{T}^3) \cong \mathbb{Z}^3$

$\exists$ linear map $\tilde{A} : \mathbb{R}^3 \to \mathbb{R}^3$

$A : \mathbb{T}^3 \to \mathbb{T}^3$

so $A_* = f_*$.

$A$ "linear part" of $f$.

$A$ is partially hyp

$A$ has eigenvalues

$$|\lambda_1| \leq |\lambda_2| \leq |\lambda_3|$$

Want $|\lambda_1| < 1 < |\lambda_3|$

and $|\lambda_1| < |\lambda_2| < |\lambda_3|$

Say $|\lambda_3| \leq 1 \implies |\lambda_i| \leq 1$

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

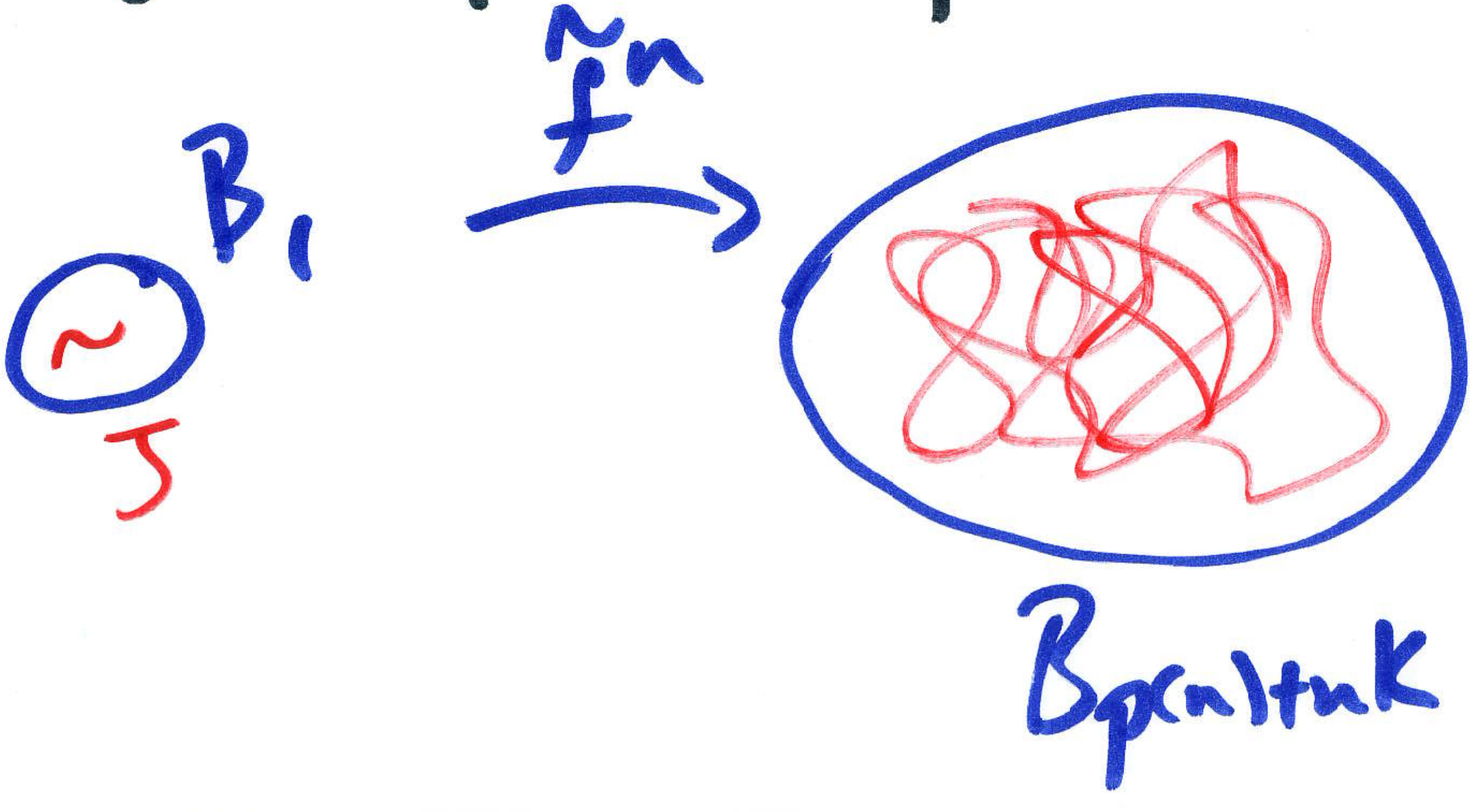
$$B_N = \{ x \in \mathbb{R}^3 : \|x\| < N \}$$

$$\tilde{A}^n B_1 \subset B_{p(n)}$$

$$A_* = f_* \qquad \|\tilde{A} - \tilde{f}\|_0 < K \text{ on } \mathbb{R}^3$$

$$\|\tilde{A}^n - \tilde{f}^n\| < nK$$

$$\tilde{f}^n B_1 \subset B_{p(n)+nK}$$



$$a \cdot \mu^n \leq C \cdot \operatorname{length} \tilde{f}^n J \leq \operatorname{volume} \mathcal{U}_1(\tilde{f}^n J)$$

$(\mu > 1)$

$$\leq \operatorname{volume} B_{p(n)+nK+1}$$

$$\sim \left(p(n)+nK+1\right)^3$$

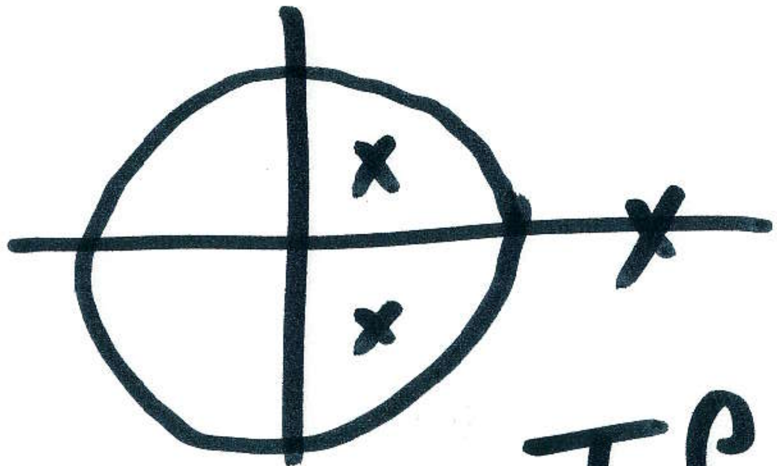
$$\Downarrow$$

$\cancel{|\lambda_3| \leq 1}$ $\qquad |\lambda_1| < 1 < |\lambda_3|$

A p.h.

$$|\lambda_1| < |\lambda_2| < |\lambda_3|$$

$$|\lambda_1| < 1 < |\lambda_3|$$



If $f : \mathbb{T}^3$ is p.h.

$A : \mathbb{T}^3 \circlearrowleft$ linear and $f_* = A_*$

then $A$ is p.h.

$M$ non-trivial circle bundle over $\mathbb{T}^2$.

nilmanifold.

$$0 \to S^1 \to M \to \mathbb{T}^2 \to 0$$

$$0 \to \mathbb{Z} \to \pi_1(M) \to \mathbb{Z}^2 \to 0$$

$\tilde{M} = \mathcal{H}$ Heisenberg space.

$\pi_1(M)$ is nilpotent

Malcev $f: M \circlearrowleft$ nilmanifold

$f_*: \pi_1(M) \circlearrowleft$

$\exists$ algebraic map $\Phi: M \circlearrowleft$

s.t. $\Phi_* = f_*$. $A \in GL(2, \mathbb{Z})$


$\Phi(v,t) = (Av, p(v,t))$

$p$ is a quadratic poly.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\pm Id"] & \pi_1(M) \arrow[r] \arrow[d, "f_* = \Phi_*"] & \mathbb{Z}^2 \arrow[r] \arrow[d, "A"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \pi_1(M) \arrow[r] & \mathbb{Z}^2 \arrow[r] & 0
\end{tikzcd}$$

$A$ has eigs $\lambda$ and $\lambda^{-1}$

If $|\lambda| \leq 1$

In Heisenberg space

$$\mathrm{vol}(B_R) \sim R^4$$

polynomial growth

~~$|\lambda| \le 1$~~

$\xrightarrow{\hat{f}^n}$

$|\lambda^{-1}| < 1 < |\lambda|$ $\Phi$ is a P.h. skew product.

$A \in GL(2,\mathbb{Z})$ hyperbolic.

$$M_A = \mathbb{T}^2 \times \mathbb{R} \big/ (Ax,t) \sim (x,t+1)$$

$$0 \to \mathbb{Z}^2 \to \pi_1(M_A) \to \mathbb{Z} \to 0$$

$$\pi_1(M_A) = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$$

solvable but not nilpotent.

$0 \longrightarrow$ 99

$f$ time one map of flow

$$\mathrm{Id} = f_* : \pi_1(M_A) \circlearrowleft$$

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$$f : M_A \circlearrowleft \text{ p.h.}$$

$\exists n \geq 1$ and a lift $\tilde{f}^n : \tilde{M}_A \circlearrowleft$

so $f^n_* = \mathrm{Id}$.