

KERNELS OF FELL BUNDLES TENSOR PRODUCTS

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ABSTRACT. We prove that if $\mathcal{A} = (A_t)_{t \in G}$ and $\mathcal{B} = (B_s)_{s \in H}$ are Fell bundles over the locally compact groups G and H respectively, then $\mathbb{k}(\mathcal{A} \otimes_{\min} \mathcal{B}) = \mathbb{k}(\mathcal{A}) \otimes_{\min} \mathbb{k}(\mathcal{B})$ and $\mathbb{k}(\mathcal{A} \otimes_{\max} \mathcal{B}) = \mathbb{k}(\mathcal{A}) \otimes_{\max} \mathbb{k}(\mathcal{B})$.

1. INTRODUCTION

In [4] we developed a theory of tensor products of Fell bundles, and we studied the relationship between the cross-sectional C*-algebras of these tensor products with the tensor products of the corresponding bundles. In this article we will do something similar but in relation to the algebras of kernels of the bundles instead of cross-sectional algebras. The C*-algebra of kernels of a Fell bundle \mathcal{A} can be described (in addition to the description we will give below) as $\mathbb{k}(\mathcal{A}) = C_r^*(\mathcal{A}) \rtimes_{\delta} \hat{G}$ where δ is the dual coaction of G on $C_r^*(\mathcal{A})$, and it helps to express a form of Takai duality for crossed products by partial actions. Besides, there is a canonical action β of G on $\mathbb{k}(\mathcal{A})$ (which corresponds to the dual action of G on $C_r^*(\mathcal{A}) \rtimes_{\delta} \hat{G}$ in the picture above) which plays an important role in the problem of globalization of partial actions (we refer the reader to [1]). For this reason one can also interpret the results of this article with an eye on the tensor products of partial actions.

We want to show that $\mathbb{k}(\mathcal{A} \otimes_{\min} \mathcal{B}) = \mathbb{k}(\mathcal{A}) \otimes_{\min} \mathbb{k}(\mathcal{B})$, and similarly for the maximal C*-tensor product. The strategy will be the following. As shown in [1], one can see the C*-algebra $I(\mathcal{A})$ of compact operators on the Hilbert module $L^2(\mathcal{A})$ as an ideal of $\mathbb{k}(\mathcal{A})$, and we know the linear β -orbit of $I(\mathcal{A})$ is dense in $\mathbb{k}(\mathcal{A})$. This means that β is the enveloping action of its restriction to $I(\mathcal{A})$ and, something we will take advantage of later, it allows to compute the norm of $k \in \mathcal{K}(\mathcal{A})$ as the supremum of the norms of k as a multiplier of the ideals $\beta_t(\mathcal{A})$ (see Proposition 2.8 below). In this way one essentially reduces the problem to proving that $I(\mathcal{A} \otimes_{\min} \mathcal{B}) = I(\mathcal{A}) \otimes_{\min} I(\mathcal{B})$ and $I(\mathcal{A} \otimes_{\max} \mathcal{B}) = I(\mathcal{A}) \otimes_{\max} I(\mathcal{B})$. For this we will serve ourselves from the results in [2] and [4].

Before we start we will recall some facts about tensor products and about Fell bundles.

Let us first recall the notion of exterior tensor product of Hilbert modules. So suppose that E is a Hilbert module over the C*-algebra A , and F is a Hilbert module over the C*-algebra B . Let $E \odot F$ be the algebraic tensor

product of E and F . Using the universal property of tensor products, it can be shown that there exists a unique bilinear map $(E \odot F) \times (E \odot F) \rightarrow A \odot B$ such that

$$(1) \quad \langle x \odot y, x' \odot y' \rangle = \langle x, x' \rangle \odot \langle y, y' \rangle, \quad \forall x, x' \in E, y, y' \in F.$$

Besides, it can be shown that this map is actually a $A \odot B$ -valued inner product (see [7, page 34 and Chapter 6]). Then, considering on $A \odot B$ the minimal C^* -norm $\| \cdot \|_{\min}$ (which is the spatial norm), one can perform the so called double completion process ([7, Chapter 1]) and obtain a Hilbert $A \otimes_{\min} B$ -module $E \otimes_{\min} F$, called the *exterior tensor product* of E and F . However, instead of $\| \cdot \|_{\min}$, one can use another C^* -norm α on $A \odot B$ to perform the double completion process, for instance the maximal C^* -norm $\| \cdot \|_{\max}$. This induces a norm $\tilde{\alpha}$ on $E \odot F$, and the corresponding completion $E \otimes_{\tilde{\alpha}} F$ turns out to be a Hilbert $A \otimes_{\alpha} B$ -module. Of course, the relation between α and $\tilde{\alpha}$ is, as usual

$$(2) \quad \tilde{\alpha}(z)^2 = \alpha(\langle z, z \rangle), z \in E \otimes_{\tilde{\alpha}} F \quad \text{and} \quad \alpha(c) = \sup\{\tilde{\alpha}(zc) : \tilde{\alpha}(z) \leq 1.\}$$

In [2] it was shown that (provided E and F are full modules) the map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism between the poset $\mathcal{N}(A \odot B)$ and the poset $\mathcal{N}(E \odot F)$ of C^* -norms on $A \odot B$ and $E \odot F$ respectively ([2, Theorem 5.12]). In particular we have $\| \cdot \|_{\min} = \| \cdot \|_{\min}$ and $\| \cdot \|_{\max} = \| \cdot \|_{\max}$. The work in [2] uses the language of C^* -trings rather than that of Hilbert modules. For the topic of C^* -trings we refer the reader to [8], where Zettl introduced and studied them.

Going now to define the tensor product of two Fell bundles $\mathcal{A} = (A_r)_{r \in G}$ and $\mathcal{B} = (B_s)_{s \in H}$, we begin by considering the algebraic tensor products $A_r \odot B_s$ and, given a C^* -norm α on $A_e \odot B_e$, we want to extend it to all of the bundle $\mathcal{A} \odot \mathcal{B} = \{A_r \odot B_s\}$, so that we obtain a Fell bundle after completing. When this is possible, we call such a C^* -norm *extendable*. The condition for a C^* -norm to be extendable is simple, and it is essentially the fact that when we perform the construction of $A_r \otimes_{\tilde{\alpha}} B_s$, the result is the same if we do it as a right or as left Hilbert $A_e \otimes_{\alpha} B_e$ -module. Fortunately, the minimal and maximal norms on $A_e \otimes B_e$ are both extendable. As shown in [4], the map $\gamma \mapsto \gamma|_{A_e \odot B_e}$ is an isomorphism of posets, between the set $\mathcal{N}(\mathcal{A} \odot \mathcal{B})$ of C^* -norms on $\mathcal{A} \odot \mathcal{B}$ and the set $\mathcal{N}(A_e \odot B_e)$ of extendable C^* -norms on $A_e \odot B_e$.

We end this section with the following result about Hilbert modules, which is certainly well-known. A proof, in terms of C^* -trings, can be found in [1, Proposition 4.1].

Proposition 1.1. *Let E be a full right Hilbert A -module and F a right Hilbert B -module. If $\pi : E \rightarrow F$ is a linear map such that $\pi(x\langle y, z \rangle) = \pi(x)\langle \pi(y), \pi(z) \rangle \quad \forall x, y, z \in E$, then there exists a unique homomorphism $\pi^r : A \rightarrow B$ of C^* -algebras such that $\pi^r(\langle y, z \rangle) = \langle \pi(y), \pi(z) \rangle \quad \forall y, z \in E$,*

and $\pi(xa) = \pi(x)\pi^r(a)$, $\forall x \in E$, $a \in A$. Moreover π^r is injective if and only if so is π . An analogous result holds for left Hilbert modules instead of right ones.

Remark 1.2. A particular case that we will encounter later is the following. Suppose that E is a right Hilbert A -module, and let $\mathcal{K}(E)$ be the C^* -algebra of compact operators. Then, if $\theta_{x,y} : E \rightarrow E$ represents the compact operator on E , so that $\theta_{x,y}(z) := z\langle x, y \rangle$, it is well known that (E, θ, \cdot) is a full left Hilbert $\mathcal{K}(E)$ -module. Then, if F is another Hilbert A -module and $\pi : E \rightarrow F$ is a linear map such that $\pi(\langle x, y \rangle) = \langle \pi(x), \pi(y) \rangle$, $\forall x, y \in E$, by the left version of Proposition 1.1 we get a homomorphism of C^* -algebras $\pi^l : \mathcal{K}(E) \rightarrow \mathcal{K}(F)$ such that $\pi^l(\theta_{x,y}) = \theta_{\pi(x), \pi(y)}$, $\forall x, y \in E$. In particular π^l is an isomorphism whenever so is π (we note in passing that $\pi^r = id$).

2. KERNELS OF A FELL BUNDLE

Given a Fell bundle $\mathcal{A} = (A_r)_{r \in G}$ over the locally compact group G , by a continuous kernel of compact support of \mathcal{A} we mean a continuous map $k : G \times G \rightarrow \mathcal{A}$ of compact support such that $k(r, s) \in A_{rs^{-1}}$, $\forall r, s \in G$. The vector space $\mathbb{k}_c(\mathcal{A})$ of kernels of compact support of \mathcal{A} is actually an A_e -module with $ka(r, s) := k(r, s)a$, $\forall r, s \in G$, $a \in A_e$, $k \in \mathbb{k}_c(\mathcal{A})$. We also have operations making $\mathbb{k}_c(\mathcal{A})$ a $*$ -algebra, namely: a product given by $k_1 * k_2(r, s) := \int_G k_1(r, t)k_2(t, s)dt$, and an involution given by $k^*(r, s) := k(s, r)^*$. In addition, $\mathbb{k}_c(\mathcal{A})$ is a normed $*$ -algebra with $\|k\|_2 := (\int_{G \times G} \|k(r, s)\|^2 d(r, s))^{1/2}$, where the integration is with respect to the Haar measure of $G \times G$. The completion of $\mathbb{k}_c(\mathcal{A})$, which is a Banach $*$ -algebra denoted by $\mathcal{HS}(\mathcal{A})$, is called the algebra of the kernels of Hilbert-Schmidt of \mathcal{A} , and its universal enveloping C^* -algebra $\mathbb{k}(\mathcal{A}) := C^*(\mathcal{HS}(\mathcal{A}))$, is the C^* -algebra of kernels of \mathcal{A} .

2.1. Canonical action on the kernels. It is clear that if $k \in \mathbb{k}_c(\mathcal{A})$ and $t \in G$, then $\beta_t(k) : G \times G \rightarrow \mathcal{A}$, such that $\beta_t(k)(r, s) := \Delta(t)k(rt, st)$ $\forall r, s \in G$, is also an element of $\mathbb{k}_c(\mathcal{A})$ (here Δ is the modular function of G). It follows that $k \mapsto \beta_t(k)$ is an action of G on $\mathbb{k}_c(\mathcal{A})$. Moreover, we have $\|\beta_t(k)\|_2 = \|k\|_2$, so the action extends to a an isometric action of G on $\mathcal{HS}(\mathcal{A})$, and hence to an action of G on $\mathbb{k}(\mathcal{A})$. If we need to remark that the underlying Fell bundle is \mathcal{A} , we write $\beta^{\mathcal{A}}$ instead of β .

2.2. Hilbert modules of a Fell bundle. Given $t \in G$, let $C_c^t(\mathcal{A})$ be the vector space of those continuous functions $\xi : G \rightarrow \mathcal{A}$ with compact support such that $\xi(r) \in A_{rt}$, $\forall r \in G$ (that is: $C_c^t(\mathcal{A}) = C_c(\mathcal{A}^t)$, where \mathcal{A}^t is the retraction of \mathcal{A} by the left translation $r \mapsto rt$ on G).

We have natural actions $C_c^t(\mathcal{A}) \times A_e \rightarrow C_c^t(\mathcal{A})$ and $\mathbb{k}_c(\mathcal{A}) \times C_c^t(\mathcal{A}) \rightarrow C_c^t(\mathcal{A})$, given by $(\xi a)(r) := \xi(r)a$ and $k\xi(r) := \int_G k(r, s)\xi(s)ds$, $\forall \xi \in C_c^t(\mathcal{A})$, $r \in G$, $k \in \mathbb{k}_c(\mathcal{A})$. In this way $C_c^t(\mathcal{A})$ is a $(\mathbb{k}_c(\mathcal{A}) - A_e)$ -bimodule. We also have an inner product $\langle \cdot, \cdot \rangle_r^t : C_c^t(\mathcal{A}) \times C_c^t(\mathcal{A}) \rightarrow A_e$, given by $\langle \xi, \eta \rangle_r^t := \int_G \xi(r)^* \eta(r) dr$.

We denote by $L_t^2(\mathcal{A})$ the Hilbert A_e -module obtained by completing $C_c^t(\mathcal{A})$ with respect to $\langle \cdot, \cdot \rangle_r^t$. Note that $L_e^2(\mathcal{A})$ is the usual Hilbert module $L^2(\mathcal{A})$, where the left regular representation of \mathcal{A} takes place. When $t = e$ we just put $\langle \xi, \eta \rangle_r$ and $L^2(\mathcal{A})$ instead of $\langle \xi, \eta \rangle_r^e$ and $L_e^2(\mathcal{A})$ respectively.

Definition 2.1. We denote by $I_t(\mathcal{A}) := \mathcal{K}(L_t^2(\mathcal{A}))$ (with the usual exception for $t = e$, when we will put in general $I(\mathcal{A})$ instead of $I_e(\mathcal{A})$).

Proposition 2.2. *The map $\rho_t : C_c(\mathcal{A}) \rightarrow C_c^t(\mathcal{A})$, such that $\rho_t(\xi)(r) := \Delta(t)^{1/2}(\xi)(rt) \forall \xi \in C_c(\mathcal{A}), r \in G$, extends to a unitary operator of Hilbert A_e -modules $L^2(\mathcal{A}) \rightarrow L_t^2(\mathcal{A})$, and it is also an isomorphism of C^* -trings.*

Proof. It is clear that ρ_t is an A_e -linear map. Let $\xi, \eta \in C_c(\mathcal{A})$. Then

$$\begin{aligned} \langle \rho_t(\xi), \rho_t(\eta) \rangle_r^t &= \int_G \rho_t(\xi)(r)^* \rho_t(\eta)(r) dr = \int_G \Delta(t)^{1/2} \xi(rt)^* \Delta(t)^{1/2} \eta(rt) dr \\ &= \Delta(t) \int_G \xi(rt)^* \eta(rt) dr = \Delta(t) \int_G \Delta(t)^{-1} \xi(s)^* \eta(s) ds = \langle \xi, \eta \rangle_r. \end{aligned}$$

Then ρ_t is isometric and can be extended by continuity to an isometry on $L^2(\mathcal{A})$. Thus ρ_t has closed range, which clearly contains $C_c^t(\mathcal{A})$, and therefore ρ_t is surjective. Hence ρ_t is unitary [7, Theorem 3.5]. The last statement is a direct consequence of the first one. \square

As a consequence of Corollary 2.3 and Proposition 1.1 we get:

Corollary 2.3. *The map $\rho_t^l : I(\mathcal{A}) \rightarrow I_t(\mathcal{A})$ is an isomorphism of C^* -algebras. It is determined by the fact that $\rho_t^l(\langle \xi, \eta \rangle_l) = \langle \rho_t(\xi), \rho_t(\eta) \rangle_l, \forall \xi, \eta \in L^2(\mathcal{A})$.*

2.3. Producing kernels. For each $t \in G$, it is possible to obtain kernels of \mathcal{A} using elements of $C_c^t(\mathcal{A})$. In fact, if $\xi, \eta \in C_c^t(\mathcal{A})$, it is clear that $\langle \xi, \eta \rangle_l^t : G \times G \rightarrow \mathcal{A}$ such that $\langle \xi, \eta \rangle_l^t(r, s) = \xi(r)\eta(s)^*$ is an element of $\mathbb{k}_c(\mathcal{A})$. We define $I_c^t(\mathcal{A}) := \text{span}\{\langle \xi, \eta \rangle_l^t : \xi, \eta \in C_c^t(\mathcal{A})\} \subseteq \mathbb{k}_c(\mathcal{A})$. When $t = e$ we just put $\langle \xi, \eta \rangle_l$ and $I_c(\mathcal{A})$ instead of $\langle \xi, \eta \rangle_l^e$ and $I_c^e(\mathcal{A})$ respectively.

Note that if $\xi, \eta \in C_c(\mathcal{A})$, we have $\beta_t(\langle \xi, \eta \rangle_l)(r, s) = \Delta(t)\xi(rt)\eta(st)^* = \Delta(t)^{1/2}\xi(rt)\Delta(t)^{1/2}\eta(st)^* = \langle \rho_t\xi, \rho_t\eta \rangle_l^t(r, s)$, so $\beta_t(\langle \xi, \eta \rangle_l) = \langle \rho_t\xi, \rho_t\eta \rangle_l^t$. It follows that $\beta_t(I_c(\mathcal{A})) = I_c^t(\mathcal{A}), \forall t \in G$. Observe also that each $I_c^t(\mathcal{A})$ is an ideal in $\mathbb{k}_c(\mathcal{A}), \forall t \in G$, because $I_c^t(\mathcal{A})$ is selfadjoint and $k * \langle \xi, \eta \rangle_l^t = \langle k\xi, \eta \rangle_l^t$, as it is easy to check. In a sense, the family $I_c^t(\mathcal{A})$ contains all the necessary information about $\mathbb{k}_c(\mathcal{A})$, since $\text{span}\{\beta_t I_c(\mathcal{A}) : t \in G\} = \text{span}\{I_c^t(\mathcal{A}) : t \in G\}$ is dense in $\mathbb{k}_c(\mathcal{A})$ in the inductive limit topology ([1, Proposition 5.3]).

Remark 2.4. Lemma 5.3 of [1] shows that $\langle \xi, \xi \rangle_l$ is a positive element of $\mathbb{k}(\mathcal{A}), \forall \xi \in C_c(\mathcal{A})$. This implies that $\langle \cdot, \cdot \rangle_l$ is an inner product on $C_c(\mathcal{A})$, and allows to complete $C_c(\mathcal{A})$ using $\langle \cdot, \cdot \rangle_l$, as done in [1]. Let us call E to such a completion, and γ to its norm. Then γ induces a C^* -norm γ^r on the dense ideal $\text{span}\langle C_c(\mathcal{A}), C_c(\mathcal{A}) \rangle_r$ of A_e , which has a unique extension to a C^* -norm on A_e , which in turn is unique, say α (see for instance [2,

Proposition 3.7, Corollary 3.8]). Since the relation between γ and γ^r is given by $\gamma(\xi)^2 = \gamma^r(\langle \xi, \xi \rangle_r)$ and, in the other hand, the norm defined on $C_c(\mathcal{A})$ by α is given by $\tilde{\alpha}(\xi)^2 = \alpha(\langle \xi, \xi \rangle_r)$, we see that γ and $\tilde{\alpha}$ agree on $C_c(\mathcal{A})$, and therefore we conclude that $E = L^2(\mathcal{A})$. In turn, this implies that $I(\mathcal{A})$ is contained in $\mathbb{k}(\mathcal{A})$.

Proposition 2.5. *Let $\xi, \eta \in C_c^t(\mathcal{A})$. Then the map $\zeta \mapsto \langle \xi, \eta \rangle_l^t \zeta$ on $C_c^t(\mathcal{A})$ extends by continuity to an adjointable (and compact) operator on $L_t^2(\mathcal{A})$.*

Proof. Recall that any couple of elements $\eta, \zeta \in L_t^2(\mathcal{A})$ define a compact operator $\theta_{\eta, \zeta}$ on $L_t^2(\mathcal{A})$, via the formula $\theta_{\eta, \zeta}(\xi) = \xi \langle \eta, \zeta \rangle_r^t$. So it is enough to see that for $\zeta \in C_c^t(\mathcal{A})$, we have $\langle \xi, \eta \rangle_l^t \zeta = \theta_{\xi, \eta}^t(\zeta)$, that is $\langle \xi, \eta \rangle_l^t \zeta = \xi \langle \eta, \zeta \rangle_r^t$, $\forall \xi, \eta$ and $\zeta \in C_c(\mathcal{A})$. Now,

$$\begin{aligned} \langle \xi, \eta \rangle_l^t \zeta(r) &= \int_G \langle \xi, \eta \rangle_l^t(r, s) \zeta(s) ds = \int_G \xi(r) \eta(s)^* \zeta(s) ds = \xi(r) \int_G \eta(s)^* \zeta(s) ds \\ &= \xi(r) \langle \eta, \zeta \rangle_r^t = (\xi \langle \eta, \zeta \rangle_r^t)(r), \end{aligned}$$

as we wanted to verify. \square

According to Proposition 2.5, $I_t(\mathcal{A})$ is a completion of $I_c^t(\mathcal{A}) = \beta_t(I_c(\mathcal{A}))$. More precisely, we have

Proposition 2.6. $I_t(\mathcal{A}) = \beta_t(I(\mathcal{A}))$.

Proof. Proposition 2.2 provides a unitary operator $\rho_t : L^2(\mathcal{A}) \rightarrow L_t^2(\mathcal{A})$, given on $\xi \in C_c(\mathcal{A})$ by $\rho_t \xi(r) := \Delta(t)^{1/2} \xi(rt)$. Since ρ_t is an isomorphism of C^* -trings, it induces an isomorphism $\rho_t^l : I(\mathcal{A}) \rightarrow I_t(\mathcal{A})$, unique such that $\rho_t^l(\langle \xi, \eta \rangle_r) = \langle \rho_t \xi, \rho_t \eta \rangle_r^t$ (Proposition 1.1). On the other hand, it was shown above that $\langle \rho_t \xi, \rho_t \eta \rangle_l^t = \beta_t(\langle \xi, \eta \rangle_l)$, $\forall \xi, \eta \in C_c(\mathcal{A})$, which entails $\beta_t = \rho_t^l|_{I_c(\mathcal{A})}$, and therefore

$$I_c^t(\mathcal{A}) = \text{span}\langle C_c^t(\mathcal{A}), C_c^t(\mathcal{A}) \rangle_l^t = \text{span}\beta_t(\langle C_c(\mathcal{A}), C_c(\mathcal{A}) \rangle_l) = \beta_t(I_c(\mathcal{A})).$$

Then $I_t(\mathcal{A}) = \overline{I_c^t(\mathcal{A})} = \overline{\beta_t(I_c(\mathcal{A}))} = \beta_t(\overline{I_c(\mathcal{A})}) = \beta_t(I(\mathcal{A}))$. \square

For the convenience of the reader we recall Lemma 2.1 of [1]:

Lemma 2.7. *Suppose that $\{J_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals in a C^* -algebra A , and let $\|\cdot\|_\Lambda : A \rightarrow \mathbb{R}$ be such that $\|a\|_\Lambda = \sup_{\lambda \in \Lambda} \{\|ax\| : x \in J_\lambda, \|x\| \leq 1\}$. Then $\|\cdot\|_\Lambda$ is a C^* -seminorm on A , such that $\|\cdot\|_\Lambda \leq \|\cdot\|$, and $\|\cdot\|_\Lambda$ is a norm iff $\overline{\text{span}}\{x \in J_\lambda : \lambda \in \Lambda\}$ is an essential ideal of A . In this case, $\|\cdot\|_\Lambda = \|\cdot\|$.*

As easy consequences we obtain:

Proposition 2.8. *Let $k \in \mathbb{k}(\mathcal{A})$. Then*

$$(3) \quad \|k\| = \sup_{t \in G} \{\|k * h\| : h \in I_t(\mathcal{A}), t \in G \text{ and } \|h\| \leq 1\}.$$

Proposition 2.9. *Let α be either the minimal or the maximal C^* -norm on $\mathbb{k}(\mathcal{A}) \odot \mathbb{k}(\mathcal{B})$, and consider the action $\beta : (G \times H) \times \mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B}) \rightarrow \mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B})$ be the corresponding tensor product action $\beta^{\mathcal{A}} \otimes \beta^{\mathcal{B}}$, so that $\beta_{(r,s)}(k \otimes k') = \beta_r^{\mathcal{A}}(k) \otimes \beta_s^{\mathcal{B}}(k')$. Then $I(\mathcal{A}) \otimes_{\alpha} I(\mathcal{B})$ is an ideal of $\mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B})$, whose linear β -orbit is dense in $\mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B})$. Besides,*

$$(4) \quad \|k\| = \sup_{t \in G} \{ \|k * h\| : h \in I_r(\mathcal{A}) \otimes_{\alpha} I_s(\mathcal{B}), r \in G, s \in H \text{ and } \|h\| \leq 1 \}.$$

Proof. It is clear that $I(\mathcal{A}) \otimes_{\alpha} I(\mathcal{B})$ is an ideal of $\mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B})$, and the fact that the linear span of the orbit of $I(\mathcal{A}) \otimes_{\alpha} I(\mathcal{B})$ is dense in $\mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B})$ follows at once from the density of the linear orbits of $I(\mathcal{A})$ and $I(\mathcal{B})$ in $\mathbb{k}(\mathcal{A})$ and $\mathbb{k}(\mathcal{B})$ respectively. Then (4) follows from Lemma 2.7. \square

3. KERNELS OF TENSOR PRODUCTS

Let $\mathcal{A} = (A_r)_{r \in G}$ and $\mathcal{B} = (B_s)_{s \in H}$ be Fell bundles over the locally compact groups G and H respectively. Then $L^2(\mathcal{A})$ and $L^2(\mathcal{B})$ are full right Hilbert modules over A_e and B_e respectively. If α is a C^* -norm on $\mathcal{A} \odot \mathcal{B}$, then $\alpha|_{(A_e \odot B_e)}$ is a C^* -norm on $A_e \odot B_e$. As shown in [4], if α is the minimal (maximal) norm on $\mathcal{A} \odot \mathcal{B}$, then $\alpha|_{(A_e \odot B_e)}$ is the minimal (respectively: maximal) norm on $A_e \odot B_e$. On the other hand, $\alpha|_{(A_e \odot B_e)}$ defines a C^* -norm $\tilde{\alpha}$ on $L^2(\mathcal{A}) \odot L^2(\mathcal{B})$, given by $\tilde{\alpha}(\mu) := \sqrt{\alpha(\langle \mu, \mu \rangle)}$, $\forall \mu \in L^2(\mathcal{A}) \odot L^2(\mathcal{B})$. Again, if $\alpha|_{(A_e \odot B_e)}$ is the minimal (maximal) norm on $A_e \odot B_e$, then $\tilde{\alpha}$ is the minimal (maximal) norm on $L^2(\mathcal{A}) \odot L^2(\mathcal{B})$. The completion $L^2(\mathcal{A}) \otimes_{\tilde{\alpha}} L^2(\mathcal{B})$ of $L^2(\mathcal{A}) \odot L^2(\mathcal{B})$ with respect to $\tilde{\alpha}$ is a full right Hilbert module over $A_e \otimes_{\alpha|_{A_e \odot B_e}} B_e$, where the corresponding inner product is determined by $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \otimes \langle \eta_1, \eta_2 \rangle$, $\forall \xi_1, \xi_2 \in L^2(\mathcal{A}), \eta_1, \eta_2 \in L^2(\mathcal{B})$.

In the remaining of the paper we will consider only the minimal norm $\alpha = \| \cdot \|_{\min}$ or the maximal norm $\alpha = \| \cdot \|_{\max}$ on $\mathcal{A} \odot \mathcal{B}$. Since in both cases we will have correspondingly that $\tilde{\alpha} = \| \cdot \|_{\min}$ and $\tilde{\alpha} = \| \cdot \|_{\max}$, we will use just the symbol α , which will therefore be either $\| \cdot \|_{\min}$ or $\| \cdot \|_{\max}$.

We fix elements $r \in G$ and $s \in H$. Consider the map $j^{(r,s)} : C_c^r(\mathcal{A}) \odot C_c^s(\mathcal{B}) \rightarrow C_c^{(r,s)}(\mathcal{A} \otimes_{\min} \mathcal{B})$ such that $\xi \odot \eta \mapsto \xi \otimes \eta$, where $\xi \otimes \eta(r', s') := \xi(r') \otimes \eta(s')$, $\forall (r', s') \in G \times H$.

For the proof of the following result we refer the reader to [1, Lemma 4.4].

Lemma 3.1. *Let $\mathcal{A} = (A_t)_{t \in G}$ and $\mathcal{B} = (B_s)_{s \in H}$ be Fell bundles, and let α represent either the minimal or maximal C^* -norm. Then there exists a unique isomorphism $j_2 : L^2(\mathcal{A}) \otimes_{\alpha} L^2(\mathcal{B}) \rightarrow L^2(\mathcal{A} \otimes_{\alpha} \mathcal{B})$, such that $j_2(\xi \otimes \eta) = \xi \otimes \eta$, $\forall \xi \in C_c(\mathcal{A}) \subseteq L^2(\mathcal{A}), \eta \in C_c(\mathcal{B}) \subseteq L^2(\mathcal{B})$. Thus we have $L^2(\mathcal{A}) \otimes_{\min} L^2(\mathcal{B}) \cong L^2(\mathcal{A} \otimes_{\min} \mathcal{B})$ and $L^2(\mathcal{A}) \otimes_{\max} L^2(\mathcal{B}) \cong L^2(\mathcal{A} \otimes_{\max} \mathcal{B})$.*

Remark 3.2. Note that the isomorphism j_2^l induced by j_2 according to Proposition 1.1 (see also Remark 1.2) is determined by the fact that $\forall \xi_1, \xi_2 \in C_c(\mathcal{A})$ and $\forall \eta_1, \eta_2 \in C_c(\mathcal{B})$ we have $j_2(\langle \xi_1, \xi_2 \rangle_l \odot \langle \eta_1, \eta_2 \rangle_l) = \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_l$.

Proposition 3.3. *The map $j_2^{(r,s)} : C_c^r(\mathcal{A}) \odot C_c^s(\mathcal{B}) \rightarrow C_c^{(r,s)}(\mathcal{A} \otimes_\alpha \mathcal{B})$ has a unique extension to an isomorphism of Hilbert $A_e \otimes_\alpha B_e$ -modules such that the following diagram is commutative:*

$$\begin{array}{ccc} L^2(\mathcal{A}) \otimes_\alpha L^2(\mathcal{B}) & \xrightarrow[\cong]{\rho_r^{\mathcal{A}} \otimes \rho_s^{\mathcal{B}}} & L_r^2(\mathcal{A}) \otimes_\alpha L_s^2(\mathcal{B}) \\ j_2 \downarrow \cong & & \downarrow j_2^{(r,s)} \\ L^2(\mathcal{A} \otimes_\alpha \mathcal{B}) & \xrightarrow[\rho_{(r,s)}^{\mathcal{A} \otimes_\alpha \mathcal{B}}]{\cong} & L_{(r,s)}^2(\mathcal{A} \otimes_\alpha \mathcal{B}) \end{array}$$

The unique isomorphism $(j_2^{(r,s)})^l : I_r(\mathcal{A}) \otimes_\alpha I_s(\mathcal{B}) \rightarrow I_{(r,s)}(\mathcal{A} \otimes_\alpha \mathcal{B})$ given by Proposition 1.1 is determined by the fact that $\langle \xi_1, \xi_2 \rangle_l \odot \langle \eta_1, \eta_2 \rangle_l \mapsto \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_l^{(r,s)}$, $\forall \xi \in C_c^r(\mathcal{A})$, $\eta \in C_c^s(\mathcal{B})$.

Proof. The first vertical arrow exists and is an isomorphism by Lemma 3.1. The bottom horizontal arrow exists and is an isomorphism due to Proposition 2.2. For the same reason $\rho_r^{\mathcal{A}}$ and $\rho_s^{\mathcal{B}}$ are isomorphisms, and hence the top horizontal arrow is an isomorphism. The commutativity of the diagram at the level of compactly supported continuous functions is clear, so $j_2^{(r,s)}$ extends to the second vertical arrow which, in view of the previous remarks, must be an isomorphism. Finally, note that applying Proposition 1.1 to the objects of the diagram above produces a commutative diagram of C^* -algebras and isomorphisms (the compact operators and the maps between them induced in each case), so the last statement follows now from Remark 3.2 \square

Theorem 3.4. *Let $\mathcal{A} = (A_r)_{r \in G}$ and $\mathcal{B} = (B_s)_{s \in H}$ be Fell bundles over the locally compact groups G and H respectively. Then*

$$\mathbb{k}(\mathcal{A} \otimes_{\min} \mathcal{B}) = \mathbb{k}(\mathcal{A}) \otimes_{\min} \mathbb{k}(\mathcal{B}) \quad \text{and} \quad \mathbb{k}(\mathcal{A} \otimes_{\max} \mathcal{B}) = \mathbb{k}(\mathcal{A}) \otimes_{\max} \mathbb{k}(\mathcal{B}).$$

Proof. Let α be either $\| \cdot \|_{\min}$ or $\| \cdot \|_{\max}$. By the universal property of tensor products of vector spaces, we have a map $\nu : \mathbb{k}_c(\mathcal{A}) \odot \mathbb{k}_c(\mathcal{B}) \rightarrow \mathbb{k}(\mathcal{A} \otimes_\alpha \mathcal{B})$, such that $\nu(k_1 \odot k_2)((r_1, s_1), (r_2, s_2)) = k_1(r_1, r_2) \otimes k_2(s_1, s_2)$, $\forall (r_1, s_1), (r_2, s_2) \in G \times H$. When restricted to each $I_c^r(\mathcal{A}) \odot I_c^s(\mathcal{B})$, ν agrees with the restriction to this set of the isomorphism $(j_2^{(r,s)})^l : I_r(\mathcal{A}) \otimes_\alpha I_s(\mathcal{B}) \rightarrow$

$I_{(r,s)}(\mathcal{A} \otimes_{\alpha} \mathcal{B})$, for if $\xi_1, \xi_2 \in C_c^r(\mathcal{A})$ and $\eta_1, \eta_2 \in C_c^s(\mathcal{B})$, then

$$\begin{aligned} \nu(\langle \xi_1, \xi_2 \rangle_l^r \odot \langle \eta_1, \eta_2 \rangle_l^s)((r_1, s_1), (r_2, s_2)) &= \langle \xi_1, \xi_2 \rangle_l^r(r_1, r_2) \otimes \langle \eta_1, \eta_2 \rangle_l^s(s_1, s_2) \\ &= \xi_1(r_1) \xi_2(r_2)^* \otimes \eta_1(s_1) \eta_2(s_2)^* = (\xi_1(r_1) \otimes \eta_1(s_1)) (\xi_2(r_2)^* \otimes \eta_2(s_2)^*) \\ &= (\xi_1 \otimes \eta_1(r_1, s_1)) (\xi_2 \otimes \eta_2(r_1, s_1))^* = \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_l^{(r,s)}((r_1, s_1), (r_2, s_2)) \\ &= (j_2^{(r,s)})^l(\langle \xi_1, \xi_2 \rangle_l^r \odot \langle \eta_1, \eta_2 \rangle_l^s)((r_1, s_1), (r_2, s_2)). \end{aligned}$$

Now let $k \in \mathbb{k}_c(\mathcal{A}) \odot \mathbb{k}_c(\mathcal{B}) \subseteq \mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B})$. If $k \in I_c^r(\mathcal{A}) \odot I_c^s(\mathcal{B})$, according with the computations above we have $\|\nu(k)\| = \|(j_2^{(r,s)})^l(k)\| = \|k\| = \alpha(k)$, because $(j_2^{(r,s)})^l$ is isometric. Now if k is arbitrary, using this and that $(j_2^{(r,s)})^l : I_r(\mathcal{A}) \otimes_{\alpha} I_s(\mathcal{B}) \rightarrow I_{(r,s)}(\mathcal{A} \otimes_{\alpha} \mathcal{B})$ is an isomorphism, plus Proposition 2.9 and the formulas (3) and (4) in Proposition 2.8 and Proposition 2.9 respectively, we have:

$$\begin{aligned} \|\nu(k)\| &\stackrel{(3)}{=} \sup_{(r,s) \in G \times H} \{ \|\nu(k) * h'\| : h' \in I_{(r,s)}(\mathcal{A} \otimes_{\alpha} \mathcal{B}) \text{ and } \|h'\| \leq 1 \} \\ &= \sup_{(r,s) \in G \times H} \{ \|\nu(k) * (j_2^{(r,s)})^l(h)\| : h \in I_r(\mathcal{A}) \otimes_{\alpha} I_s(\mathcal{B}) \text{ and } \|h\| \leq 1 \} \\ &= \sup_{(r,s) \in G \times H} \{ \|\nu(k) * \nu(h)\| : h \in I_r(\mathcal{A}) \otimes_{\alpha} I_s(\mathcal{B}) \text{ and } \|h\| \leq 1 \} \\ &= \sup_{(r,s) \in G \times H} \{ \|\nu(k * h)\| : h \in I_r(\mathcal{A}) \otimes_{\alpha} I_s(\mathcal{B}) \text{ and } \|h\| \leq 1 \} \\ &\stackrel{2.9}{=} \sup_{(r,s) \in G \times H} \{ \|(j_2^{(r,s)})^l(k * h)\| : h \in I_r(\mathcal{A}) \otimes_{\alpha} I_s(\mathcal{B}) \text{ and } \|h\| \leq 1 \} \\ &= \sup_{(r,s) \in G \times H} \{ \|k * h\| : h \in I_r(\mathcal{A}) \otimes_{\alpha} I_s(\mathcal{B}) \text{ and } \|h\| \leq 1 \} \\ &\stackrel{(4)}{=} \|k\|. \end{aligned}$$

Then ν is an isometric map between the dense subspace $\nu : \mathbb{k}_c(\mathcal{A}) \odot \mathbb{k}_c(\mathcal{B})$ of $\mathbb{k}(\mathcal{A}) \otimes_{\alpha} \mathbb{k}(\mathcal{B})$ and its image, so it extends by continuity to an isomorphism. \square

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