TENSOR PRODUCTS OF FELL BUNDLES OVER GROUPS

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ABSTRACT. We extend the theory of tensor products of C*-algebras to the larger category of Fell bundles over locally compact groups. We prove that, like in the case of C*-algebras, there exist maximal and minimal tensor products. Given two Fell bundles, we compare the tensor products of their cross-sectional algebras with the cross-sectional algebras of their tensor products. As applications we prove that, under certain conditions, the cross-sectional C*-algebra of a Fell bundle is nuclear or exact whenever so is its fiber over the unit element of the group.

CONTENTS

1. INTRODUCTION

The original motivation for the present work was to study nuclearity and exactness of crossed products by partial actions, both important properties of C^*-algebras related with tensor products.

The best way to define and study crossed products by partial actions is through the theory of C^*-algebraic bundles, today also called Fell bundles (for a comprehensive treatment of such theory see [13]). According to [11], given a partial action α of the locally compact group G on the C^*-algebra A, or even a twisted partial action, a Fell bundle $\mathcal{B}_\alpha$ over G is associated to α. The cross-sectional algebra of $\mathcal{B}_\alpha$ is called the crossed-product of A by the partial action α, and it is denoted by $A \rtimes_\alpha G$. Similarly, the reduced cross-sectional algebra of $\mathcal{B}_\alpha$ is called the reduced crossed-product of A by the partial action α, and it is denoted by $A \rtimes_{\alpha,r} G$ (in Section 4 we recall the definition of the reduced cross-sectional algebra of a Fell bundle; for additional information the reader is referred to [9] and [3]). On the other hand, Fell bundles are closely related to partial actions, since not only many of them can be described as associated to twisted partial actions ([11]), but any Fell bundle carries a natural partial action of its underlying group on the spectrum of its unit fiber ([1]) and, in a sense, it is equivalent to the Fell bundle associated to a partial action (see [5], [19] and [4]).

A point exploited in this paper is that some properties of the cross-sectional algebras of a Fell bundle are in part just consequences of properties of the fibers of the bundle itself, which in turn are many times directly related to those of the unit fiber. Moreover, some constructions with these algebras are better understood when they are made directly on the bundle. In particular this viewpoint applies to tensor products. Thus we were led to define and study tensor products of Fell bundles. So posed in terms of Fell bundles, what we are interested in studying are the tensor products of cross-sectional algebras of Fell bundles, and the strategy we follow is to permute the order in which we consider such constructions, i.e., first define the tensor products of Fell bundles and then consider the cross-sectional algebras of the resulting bundles. In fact, what we will show is that these constructions "commute", in the sense that, starting from two Fell bundles, the result is independent of the order in which we take the tensor product and the cross-sectional algebras. Furthermore, we will see that there is a perfect harmony in relation to the type of construction we choose in each case, i.e., maximal tensor products and full cross-sectional algebras, or spatial tensor products and reduced cross-sectional algebras (see below).

Let us describe briefly the contents and structure of the paper.

Since the fibers of a Fell bundle are C*-ternary rings (C^*-trings for short), the study carried out in [6] (in particular Section 5.2) can be considered as a preliminary step in the direction of studying tensor products of Fell bundles. In the present work we will make considerable use of the results of [6], so, for the reader's convenience, in the next section we will recall and expand on some of the aspects that interest us most. Also briefly discussed in this section will be the possibility of extending a C*-norm on the unit fiber of a *-algebraic bundle to a C*-norm on the entire bundle, which will lead to consideration of the notion of positive *-algebraic bundle.

In the third section we deal with tensor products of Fell bundles. If $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ are Fell bundles over the locally compact groups G and H respectively, then a tensor product $\mathcal{A} \bigotimes_\alpha \mathcal{B}$ will be a Fell bundle over $G \times H$, with fibers $A_t \bigotimes_\alpha B_s$. As in the case of C^*-algebras and C^*-trings there are a maximal and a minimal tensor products, which correspond respectively to the maximal and minimal tensor products of the corresponding unit fibers of the bundles. First we consider bundles over discrete groups, and show that the algebraic tensor product $\mathcal{A} \bigodot \mathcal{B}$ is a positive *-algebraic bundle. Then we define the tensor product $\mathcal{A} \bigotimes_\alpha \mathcal{B}$ for any C*-norm on $\mathcal{A} \bigodot \mathcal{B}$. Finally, we topologize $\mathcal{A} \bigotimes_\alpha \mathcal{B}$ for the case the base groups of $\mathcal{A}$ and $\mathcal{B}$ are general locally compact groups. We end the section by generalizing some results on representations of tensor products of C^*-algebras to the case of Fell bundles.

The fourth section is devoted to comparing the cross-sectional algebras of tensor products. Let $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$ be the full and the reduced cross-sectional algebras of the Fell bundle $\mathcal{B}$ respectively. On one hand we prove that $C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B}) \cong C^*(\mathcal{A})\bigotimes_{\max} C^*(\mathcal{B})$, and in the other hand we show that also $C^*_r(\mathcal{A}\bigotimes_{\min}\mathcal{B}) \cong C^*_r(\mathcal{A})\bigotimes_{\min} C^*_r(\mathcal{B})$, which reflects the harmony between universal constructions on one hand and between spatial ones on the other hand. Perhaps it is appropriate to comment here that, in reality, the most useful results we obtain refer to these two norms. However, we have tried to develop the theory in general, which could be useful for example if a theory of "exotic tensor products", in the style of exotic crossed products, were to be developed in the future.

In the final section we consider some applications. We consider Fell bundles with certain approximation properties and we prove that these approximation properties are preserved by taking tensor products. We show how to apply our results to prove the nuclearity or exactness of cross-sectional C*-algebras of Fell bundles under suitable conditions.

This paper corresponds to the first part of [2], and is an expanded version of the previous work "*Tensor products of Fell bundles over discrete groups*" (`http://xxx.if.usp.br/abs/funct-an/9712006`), which circulated as a preprint, and where only Fell bundles over discrete groups were considered. It should also be mentioned that in his 2017 book [8], Exel developed a minimal theory of tensor products between C*-algebras and Fell bundles over discrete groups.

2. C*-trings and Fell bundles

In the first two parts of this section we will recall from [22], [3] and [6] some aspects of the theory of C*-ternary rings and their tensor products that will be needed later. Since in [6] the context is more general than that of tensor products, we have tried to outline the proofs concerning to our setting, mainly those leading to Theorem 2.7. The occasion will also serve to prove some new results and to introduce some of the notation to be used later. In the third part of the section we will begin the preparation for defining tensor products of Fell bundles in the next section.

2.1. **C*-trings and the functors of Zettl.** A $*$-ternary ring, or $*$-tring for short, is a complex vector space E with a map (called $*$-ternary product) $\mu : E\times E\times E \to E$, which is linear in the odd variables and conjugate linear in the second one, and satisfies: $\mu\big(\mu(x,y,z),u,v\big) = \mu\big(x,\mu(u,z,y),v\big) = \mu\big(x,y,\mu(z,u,v)\big)$, $\forall x,y,z,u,v \in E$. A C*-seminorm on E is a seminorm that satisfies $\|\mu(x,y,z)\| \le \|x\|\|y\|\|z\|$, and $\|\mu(x,x,x)\| = \|x\|^3$ $\forall x,y,z \in E$. A $*$-tring E with a C*-norm making it a Banach space is called a C*-ternary ring, or just a C*-tring. In general we write just (x,y,z) instead of $\mu(x,y,z)$. Note that if (E,μ) is a C*-tring, its opposite $E^{\rm op} := (E,-\mu)$ also is a C*-tring.

In [22] Zettl proved that if E is a C*-tring, then there exist a C*-algebra E^r (unique up to isomorphism) and an E^r-valued sesquilinear map $\langle\ ,\ \rangle_r : E\times E\to E^r$ such that E is a right E^r-module and $\langle\ ,\ \rangle_r$ satisfies all the properties of a right inner product except possibly that of positivity, with $(x,y,z)=x\langle y,z\rangle_r$, and $\|x\|^2=\|\langle x,x\rangle_r\|\ \forall x,y,z\in E$, and in addition $\mathrm{span}\{\langle y,z\rangle_r : y,z\in E\}$ is dense in E^r. Moreover, he showed that, if $E_+ := \{x\in E : \langle x,x\rangle_r\in E^{r,+}\}$ and $E_- := \{x\in E : -\langle x,x\rangle_r\in E^{r,+}\}$ (here $E^{r,+}$ is the positive cone of the C*-algebra E^r), then E_+ and E_- are sub-C*-trings of E such that $\langle E_+,E_-\rangle=0$ and $E=E_+\oplus E_-$ as C*-trings. When $E=E_+$ we say that E is a positive C*-tring (so in this case the sesquilinear map $\langle\ ,\ \rangle_r$ is an inner product). When $E=E_-$, so E is the opposite of a positive C*-tring, we say that E is a negative C*-tring. Besides, $(E_+,\langle\ ,\ \rangle_r)$ and $(E_-,-\langle\ ,\ \rangle_r)$ are full right Hilbert modules over $(E_+)^r$ and $(E_-)^r$ respectively, and $E^r=(E_+)^r\oplus(E_-)^r$ as C*-algebras. Note that, conversely, Hilbert modules provide examples of C*-trings: if $(F,\langle\ ,\ \rangle)$ is a right Hilbert module, and we define $\mu(x,y,z):=x\langle y,z\rangle$, then both (F,μ) and $(F,-\mu)$ are C*-trings, the former positive.

Actually, C*-trings are the objects of a category, which we denote Ct, in which the morphisms are linear maps $\pi : E\to F$ that preserve the ternary product, that is $\pi(x,y,z)=(\pi x,\pi y,\pi z)$, $\forall x,y,z\in E$. As shown in [3], in this case there exists a unique homomorphism $\pi^r : E^r\to F^r$ such that

$$\pi^r(\langle x,y\rangle_r)=\langle \pi x,\pi y\rangle\quad \forall x,y\in E, \tag{1}$$

so the the correspondence $E\mapsto E^r$ is in fact the object part of a functor from the category Ct of C*-trings to the category C of C*-algebras. In particular, if E is a full right Hilbert module over the C*-algebra A, and we define on E the ternary product $(x,y,z):=x\langle y,z\rangle_A$ as above, then we have an isomorphism $E^r\cong A$, such that $\langle x,y\rangle_r\mapsto\langle x,y\rangle_A$, $\forall x,y\in E$. It is easily seen that, as is the case with homomorphisms of C*-algebras, morphisms of C*-trings are automatically contractive and have closed range, and are isometric exactly when they are injective. In passing, we note that a C*-algebra is also a C*-tring with the $*$-ternary product given by $(x,y,z):=xy^*z$. Then any homomorphism of C*-algebras is also a morphism or $*$-ternary rings, so the category of C*-algebras embedds into the category of C*-trings.

Finally, let us mention that, just as we have a Zettl functor $\big(E\xrightarrow{\pi}F\big)\mapsto\big(E^r\xrightarrow{\pi^r}F^r\big)$ on the right, we also have one on the left: $\big(E\xrightarrow{\pi}F\big)\mapsto\big(E^l\xrightarrow{\pi^l}F^l\big)$. Of course, here E^l is a C*-algebra and we have an E^l-sesquilinear map $\langle\ ,\ \rangle_l : E\times E\to E^l$ such that E is a left E^l-module and $\langle\ ,\ \rangle_l$ satisfies all the properties of a left inner product, except possibly that of positivity, with $(x,y,z)=\langle x,y\rangle_l z$, $\forall x,y,z\in E$, and also $E^l=\overline{\mathrm{span}}\{\langle y,z\rangle_r : y,z\in E\}$. Combining both Zettl functors we conclude that a positive C*-tring E is an E^l-E^r Morita-Rieffel equivalence bimodule. In fact, if $E=E_+\oplus E_-$ is the Zettl's decomposition of E, then $E^p:=E_+\oplus(E_-)^{\mathrm{op}}$ is a positive C*-tring, and we have $E^r=(E^p)^r$ and $E^l=(E^p)^l$, so an arbitrary C*-tring E is close to being an equivalence bimodule, and in any case its associated C*-algebras E^l and E^r are Morita-Rieffel equivalent. For this reason, many properties of these C*-algebras can be considered as inherited from the C*-tring. This is the case of nuclearity and exactness for example, as shown in [6].

2.2. **Tensor products of C*-ternary rings and of Hilbert modules.** Suppose that E and F are right Hilbert modules over the C*-algebras A and B respectively. Then one can form its *exterior tensor product* $E \bigotimes F$, which is a right Hilbert module over the C*-algebra $A \bigotimes B$, where the latter is the spatial tensor product of A and B (see [15]). However, as shown below, it is possible to make the same construction using other tensor products between A and B and without major modifications..

In what follows we denote by $\mathcal{SN}(E)$ and by $\mathcal{N}(E)$ the sets of C*-seminorms and C*-norms respectively on the $*$-tring or $*$-algebra E. Note that $\mathcal{SN}(E)$ is a partially ordered set with the pointwise order: $\gamma_1 \leq \gamma_2 \iff \gamma_1(x) \leq \gamma_2(x)\ \forall x \in E$.

Recall that if $(G, \|\ \|)$ is a seminormed space, and $N := \{x \in G : \|x\| = 0\}$, the Hausdorff completion of G is the completion of the quotient space G/N with respect to the quotient norm $\|x + N\| := \|x\|$.

Suppose that A is a $*$-algebra and that α is a C*-seminorm on A. Then the Hausdorff completion of A is a C*-algebra, which we denote by A_α. Let $p_\alpha : A \to A_\alpha$ be the canonical map. If A_α^+ is the set of positive elements of the C*-algebra A_α, the set $p_\alpha^{-1}(A_\alpha^+)$ is a cone in A, whose elements will be called α-positive elements of A. Note that if $\alpha \geq \beta$ are C*-seminorms, then the identity $id : (A, \alpha) \to (A, \beta)$ is continuous, so it defines a surjective homomorphism of C*-algebras $\sigma_\beta^\alpha : A_\alpha \to A_\beta$ such that $p_\beta = \sigma_\beta^\alpha p_\alpha$. Thus any α-positive element is also a β-positive element.

Definition 2.1. Let A be a $*$-algebra. We define the set of positive elements of A to be the set $A^+ := \cap_{\alpha \in \mathcal{SN}(A)} p_\alpha^{-1}(A_\alpha)$, where A_α is the Hausdorff completion of A with respect to the C*-seminorm α.

Note that elements of the set $C_A := \{\sum_{i=1}^n a_i^* a_i : n \in \mathbb{N}, a_1, \dots, a_n \in A\}$ are α-positive, $\forall \alpha \in \mathcal{SN}(A)$.

Remark 2.2. If $\mathcal{N}(A) \neq \emptyset$, then $A^+ := \cap_{\alpha \in \mathcal{N}(A)} p_\alpha^{-1}(A_\alpha)$, that is, we only need C*-norms rather than C*-seminorms to determine the positive elements. To see this, let β be any C*-seminorm on A, and α a C*-norm that it is supposed to exist on A. Then $\gamma := \max\{\alpha, \beta\} \in \mathcal{N}(A)$ and $\gamma \geq \beta$. Therefore, as observed before the definition, $p_\gamma^{-1}(A_\gamma) \subseteq p_\beta^{-1}(A_\beta)$. Then $\cap_{\alpha \in \mathcal{N}(A)} p_\alpha^{-1}(A_\alpha) \subseteq \cap_{\beta \in \mathcal{SN}(A)} p_\beta^{-1}(A_\beta) \subseteq \cap_{\alpha \in \mathcal{N}(A)} p_\alpha^{-1}(A_\alpha)$, so they are equal.

We will need the following result, which is exactly [15, Lemma 4.3], except that the C*-norm considered here is arbitrary, while Lance's version is only stated for the minimal norm. Since the proof is also the same, we omited it.

Lemma 2.3. *Let A and B be C^*-algebras, and suppose that $\mathfrak{a} = (a_{ij})$, $\mathfrak{c} = (c_{ij}) \in M_n(A)$, $\mathfrak{b} = (b_{ij})$, $\mathfrak{d} = (d_{ij}) \in M_n(B)$. Let $A \bigotimes_\alpha B$ be a C^*-tensor product of A and B. Then:*

1. *If $0 \leq \mathfrak{a} \leq \mathfrak{c}$ and $0 \leq \mathfrak{b} \leq \mathfrak{d}$, we have $0 \leq (a_{ij} \otimes b_{ij}) \leq (c_{ij} \otimes d_{ij})$ in $M_n(A \bigotimes_\alpha B)$.*
2. *If $\mathfrak{a}$, $\mathfrak{b} \geq 0$, then $\sum_{i,j=1}^n a_{ij} \otimes b_{ij} \geq 0$ in $A \bigotimes_\alpha B$.*

Let E and F be right Hilbert modules over the C*-algebras A and B, and let $E \odot F$ and $A \odot B$ their corresponding algebraic tensor products. Using the universal property of the algebraic tensor product, we easily see that $E \bigodot F$ is a right module over $A \bigodot B$ and that we have an $A \bigodot B$-valued sesquilinear form on $E \bigodot F$. On elementary tensors the action and the form are given by $(x \odot y)(a \odot b) = xa \odot yb$ and $\langle x \odot y, x' \odot y' \rangle = \langle x, x' \rangle_E \odot \langle y, y' \rangle_F$. We want to see that this sesquilinear form is positive, that is, that $\langle z, z \rangle \in (A \odot B)^+$ according to Definition 2.1.

Proposition 2.4. *The sesquilinear map $\langle\, ,\rangle : (E \bigodot F) \times (E \bigodot F) \to A \bigodot B$, given by $\langle z, z' \rangle = \sum_{i=1}^n \sum_{j=1}^m \langle x_i, x_j' \rangle_E \odot \langle y_i, y_j' \rangle_F$ for $z = \sum_{i=1}^n x_i \odot y_i$, $z' = \sum_{j=1}^m x_j' \odot y_j'$ is positive, that is $\langle z, z \rangle \in (A \odot B)^+$ $\forall z \in E \bigodot F$.*

Proof. Since $\mathcal{N}(A \bigodot B) \neq \emptyset$, by Remark 2.2 it is enough to show that $\langle z, z \rangle \in (A \otimes_\alpha B)^+$ for every $\alpha \in \mathcal{N}(A \bigodot B)$ (in fact it would be enough to do so just for $\|\ \|_{\max}$, but the proof is the same).

So let α be any C*-norm on $A \bigodot B$, and $z = \sum_{i=1}^n x_i \odot y_i \in E \bigodot F$. By [15, Lemma 4.2] the Gramian matrices $X = (\langle x_i, x_j \rangle_E)$ and $Y = (\langle y_i, y_j \rangle_F)$ are positive elements of $M_n(A)$ and $M_n(B)$ respectively. Therefore $\langle z, z \rangle = \sum_{i,j=1}^n \langle x_i, x_j \rangle_E \odot \langle y_i, y_j \rangle_F$ is a positive element of $A \bigotimes_\alpha B$ by (2) of Lemma 2.3, which ends the proof. $\square$

Let α be a C*-norm on $A \bigodot B$. Since the sesquilinear map just defined $\langle\, ,\rangle : (E \bigodot F) \times (E \bigodot F) \to A \bigodot B$ is positive, we can perform the double completion process described in [15, top of page 5] to obtain a Hilbert module $E \bigotimes_{\tilde{\alpha}} F$, which is the completion of $E \bigodot F$ with respect to the norm $\tilde{\alpha} : E \bigodot F \to \mathbb{R}$ given by

$$\tilde{\alpha}(z) := \sqrt{\alpha(\langle z, z \rangle)}, \quad \forall z \in E \bigodot F. \tag{2}$$

Remark 2.5. Lance proves along [15] that for z as in Proposition 2.4 we have $z = 0$ in $E \bigodot F$ if and only if $\langle z, z \rangle = 0$ in $A \bigodot B$. This shows that the sesquilinear maps above are actually inner products.

Definition 2.6. We call the right Hilbert $A \bigotimes_\alpha B$-module $E \bigotimes_{\tilde{\alpha}} F$ the α-exterior product corresponding to the C*-norm $\alpha \in \mathcal{N}(E \bigodot F)$.

Note that $E \bigotimes_{\tilde{\alpha}} F$ is full whenever E and F are full Hilbert modules.

We turn again to the C*-trings perspective. Suppose that E and F are positive C*-trings, so they are full right Hilbert modules over the C*-algebras

E^r and F^r respectively. Note that $E\bigodot F$ has a structure of $*$-tring with the ternary product given by $(x\odot y, x'\odot y', z\odot z') := (x,y,z)\odot(x',y',z')$ on elementary tensors, which in terms of our just defined sesquilinear form can be written as $(x\odot y, x'\odot y', z\odot z') = (x\odot y)\langle x'\odot y', x''\odot y''\rangle$. So we have just seen that every C*-norm α on $E^r\bigodot F^r$ defines a C*-norm $\tilde{\alpha}$ on the $*$-tring $E\bigodot F$ (given by (2)), whose completion is the positive C*-tring $E\bigotimes_{\tilde{\alpha}} F$, and $(E\bigotimes_{\tilde{\alpha}} F)^r$ turns out to be $E^r\bigotimes_\alpha F^r$ (recall (1) and subsequent comments).

Suppose conversely that γ is a C*-norm on the $*$-tring $E\bigodot F$, and let $E\bigotimes_\gamma F$ be the corresponding completion, which is a C*-tring. Let $E_0^r := \mathrm{span}\{\langle x,x'\rangle_E : x,x'\in E\}$ and $F_0^r := \mathrm{span}\{\langle y,y'\rangle_F : y,y'\in F\}$. Then E_0^r and F_0^r are dense two-sided ideals of E^r and F^r respectively. Let $z = \sum_{i=1}^n x_i\odot y_i\in E\bigodot F$ and $c := \sum_{j=1}^m \langle x_j',x_j''\rangle_E\odot\langle y_j',y_j''\rangle_F\in E^r\bigodot F^r$. We have

$$\begin{aligned} zc &= \sum_{j=1}^m\sum_{i=1}^n x_i\langle x_j',x_j''\rangle_E\odot y_i\langle y_j',y_j''\rangle_F = \sum_{j=1}^m\sum_{i=1}^n (x_i,x_j',x_j'')_E\odot(y_i,y_j',y_j'')_F\\ &= \sum_{j=1}^m\Big(\sum_{i=1}^n (x_i\odot y_i, x_j'\odot y_j', x_j''\odot y_j'') = \sum_{j=1}^m (z, x_j'\odot y_j', x_j''\odot y_j'').\end{aligned}$$

So, since γ is a C*-norm: $\gamma(zc) = \gamma(\sum_{j=1}^m (z,x_j'\odot y_j', x_j''\odot y_j'')) \le \sum_{j=1}^m \gamma(x_j'\odot y_j')\gamma(x_j''\odot y_j'')\gamma(z)$, $\forall z\in E\bigodot F$. Therefore the action of multiplication by c is γ-bounded on $E\bigodot F$, and hence it extends to a bounded operator on $E\bigotimes_\gamma F$. In fact, recalling that $(x\odot y, x'\odot y', x''\odot y'') = (x\odot y)\langle x'\odot y', x''\odot y''\rangle$, where the latter is the inner product that Zettl's associates to the C*-tring $E\bigotimes_\gamma F$, we can continue our computations above, and get:

$$\begin{aligned} zc &= \sum_{j=1}^m (z, x_j'\odot y_j', x_j''\odot y_j'') = \sum_{j=1}^m z\langle x_j'\odot y_j', x_j''\odot y_j''\rangle\\ &= z\Big(\sum_{j=1}^m \langle x_j'\odot y_j', x_j''\odot y_j''\rangle\Big).\end{aligned}$$

Thus we get an injective homomorphism of $*$-algebras $E^r\bigodot F^r\to(E\bigotimes_\gamma F)^r$, given by $c = \sum_{j=1}^m\langle x_j',x_j''\rangle_E\odot\langle y_j',y_j''\rangle_F\mapsto\sum_{j=1}^m\langle x_j'\otimes y_j', x_j''\otimes y_j''\rangle$. So we can define on $E_0^r\bigodot F_0^r$ the operator norm, namely $\gamma^r : E_0^r\bigodot F_0^r\to\mathbb{R}$ such that

$$\gamma^r(c) := \sup\{\gamma(zc) : z\in E\bigodot F, \gamma(z)\le 1\}. \tag{3}$$

Now observe that, since E_0^r and F_0^r are dense ideals in E^r and F^r respectively, this C*-norm uniquely extends to a C*-norm on $E^r\bigodot F^r$ (because of [6, Lemma 5.12]) by the same formula (3).

In conclusion, given two positive C*-trings, we have two maps

$$\Psi_r : \mathcal{N}(E^r \bigodot F^r) \to \mathcal{N}(E \bigodot F) \text{ such that } \alpha \mapsto \tilde{\alpha} \text{ given by (2)}$$
$$\Phi_r : \mathcal{N}(E \bigodot F) \to \mathcal{N}(E^r \bigodot F^r) \text{ such that } \gamma \mapsto \gamma^r \text{ given by (3).}$$

And these correspondences satisfy

$$(4) \qquad (E \bigotimes_{\tilde{\alpha}} F)^r = E^r \bigotimes_{\alpha} F^r \qquad \textit{and} \qquad E^r \bigotimes_{\gamma^r} F^r = (E \bigotimes_{\gamma} F)^r.$$

It is easily checked that Ψ_r is order preserving and $\Psi_r\Phi_r$ is the identity on $\mathcal{N}(E \bigodot F)$. Moreover, due to the uniqueness of the C*-algebra E^r, $\Phi_r\Psi_r$ is the identity on $\mathcal{N}(E^r \bigodot F^r)$. Finally, Φ_r also is order preserving: if $\gamma_1 \geq \gamma_2$ are C*-norms on $E \bigodot F$, then $id : (E \bigodot F, \gamma_1) \to (E \bigodot F, \gamma_2)$ is continuous, and therefore it extends by continuity to a homomorphism of C*-trings $\pi : E \bigotimes_{\gamma_1} F \to E \bigotimes_{\gamma_2} F$, which induces a homomorphism of C*-algebras $\pi^r : E^r \bigotimes_{\gamma_1^r} F^r \to E^r \bigotimes_{\gamma_2^r} F^r$, thus contractive; therefore $\gamma_1^r \geq \gamma_2^r$. In conclusion the maps Φ_r and Ψ_r are mutually inverse isomorphisms between the posets $\mathcal{N}(E \bigodot F)$ and $\mathcal{N}(E^r \bigodot F^r)$. We record this fact:

Theorem 2.7. *Let E and F be positive C*-trings. Then the maps $\Psi_r : \mathcal{N}(E^r \bigodot F^r) \to \mathcal{N}(E \bigodot F)$ such that $\alpha \mapsto \tilde{\alpha}$, given by* (2)*, and $\Phi_r : \mathcal{N}(E \bigodot F) \to \mathcal{N}(E^r \bigodot F^r)$ such that $\gamma \mapsto \gamma^r$, given by* (3)*, are mutually inverse isomorphisms of partially ordered sets. Moreover, if $\alpha \in \mathcal{N}(E^r \bigodot F^r)$ and $\gamma \in \mathcal{N}(E \bigodot F)$, then $E \bigotimes_{\tilde{\alpha}} F$ and $E \bigotimes_{\gamma} F$ are full right Hilbert modules over $E^r \bigotimes_{\alpha} F^r$ and $E^r \bigotimes_{\gamma^r} F^r$ respectively, so $\big(E \bigotimes_{\tilde{\alpha}} F\big)^r \cong E^r \bigotimes_{\alpha} F^r$ and $\big(E \bigotimes_{\gamma} F\big)^r \cong E^r \bigotimes_{\gamma^r} F^r$, where the isomorphisms extend the map $\langle x \odot y, x' \odot y' \rangle \mapsto \langle x, x' \rangle \odot \langle y, y' \rangle$, $\forall x, x' \in E$, $y, y' \in F$.*

In fact in [6] it is proved that the correspondences above extend to isomorphisms between $\mathcal{SN}(E \bigodot F)$ and $\mathcal{SN}(E^r \bigodot F^r)$ for abitrary C*-trings.

In particular, since Ψ_r and Φ_r are order preserving maps, and $\mathcal{N}(E^r \bigodot F^r)$ has a maximum and a minimum elements $\| \, \|_{\max}$ and $\| \, \|_{\min}$ respectively, we have:

Corollary 2.8. *(cf* [6, Corollary 5.13]*). Let E and F be positive C*-trings. Then there exist a maximum C*-norm $\| \cdot \|_{\max}$ on $E \bigodot F$, and a minimum C*-norm $\| \cdot \|_{\min}$ on $E \bigodot F$, and*

$$\big(E \bigotimes_{\max} F\big)^l = E^l \bigotimes_{\max} F^l, \qquad \big(E \bigotimes_{\max} F\big)^r = E^r \bigotimes_{\max} F^r,$$
$$\big(E \bigotimes_{\min} F\big)^l = E^l \bigotimes_{\min} F^l \qquad \big(E \bigotimes_{\min} F\big)^r = E^r \bigotimes_{\min} F^r.$$

Recall that the minimum norm on the $*$-algebras $E^r \bigodot F^r$ and $E^l \bigodot F^l$ agrees with the so called spatial one.

Remark 2.9. Let α be a C*-norm on $E^r \bigodot F^r$. The well-known fact that α is cross-norm, that is $\alpha(a \odot b) = \|a\|_{E^r} \|b\|_{F^r}$ $\forall a \in E^r$ and $b \in F^r$, implies that $\tilde{\alpha}$ also is cross-norm, for if $x \in E$, $y \in F$:

$$\begin{aligned}\tilde{\alpha}(x \odot y)^2 &= \alpha(\langle x \odot y, x \odot y\rangle) = \alpha(\langle x, x\rangle \odot \langle y, y\rangle)\\ &= \|\langle x, x\rangle\|_E \|\langle y, y\rangle\|_F = \tilde{\alpha}(x)^2 \tilde{\alpha}(y)^2.\end{aligned}$$

To end this part of the section we prove the following two results, which will be needed later. To prove the first of them, recall first from [3, Proposition 4.1] that if $\pi : E \to F$ is an injective homomorphism of C*-trings, then the induced homomorphism of C*-algebras $\pi^r : E^r \to F^r$ also is injective. We remark that the converse is easily proved as well: if $\pi(x) = 0$, then $0 = \langle \pi(x), \pi(x)\rangle = \pi^r(\langle x, x\rangle)$, so $\langle x, x\rangle = 0$ if π^r is injective, and in this case $x = 0$.

Proposition 2.10. *Let $\pi_1 : E_1 \to F_1$ and $\pi_2 : E_2 \to F_2$ be homomorphisms of positive C*-trings. Then $\pi_1 \odot \pi_2 : E_1 \bigodot E_2 \to F_1 \bigodot F_2$ is $\| \ \|_{min}$-continuous, so it extends to a homomorphism $\pi_1 \bigotimes_{\min} \pi_2 : E_1 \bigotimes_{\min} E_2 \to F_1 \bigotimes_{\min} F_2$. Moreover, if π_1 and π_2 are injective, then so is $\pi_1 \bigotimes_{\min} \pi_2$.*

Proof. Applying the right Zettl functor we obtain homomorphisms $\pi_1^r : E_1^r \to F_1^r$ and $\pi_2^r : E_2 \to F_2^r$, so by [20, T.5.19] we get a homomorphism $\pi_1^r \bigotimes_{\min} \pi_2^r : E_1^r \bigotimes_{\min} E_2^r \to F_1^r \bigotimes_{\min} F_2^r$, which by definition extends $\pi_1^r \bigodot \pi_2^r : E_1^r \bigodot E_2^r \to F_1^r \bigodot F_2^r$. Now, if $z = \sum_{i=1}^n x_i \odot y_i \in E \bigodot F$:

$$\begin{aligned}\langle(\pi_1 \odot \pi_2)z, (\pi_1 \odot \pi_2)z\rangle &= \sum_{i,j=1}^n \langle \pi_1(x_i) \odot \pi_2(y_i), \pi_1(x_j) \odot \pi_2(y_j)\rangle\\ &= \sum_{i,j=1}^n \langle \pi_1(x_i), \pi_1(x_j)\rangle \odot \langle \pi_2(y_i), \pi_2(y_j)\rangle\\ &= \sum_{i,j=1}^n \pi_1^r(\langle x_i, x_j\rangle) \odot \pi_2^r(\langle y_i, y_j\rangle) = (\pi_1^r \odot \pi_2^r)(\langle z, z\rangle).\end{aligned}$$

Therefore, since $\|(\pi_1 \odot \pi_2)z\|_{\min}^2 = \|\langle(\pi_1 \odot \pi_2)z, (\pi_1 \odot \pi_2)z\rangle\|_{\min}$, we get

$$\|(\pi_1 \odot \pi_2)z\|_{\min}^2 = \|(\pi_1^r \odot \pi_2^r)(\langle z, z\rangle)\|_{\min} = \|(\pi_1^r \otimes_{\min} \pi_2^r)(\langle z, z\rangle)\|_{\min} \leq \|z\|_{\min}^2,$$

which ends the proof of the first statement. As for the second one, it follows from the last assertion of [20, T.6.9] and the remark preceding the present Proposition. □

In the same way, but using [20, T.6.9] instead of [20, T.5.19], we obtain

Proposition 2.11. *Let $\pi_1 : E_1 \to F_1$ and $\pi_2 : E_2 \to F_2$ be homomorphisms of positive C*-trings. Then $\pi_1 \odot \pi_2 : E_1 \bigodot E_2 \to F_1 \bigodot F_2$ is $\| \ \|_{max}$-continuous, so it extends to a homomorphism $\pi_1 \bigotimes_{\max} \pi_2 : E_1 \bigotimes_{\max} E_2 \to F_1 \bigotimes_{\max} F_2$.*

It follows from Propositions 2.10 and 2.11 that both the minimal and maximal tensor products are bifunctors $\mathsf{Ct}\times\mathsf{Ct}\to\mathsf{Ct}$, where Ct is the category of C*-trings.

Lemma 2.12. *Let E and F be full right Hilbert modules over the C*-algebras A and B respectively, and $S\in\mathcal{L}(E)$, $T\in\mathcal{L}(F)$. If γ is a C*-norm on $E\bigodot F$, then the map $S\odot T: E\bigodot F\to E\bigodot F$ is γ-continuous, with $\|S\odot T\|\leq\|S\|\|T\|$, and it extends to an adjointable map $S\otimes T\in\mathcal{L}(E\bigotimes_\gamma F)$, whose adjoint is $S^*\otimes T^*$.*

Proof. First recall that the C*-norm γ induces a C*-norm γ^r on $A\bigodot B$, namely the operator norm

$$\gamma^r(c)=\sup\{\gamma(zc): z\in E\bigodot F:\gamma(z)\leq 1\}.$$

Let $z=\sum_{i=1}^n x_i\odot y_i\in E\bigodot F$, and consider the matrices $X=(\langle x_i,x_j\rangle)$, $X_S=(\langle Sx_i,Sx_j\rangle)$, $Y=(\langle y_i,y_j\rangle)$ and $Y_T=(\langle Ty_i,Ty_j\rangle)$. By [15, Lemmas 4.1 and 4.2], we see that $0\leq X_S\leq\|S\|^2X$ and $0\leq Y_T\leq\|T^2\|Y$. Then, using the last assertion of Lemma 2.3, we get:

$$\begin{aligned}\langle (S\odot T)z,(S\odot T)z\rangle &=\sum_{i,j=1}^n\langle Sx_i\odot Ty_i, Sx_j\odot Ty_j\rangle\\ &=\sum_{i,j=1}^n\langle Sx_i,Sx_j\rangle\odot\langle Ty_i,Ty_j\rangle\\ &\leq\sum_{i,j=1}^n\|S\|^2\langle x_i,x_j\rangle\odot\|T\|^2\langle y_i,y_j\rangle=\|S\|^2\|T\|^2\langle z,z\rangle.\end{aligned}$$

Therefore:

$$\begin{aligned}\gamma((S\odot T)z)^2&=\gamma^r(\langle (S\odot T)z,(S\odot T)z\rangle)\leq\gamma^r(\|S\|^2\|T\|^2\langle z,z\rangle)\\ &=\|S\|^2\|T\|^2\gamma^r(\langle z,z\rangle)=\|S\|^2\|T\|^2\gamma(z)^2.\end{aligned}$$

We conclude that $S\odot T$ is bounded, with $\|S\odot T\|\leq\|S\|\|T\|$ as claimed. Thus $S\odot T$ extends by continuity to a map $S\otimes T$. It is now easy to verify that $S\otimes T$ is adjointable, and that $(S\otimes T)^*=S^*\otimes T^*$. □

Corollary 2.13. *Let E and F be Hilbert modules, and γ be a C*-norm on $E\bigodot F$. Then there exists a (unique) C*-norm $\overline{\gamma}$ on $\mathcal{L}(E)\bigodot\mathcal{L}(F)$ such that $\mathcal{L}(E)\bigotimes_{\overline{\gamma}}\mathcal{L}(F)$ is a C*-subalgebra of $\mathcal{L}(E\bigotimes_\gamma F)$. In case $\gamma=\|\ \|_{min}$, also is $\overline{\gamma}=\|\ \|_{min}$.*

Proof. It follows from Lemma 2.12 that we have a $*$-homomorphism $\varphi_\gamma:\mathcal{L}(E)\bigodot\mathcal{L}(F)\to\mathcal{L}(E\bigotimes_\gamma F)$. In case $\gamma=\|\ \|_{\min}$, in [15, page 37] it is shown that this homomorphism extends to an isometric homomorphism $\varphi_{\min}:\mathcal{L}(E)\bigotimes_{\min}\mathcal{L}(F)\to\mathcal{L}(E\bigotimes_{\min}F$ (which in particular proves our last statement). Suppose $T\in\mathcal{L}(E)\bigodot\mathcal{L}(F)$ is such that $\varphi_\gamma(T)=0$. Since φ_γ and $\varphi_{\min}$ agree on $E\bigodot F$, the fact $\varphi_\gamma(T)|_{E\bigodot F}=0$, implies

$\varphi_{\min}(T)|_{E\bigodot F} = 0$ and, since $E \bigodot F$ is dense in $E \bigotimes_{\min} F$, this entails $\varphi_{\min}(T) = 0$. Since $\varphi_{\min}$ is injective, we conclude that $T = 0$. Consequently φ_γ is injective, and therefore, identifying $\mathcal{L}(E) \bigodot \mathcal{L}(F)$ with the *-subalgebra $\varphi_\gamma(\mathcal{L}(E) \bigodot \mathcal{L}(F))$, it is enough (and necessary) to take $\overline{\gamma}$ as the restriction of the norm on $\mathcal{L}(E \bigotimes_\gamma F)$ to $\mathcal{L}(E) \bigodot \mathcal{L}(F)$. □

2.3. **Positive $*$-algebraic bundles and Fell bundles.**

Definition 2.14. Let G be a discrete group, and suppose that $\mathcal{C} = (C_t)_{t\in G}$ is a family of complex vector spaces. We identify $\mathcal{C}$ with the disjoint union of the spaces C_t. We then say that $\mathcal{C}$ is a $*$-algebraic bundle over G, with product $\cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and involution $* : \mathcal{C} \to \mathcal{C}$ if, $\forall a, b \in \mathcal{C}$, $t, s \in G$, the following holds:

1) $C_s C_t \subseteq C_{st}$
2) The product $\cdot$ is bilinear on $C_s \times C_t \to C_{st}$
3) The product on $\mathcal{C}$ is associative.
4) $*$ is conjugate linear from C_t into $C_{t^{-1}}$.
5) $(C_t)^* \subseteq C_{t^{-1}}$
6) $(ab)^* = b^* a^*$.
7) $a^{**} = a$.

The vector spaces C_t are called the fibers of the bundle. Note that each C_t is a $*$-tring with the product $(a, b, c) := ab^*c$, and in particular C_e is a $*$-algebra (here and in the rest of the paper e will denote the unit of a group).

Suppose that $\mathcal{C} = (C_t)$ is a $*$-algebraic bundle over G, and that $\mathcal{I} = (I_t)$ is a subset of $\mathcal{C}$ such that $\mathcal{I}$ is also a $*$-algebraic bundle with the operations inherited from $\mathcal{C}$, which moreover satisfies $\mathcal{C}\mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I}\mathcal{C} \subseteq \mathcal{I}$. Then we say that $\mathcal{I}$ is a (two-sided) ideal of $\mathcal{C}$. It is easy to see $\mathcal{C}/\mathcal{I} := (C_t/I_t)$ is also a $*$-algebraic bundle with the obviuos operations naturally induced on the quotients by the operations on $\mathcal{C}$.

Definition 2.15. Let $\mathcal{C} = (C_t)_{t\in G}$ be a $*$-algebraic bundle, and α a C^*-seminorm on C_e. We say that $\mathcal{C}$ is an α-positive $*$-algebraic bundle if for each $c \in \mathcal{C}$ the element c^*c is positive in the Hausdorff completion $(C_e)_\alpha$ of C_e. We say that $\mathcal{C}$ is a positive $*$-algebraic bundle if it is α-positive $\forall \alpha \in \mathcal{SN}(C_e)$.

In other words, $\mathcal{C}$ is positive if $c^*c \in C_e^+$ in the meaning of C_e^+ according to Definition 2.1.

Definition 2.16. Let $\mathcal{C} = (C_t)_{t\in G}$ be a $*$-algebraic bundle. Let $\|\cdot\| : \mathcal{C} \to \mathbb{R}$ be such that:

8) $(C_t, \|\cdot\|)$ is a seminormed space, $\forall t \in G$.
9) $\|c_1 c_2\| \leq \|c_1\| \, \|c_2\| \ \forall c_1, c_2 \in \mathcal{C}$.
10) $\|c^*c\| = \|c\|^2$.

We then say that $\|\ \|$ is a C^*-seminorm on $\mathcal{C}$, and that it is a C^*-norm if each $(C_t, \|\cdot\|)$ is a normed space. We represent respectively by $\mathcal{SN}(\mathcal{C})$ and $\mathcal{N}(\mathcal{C})$ the sets of of C^*-seminorms and C^*-norms on $\mathcal{C}$.

If, moreover,

11) $\mathcal{C}$ is α-positive, where α is the restriction of $\|\ \|$ to the unit fiber C_e,

we will say that $(\mathcal{C}, \cdot, *, \|\cdot\|)$ is a semi-pre-Fell bundle over the discrete group G, and that it is a pre-Fell bundle if $\| \ \|$ is a C^*-norm. A pre-Fell bundle $\mathcal{C}$ is called a Fell bundle if each $(C_t, \|\cdot\|)$ is complete for all $t \in G$.

Note that 9) and 10) imply that $\|c^*\| = \|c\|$, $\forall c \in \mathcal{C}$, and also that $\| \ \|$ is a C*-norm on the $*$-tring C_t.

The proof of the following result is routine, and it is left to the reader.

Proposition 2.17. *Let $\mathcal{C}^0 = (C_t^0)_{t\in G}$ be a pre-Fell bundle over the discrete group G, with C^*-norm $\|\cdot\|$. For $t \in G$, let C_t be the completion of C_t^0, and consider the family of Banach spaces $(C_t)_{t\in G}$ with the extended norm. Then the product and involution on $\mathcal{C}^0$ extend by continuity to $\mathcal{C}$, and with the extended operations and norm $\mathcal{C}$ is a Fell bundle over G. We say that $\mathcal{C}$ is a completion of the pre-Fell bundle $\mathcal{C}^0$.*

Given a semi-pre-Fell bundle $\mathcal{C} = (C_t)_{t\in G}$, let $\mathcal{I} := \{x \in \mathcal{C} : \|x\| = 0\}$. Then $\mathcal{I}$ can be identified with the $*$-algebraic bundle $\mathcal{I} = (I_t)_{t\in G}$, where $I_t := \mathcal{I} \cap C_t$, $\forall t \in G$. Note that $\mathcal{I}$ is also an ideal of $\mathcal{C}$, for property 9) above implies $\mathcal{C}\mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I}\mathcal{C} \subseteq \mathcal{I}$. It is easy to check that $\mathcal{C}/\mathcal{I} := (C_t/I_t)_{t\in G}$ is a pre-Fell bundle with the norm induced by the seminorm on $\mathcal{C}$: if $c \in C_t$, then $\|c_t + I_t\| := \|c_t\|$. We will say that the Fell bundle $\mathcal{C}_{\| \|}$ obtained by completing this pre-Fell bundle $\mathcal{C}/\mathcal{I}$ is the *Hausdorff completion of* $\mathcal{C}$.

Definition 2.18. Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_t)_{t\in G}$ be *-algebraic bundles over the discrete group G. A homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is a map such that $\phi(A_t) \subseteq B_t$, $\forall t \in G$, and, $\forall a, b \in A$, $t \in G$: 1) $\phi\big|_{A_t} : A_t \to B_t$ is linear; 2) $\phi(ab) = \phi(a)\phi(b)$; 3) $\phi(a^*) = \phi(a)^*$. If $\mathcal{A}$ and $\mathcal{B}$ are semi-pre-Fell bundles we also require that ϕ is continuous on each fiber A_t.

Note that if $\mathcal{A}$ and $\mathcal{B}$ are semi-pre-Fell bundles and $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of *-algebraic bundles, then ϕ is continuous if and only if $\phi : A_e \to B_e$ is continuous, because if $x \in \mathcal{A}$, then

$$\|\phi(x)\|^2 = \|\phi(x)^*\phi(x)\| = \|\phi(x^*x)\| \leq \|\phi\big|_{A_e}\| \, \|x^*x\| = \|\phi\big|_{A_e}\| \, \|x\|^2.$$

In particular every homomorphism of *-algebraic bundles between Fell bundles over discrete groups is continuous. Observe also that, with the notion of homomorphism just introduced, any two Hausdorff completions of a given semi-pre-Fell bundle are necessarily isomorphic, and therefore the Hausdorff completion of a semi-pre-Fell bundle is essentially unique.

If β is a C^*-seminorm on the $*$-algebraic bundle $\mathcal{C} = (C_t)$, and $\alpha := \beta|_{C_e}$, it is clear that $\alpha \in \mathcal{SN}(C_e)$. Besides, by properties 9) and 10) we have:

$$\alpha(c^*c) = \beta(c^*c) = \beta(c)^2 = \beta(cc^*) = \alpha(cc^*), \quad \forall c \in \mathcal{C}. \tag{5}$$

A natural question that arises is whether a C^*-seminorm on C_e can be extended to a C^*-seminorm on $\mathcal{C}$. It follows that if $\mathcal{C}$ is an α-positive $*$-algebraic bundle, the necessary condition (5) is also sufficient for this to be true:

Proposition 2.19. *Let $\mathcal{C} = (C_t)_{t\in G}$ be a $*$-algebraic bundle over the discrete group G, and $\alpha \in \mathcal{SN}(C_e)$ such that $\mathcal{C}$ is α-positive. Then α can be extended to a C^*-seminorm on $\mathcal{C}$ if and only if α satisfies the relation (5) above. In this case its extension is given by $\tilde{\alpha} : \mathcal{C} \to [0,\infty)$ such that $\tilde{\alpha}(c) := \sqrt{\alpha(c^*c)}$, $\forall c \in \mathcal{C}$. Moreover $\tilde{\alpha} \in \mathcal{N}(\mathcal{C}) \iff \alpha \in \mathcal{N}(C_e)$.*

Proof. Each fiber C_t is a right module over C_e, and $\langle\ ,\ \rangle_r^t : C_t \times C_t \to C_e$ such that $\langle c, d\rangle_r^t := c^*d$ is a right semi-inner product on C_t because $\mathcal{C}$ is α-positive. Then $\tilde{\alpha}|_{C_t}$ is a seminorm on C_t (see [15, page 3] or [6, Proposition 3.30]).Therefore we have that $\alpha(\langle c, d\rangle_r^t) \le \tilde{\alpha}(c)\tilde{\alpha}(d)$ and $\tilde{\alpha}(ca) \le \tilde{\alpha}(c)\alpha(a)$ $\forall c, d \in C_t$, $a \in C_e$ (see for instance [6, Proposition 3.30]). Similarly, C_t is a left module over C_e, and $\langle\ ,\ \rangle_l^t : C_t \times C_t \to C_e$ such that $\langle c, d\rangle_l^t := cd^*$ is a left semi-inner product on C_t, which induces the seminorm $\tilde{\tilde{\alpha}}$ such that $\tilde{\tilde{\alpha}}(c) := \alpha(cc^*)$ $\forall c \in C_t$, and we have $\alpha(\langle c, d\rangle_l^t) \le \tilde{\tilde{\alpha}}(c)\tilde{\tilde{\alpha}}(d)$ and $\tilde{\tilde{\alpha}}(ac) \le \alpha(a)\tilde{\tilde{\alpha}}(c)$ $\forall c, d \in C_t$, $a \in C_e$. Now suppose that (5) holds for α, that is $\tilde{\alpha} = \tilde{\tilde{\alpha}}$. Then, if $c \in C_s, d \in C_t$, recalling the above inequalities and observing that $c^*c \in C_e$, we have:

$$\begin{aligned}\tilde{\alpha}(cd)^2 = \alpha(d^*c^*cd) = \alpha(\langle d, c^*cd\rangle_r^t) \le \tilde{\alpha}(d)\tilde{\alpha}(c^*cd) = \tilde{\alpha}(d)\tilde{\tilde{\alpha}}(c^*cd)\\ \le \tilde{\alpha}(d)\alpha(c^*c)\tilde{\tilde{\alpha}}(d) = \tilde{\alpha}(c)^2\tilde{\alpha}(d)\tilde{\tilde{\alpha}}(d) = \tilde{\alpha}(c)^2\tilde{\alpha}(d)^2.\end{aligned}$$

On the other hand: $\tilde{\alpha}(c^*c) = \sqrt{\alpha(c^*cc^*c)} = \sqrt{\alpha(c^*c)^2} = \tilde{\alpha}(c)^2$. We conclude that $\tilde{\alpha}$ satisfies properties 8)–10) of Definition 2.16, so it is a C^*-seminorm on $\mathcal{C}$. The converse has already been observed, and the last statement is clear. □

Since C_t can be considered as both a right and a left C_e-module, condition (5) expresses the fact that the C*-seminorms induced on C_t in both cases by the C*-norms α agree.

As in the case of $*$-algebras, the sets $\mathcal{SN}(\mathcal{C})$ and $\mathcal{N}(\mathcal{C})$ of C^*-seminorms and C^*-norms on a $*$-algebraic bundle $\mathcal{C}$ are partially ordered sets. Moreover, the considerations above lead to consider also the (partially ordered) sets:

$$\begin{aligned}\mathcal{SN}_{\mathcal{C}}(C_e) := \{\alpha \in \mathcal{SN}(C_e) :\ \alpha(c^*c) = \alpha(cc^*)\,\forall c \in \mathcal{C}\}\\ \mathcal{N}_{\mathcal{C}}(C_e) := \mathcal{SN}_{\mathcal{C}}(C_e) \cap \mathcal{N}(C_e)\end{aligned}$$

Theorem 2.20. *Let $\mathcal{C} = (C_t)_{t\in G}$ be a positive $*$-algebraic bundle over the discrete group G. Then the map $\Phi : \mathcal{SN}(\mathcal{C}) \to \mathcal{SN}_{\mathcal{C}}(C_e)$ given by $\Phi(\beta) := \beta|_{C_e}$ is an isomorphism of partially ordered sets, whose inverse Ψ is given by $\Psi(\alpha) = \tilde{\alpha}$, where $\tilde{\alpha}(c) := \sqrt{\alpha(c^*c)}$, $\forall c \in \mathcal{C}$. Besides: $\Phi(\mathcal{N}(\mathcal{C})) = \mathcal{N}_{\mathcal{C}}(C_e)$.*

Proof. It is clear that both Φ and Ψ are order preserving, and Proposition 2.19 shows that $\Phi\circ\Psi = Id_{\mathcal{SN}(C_e)}$. The fact that $\Psi\circ\Phi = Id_{\mathcal{SN}(\mathcal{C})}$ follows directly from the definition of $\tilde{\alpha}$ and property 10) in Definition 2.16. □

Again as in the case of $*$-algebras, note that if $\alpha \ge \beta$ are C*-seminorms on the $*$-algebraic bundle $\mathcal{C}$, then every α-positive element of $\mathcal{C}$ is β-positive as well. Moreover, the indentity on $\mathcal{C}$ induces a surjective homomorphism of Fell bundles $\sigma^{\alpha}_{\beta}:\mathcal{C}_{\alpha}\to\mathcal{C}_{\beta}$.

We end the section with the definition of general Fell bundles and related concepts.

A *Fell bundle* (or *C^*-algebraic bundle*) $\mathcal{B}=(B_t)_{t\in G}$ over the locally compact group G is a Banach bundle $\mathcal{B}$ over G, with fiber B_t over $t\in G$, and such that there exist continuous product and involution defined on $\mathcal{B}$ and satisfying conditions 1)–11) of Definition 2.14. Recall that a *Banach bundle* ([13, II-13.4]) over a Hausdorff space X, called base space, is a pair $(\mathcal{B},p)$ formed by a Hausdorff space $\mathcal{B}$, called total space, and a continuous open surjection $p:\mathcal{B}\to X$, together with continuous maps $\|\ \|:\mathcal{B}\to\mathbb{R}$, $+:\{(b,b')\in\mathcal{B}\times\mathcal{B}:\ p(b)=p(b')\}\to\mathcal{B}$ and $\mathbb{C}\times\mathcal{B}\to\mathcal{B}$ such that each fiber $B_x:=p^{-1}(\{x\})$ becomes a complex Banach space with the restrictions of these maps, and such that it satisfies the additional property: *if $x\in X$ and $(b_i)\subseteq\mathcal{B}$ is a net such that $p(b_i)\to x$ and $\|b_i\|\to 0$, then $b_i\to 0_x\in\mathcal{B}$*, where 0_x is the zero element of B_x. A *homomorphism of Banach bundles* $\phi:\mathcal{A}\to\mathcal{B}$ over X is a continuous map such that $\phi_x:=\phi\big|_{A_x}:A_x\to B_x$ is a bounded linear operator, $\forall x\in X$, and $\|\phi\|:=\sup_{x\in X}\|\phi_x\|<\infty$.

A section of $\mathcal{B}$ is a function $\xi:X\to\mathcal{B}$ such that $p(\xi(x))=x$, $\forall x\in X$. If ξ,η are continuous sections of $\mathcal{B}$, and $\alpha\in\mathbb{C}$, then $t\mapsto\alpha\xi(t)+\eta(t)$ is again a continuous section. We will denote by $C_c(\mathcal{B})$ the vector space of continuous sections of compact support of the Banach bundle $\mathcal{B}$. If $K\subseteq X$ is a compact subset, we denote by $C_K(\mathcal{B})$ the subspace of $C_c(\mathcal{B})$ whose elements are those with support contained in K. The map $\|\ \|_K:C_K(\mathcal{B})\to\mathbb{R}$ given by $\|\xi\|_K=\max_{x\in X}\xi(x)$ is a norm and $(C_K(\mathcal{B}),\|\ \|_K)$ is a Banach space. We endow $C_c(\mathcal{B})$ with the locally convex inductive limit topology induced by the family $\{(C_K(\mathcal{B}),\iota_K)\}_K$, where K runs over the family of compact subsets of X, and $\iota_K:C_K(\mathcal{B})\hookrightarrow C_c(\mathcal{B})$ is the natural inclusion. We refer the reader to [13] for further information on Banach bundles.

If X is a topological space, X_d will denote the set X with the discrete topology and, if $\mathcal{B}$ is a Banach bundle over X, we will denote by $\mathcal{B}_d$ the Banach bundle over X_d whose fiber over $x\in X$ is the corresponding fiber of $\mathcal{B}$. That is, $\mathcal{B}_d$ is the disjoint union of the fibers B_x, $x\in X$. Since $\mathcal{B}$ is a topological space the notation just introduced is ambiguous. Thus, in order to avoid any confusion we will use calligraphic letters only to represent Banach bundles. Note that if $\mathcal{A}$ is a Fell bundle over G, then $\mathcal{A}_d$ is a Fell bundle over G_d.

Definition 2.21. Let $\mathcal{A}=(A_t)_{t\in G}$ and $\mathcal{B}=(B_t)_{t\in G}$ be Fell bundles over the locally compact group G. We say that a homomorphism of Banach bundles $\phi:\mathcal{A}\to\mathcal{B}$ is a homomorphism of Fell bundles if $\phi:\mathcal{A}_d\to\mathcal{B}_d$ is a homomorphism of Fell bundles over G_d (see Definition 2.18).

Along this work we will use repeatedly the following two results. The first one is *Cohen-Hewitt theorem: if B is a Banach algebra with approximate unit and if E is a non-degenerate Banach B-module (i.e. $\overline{\text{span}}EB=E$), then* *for each $x \in E$ there exist $y \in E$, $b \in B$, such that $x = yb$.* Although the use of this theorem is not strictly necessary for our purposes, it facilitates the exposition and allows us to avoid the repetition of similar approximation arguments. A proof of this theorem may be found in [13] (there is a nice proof for Hilbert modules in [18]).

The second of the mentioned results is the theorem of *Douady-dal Soglio Hérault*, which is fundamental in the theory of Banach bundles: *let X be a Hausdorff space, and $(\mathcal{B}, p)$ a Banach bundle over X; if X is paracompact or locally compact, then for each $b \in \mathcal{B}$ there exists a continuous section of compact support ξ of $\mathcal{B}$ such that $\xi\big(p(b)\big) = b$.* The reader is referred to [13, Apendix C] for a proof.

3. Tensor Products of Fell Bundles

Our aim in what follows is to introduce tensor products of Fell bundles. A tensor product of the Fell bundles $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ over the groups G and H will be a Fell bundle $\mathcal{C} = (C_r)_{r\in G\times H}$ over $G\times H$, and we will have that C_e is a tensor product of A_e and B_e (recall that e denotes the unit of the group). We will show that there exist, up to isomorphisms, unique tensor products $\mathcal{C}_{\max}$ and $\mathcal{C}_{\min}$ of $\mathcal{A}$ and $\mathcal{B}$, such that $(\mathcal{C}_{\max})_e = A_e \bigotimes_{\max} B_e$ and $(\mathcal{C}_{\min})_e = A_e \bigotimes_{\min} B_e$.

In the first part of the section we consider the case of bundles over discrete groups. The treatment of the general case is postponed to the the second part of the section. Finally, the end of the section is devoted to study the representations of tensor products.

3.1. Tensor products of Fell bundles over discrete groups.

Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles over the groups G and H respectively. Consider, for $t \in G$, $s \in H$, the algebraic tensor product $A_t \bigodot B_s$. When we let t, s run in G and H, we obtain a family $\{A_t \bigodot B_s\}_{(t,s)\in G\times H}$ of vector spaces. Let denote by $\mathcal{A} \bigodot \mathcal{B}$ the disjoint union of these vector spaces. For $(t,s),(t',s') \in G \times H$, we have unique linear maps $\big(A_t \bigodot B_s\big) \times \big(A_{t'} \bigodot B_{s'}\big) \to A_{tt'} \bigodot B_{ss'}$ such that $(a_t \odot b_s, a_{t'} \odot b_{s'}) \mapsto a_t a_{t'} \odot b_s b_{s'}$, and unique conjugate linear maps $A_t \bigodot B_s \to A_{t^{-1}} \bigodot B_{s^{-1}}$ such that $a_t \odot b_s \mapsto a_t^* \odot b_s^*$. Put together, these families of maps define a product $\cdot : \big(\mathcal{A} \bigodot \mathcal{B}\big) \times \big(\mathcal{A} \bigodot \mathcal{B}\big) \to \big(\mathcal{A} \bigodot \mathcal{B}\big)$ and an involution $* : \big(\mathcal{A} \bigodot \mathcal{B}\big) \to \big(\mathcal{A} \bigodot \mathcal{B}\big)$ such that the product is associative, bilinear on every $\big(A_t \bigodot B_s\big) \times \big(A_{t'} \bigodot B_{s'}\big) \to A_{tt'} \bigodot B_{ss'}$, * is conjugate linear when restricted to $A_t \bigodot B_s \to A_{t^{-1}} \bigodot B_{s^{-1}}$ and $(x \cdot y)^* = y^* \cdot x^*$, $\forall x, y \in \mathcal{A} \bigodot \mathcal{B}$. In other words, $\mathcal{A} \bigodot \mathcal{B}$ is a *-algebraic bundle, in the sense of Definition 2.14. We will say that $\mathcal{A} \bigodot \mathcal{B}$ is the *algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$.*

Proposition 3.1. *The algebraic tensor product of Fell bundles is a positive $*$-algebraic bundle (Definition 2.15).*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over G and H respectively. We have to show that for any $s \in G$, $t \in H$, and elements $a_1, \ldots, a_n, a_1', \ldots, a_n' \in A_t$,

$b_1,\dots,b_n,b'_1,\dots,b'_n\in B_t$, the element $(a_1^*a'_1+\cdots+a_n^*a'_n)\odot(b_1^*b'_1+\cdots+b_n^*b'_n)$ is a positive element of $A_e\bigodot B_e$. Since A_s and B_t are positive C^*-trings, and Hilbert modules over A_e and B_e, this fact follows from Proposition 2.4. □

Definition 3.2. Let $\mathcal{A}=(A_t)_{t\in G}$ and $\mathcal{B}=(B_s)_{s\in H}$ be Fell bundles over the discrete groups G and H, and consider their algebraic tensor product $\mathcal{A}\bigodot\mathcal{B}$. If α is a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$, we will call the completion $\mathcal{A}\bigotimes_\alpha\mathcal{B}$ of $(\mathcal{A}\bigodot\mathcal{B},\alpha)$ a tensor product of $\mathcal{A}$ and $\mathcal{B}$.

If $\mathcal{A}\bigotimes_\alpha\mathcal{B}$ is a tensor product of $\mathcal{A}$ and $\mathcal{B}$, then the unit fiber $\big(\mathcal{A}\bigotimes_\alpha\mathcal{B}\big)_e$ is a tensor product of A_e and B_e. In fact, if we know the C^*-norm determined by $\big(\mathcal{A}\bigotimes_\alpha\mathcal{B}\big)_e$ on $A_e\bigodot B_e$, then we know the norm of every $x\in\mathcal{A}\bigotimes_\alpha\mathcal{B}$, because it is equal to $\sqrt{\alpha(x^*x)}$. Therefore, two tensor products will be isomorphic if and only if their fibers on the identity element are the same tensor product of A_e and B_e. This raises the question of whether or not a given tensor product of A_e and B_e determines a tensor product of the Fell bundles $\mathcal{A}$ and $\mathcal{B}$. According to Proposition 2.19, if α is a C^*-norm on $A_e\bigodot B_e$, then α can be extended to a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$ if and only if $\alpha(x^*x)=\alpha(xx^*)$, $\forall x\in\mathcal{A}\bigodot\mathcal{B}$, and in this case the extension is unique. Writing $x=\sum_{i=1}^n a_i\odot b_i\in A_r\odot B_s$, this condition is $\alpha(\sum_{i,j=1}^n x_i^*x_i\odot y_i^*y_i)=\alpha(\sum_{i,j=1}^n x_ix_i^*\odot y_iy_i^*)$. Although we will not go deeper into this problem, we will see that this is in fact the case for the maximal and minimal tensor products (see Proposition 3.4 below). We begin with a result certainly well-known; for lack of reference we include a proof of it.

Lemma 3.3. *Let I and J be ideals of the C^*-algebras A and B respectively. Then $I\bigotimes_{\max}J$ is the closure of $I\bigodot J$ in $A\bigotimes_{\max}B$.*

Proof. Let $\pi:I\bigotimes_{\max}J\to B(H)$ be a faithful and non-degenerate representation of $I\bigotimes_{\max}J$. Then there are faithful and non-degenerate representations $\pi_I:I\to B(H)$ and $\pi_J:J\to B(H)$, such that $\pi_I(x)\pi_J(y)=\pi(x\otimes y)=\pi_J(y)\pi_I(x)$, $\forall x\in I$, $y\in J$ ([20, T.6.4]). Since π_I and π_J are non-degenerate they have unique extensions $\pi_A:A\to B(H)$ and $\pi_B:B\to B(H)$ to representations of A and B respectively ([13, VI-19.11]). If $a\in A$, $x\in I$, $b\in B$ and $y\in J$, then $\pi_A(ax)\pi_B(by)=\pi_B(by)\pi_A(ax)$, because $ax\in I$ and $by\in J$. Since π_I and π_J are non-degenerate, we conclude that $\pi_A(a)\pi_B(b)=\pi_B(b)\pi_A(a)$, $\forall a\in A$, $b\in B$. Hence there exists a representation $\tilde\pi:A\bigotimes_{\max}B\to B(H)$ such that $\tilde\pi(a\otimes b)=\pi_A(a)\pi_B(b)$, $\forall a\in A$, $b\in B$. Thus $\tilde\pi$ is an extension of $\pi\big|_{I\bigodot J}$. Since $\tilde\pi$ is contractive, we conclude that if $x\in I\bigodot J$, its norm as an element of $A\bigotimes_{\max}B$ is greater or equal to its norm in $I\bigotimes_{\max}J$, and therefore they agree. □

Proposition 3.4. *Let $\mathcal{A}=(A_t)_{t\in G}$ and $\mathcal{B}=(B_s)_{s\in H}$ be Fell bundles over the discrete groups G and H. Then the norms $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ on $A_e\bigodot B_e$ can be extended to C^*-norms on $\mathcal{A}\bigodot\mathcal{B}$.*

Proof. Let $A_t^*A_t := \overline{\operatorname{span}}\{a_t^*a_t : \ a_t \in A_t\} \subseteq A_e$ and $B_s^*B_s := \overline{\operatorname{span}}\{b_s^*b_s : b_s \in B_s\} \subseteq B_e$. Then $A_t^*A_t$ and $B_s^*B_s$ are ideals in A_e and B_e respectively, and A_t may be seen as a positive C^*-tring with $A_t^r = A_t^*A_t$ and $A_t^l = A_tA_t^*$, and similarly B_s. Recall that there exists a maximum C^*-norm μ on $A_t \bigodot B_s$. By [6, Corollary 5.13], we must have $(A_t \bigotimes_{\max} B_s)^r = A_t^*A_t \bigotimes_{\mu^r} B_s^*B_s$ and $(A_t \bigotimes_{\max} B_s)^l = A_tA_t^* \bigotimes_{\mu^l} B_sB_s^*$, where μ^r denotes the maximum norm on $A_t^*A_t \bigodot B_s^*B_s$ and μ^l denotes the maximum norm on $A_tA_t^* \bigodot B_sB_s^*$. Now, Lemma 3.3 implies that μ^r and μ^l are restrictions of the maximum norm of $A_e \bigodot B_e$. Since $A_t \bigotimes_{\max} B_s$ is a Hilbert $\big(A_tA_t^* \bigotimes_{\mu^l} B_sB_s^*\text{-}A_t^*A_t \bigotimes_{\mu^r} B_s^*B_s\big)$ -bimodule we have, for $x \in A_t \bigodot B_s$

$$\|xx^*\|_{\max} = \|xx^*\|_{\mu^l} = \|x\|_\mu^2 = \|x^*x\|_{\mu^r} = \|x^*x\|_{\max}.$$

Thus $\| \ \|_{\max}$ may be extended to all of $\mathcal{A} \bigodot \mathcal{B}$ by Proposition 2.19.

On the other hand, it is well-known that if C and D are C^*-subalgebras of the C^*-algebras A and B respectively, then the restriction of the spatial norm on $A \bigodot B$ to $C \bigodot D$ is precisely the spatial norm on $C \bigodot D$ (see for instance [6, Corollary B.14], or simply Proposition 2.10). Therefore the same arguments given above for $\| \ \|_{\max}$ also apply to the spatial norm on $A_e \bigodot B_e$ and hence $\| \ \|_{\min}$ can also be extended to $A_e \bigodot B_e$. $\square$

3.2. **Tensor products of Fell bundles over locally compact groups.** We will extend next the construction done in the previous section to the case of Fell bundles over arbitrary locally compact groups.

Suppose now that $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ are Fell bundles over the locally compact groups G and H, and let $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ be a tensor product of $\mathcal{A}_d$ and $\mathcal{B}_d$ as in the previous section. We will endow $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ with a topology such that $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ will be a Fell bundle over $G \times H$.

For $f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$, let $f \oslash g : G \times H \to \mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ be such that $(f \oslash g)(t, s) = f(t) \otimes g(s)$, $\forall t \in G,\ s \in H$. Every $f \oslash g$ is a section of $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$. We consider the vector space

$$L := \operatorname{span}\{f \oslash g : \ f \in C_c(\mathcal{A}), g \in C_c(\mathcal{B})\},$$

which is a vector subspace of the space of sections of $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$. The topology we want to define on $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ is determined by the requirement that every element of L is a continuous section:

Proposition 3.5. *With the notation above we have:*

(1) *For each $l \in L$, the map $G \times H \to \mathbb{R}$ such that $(t, s) \mapsto \alpha\big(l(t, s)\big)$ is continuous.*
(2) *For each $(t, s) \in G \times H$, the set $L(t, s) := \{l(t, s) : \ l \in L\}$ is dense in $A_t \bigotimes_\alpha B_s$.*
(3) *There exists a unique topology on $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ for which $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ is a Banach bundle over $(G \times H)$ and such that L is contained in the space of continuous sections of the bundle $\mathcal{A}_d \bigotimes_\alpha \mathcal{B}_d$ with this topology.*

The Banach bundle over $G\times H$ thus obtained will be denoted by $\mathcal{A}\bigotimes_\alpha\mathcal{B}$.

Proof. Since 3) is a consequence of 1) and 2) ([13, II-13.18]) it is enough to prove the first two assertions. We begin by 2). If $x=\sum_{i=1}^n a_i\otimes b_i\in A_t\bigotimes_\alpha B_s$, there exist continuous sections f_i, g_i of $\mathcal{A}$ and $\mathcal{B}$ respectively such that $f_i(t)=a_i$, $g_i(s)=b_i$, $\forall i=1\dots n$ ([13, C.17]). If $l=\sum_{i=1}^n f_i\oslash g_i$, then $l\in L$, and $l(t,s)=\sum_{i=1}^n f_i(t)\otimes g_i(s)=\sum_{i=1}^n a_i\otimes b_i=x$. Hence 2) follows, because $A_t\bigodot B_s$ is dense in $A_t\bigotimes_\alpha B_s$.

To prove 1), fix $l=\sum_{i=1}^n f_i\oslash g_i\in L$, and let $(t,s)\to(t_0,s_0)$. Then:

$$\alpha\big(l(t,s)\big)^2=\alpha\big(\sum_{i=1}^n f_i(t)\otimes g_i(s)\big)^2=\alpha\big(\sum_{i,j=1}^n f_i(t)^*f_j(t)\otimes g_i(s)^*g_j(s)\big)$$

Note that $t\mapsto f_i(t)^*f_j(t)$ and $s\mapsto g_i(s)^*g_j(s)$ are continuous maps, because the f_i and g_i are continuous sections, and the involutions of $\mathcal{A}$ and $\mathcal{B}$ are continuous as well. Now the "cross-norm" property (i.e.: $\alpha(a\otimes b)=\alpha(a)\alpha(b)$) of the C^*-norms on tensor products implies that $a\otimes b\to a_0\otimes b_0$ when $a\to a_0$ and $b\to b_0$. Therefore,

$$\sum_{i,j=1}^n f_i(t)^*f_j(t)\otimes g_i(s)^*g_j(s)\to\sum_{i,j=1}^n f_i(t_0)^*f_j(t_0)\otimes g_i(s_0)^*g_j(s_0)$$

when $(t,s)\to(t_0,s_0)$. Thus $\alpha\big(l(t,s)\big)\to\alpha\big(l(t_0,s_0)\big)$ if $(t,s)\to(t_0,s_0)$. □

If G is a group, * is an involution on a set X and $l:G\to X$ is a map, we define a new map $\tilde l:G\to X$ as $\tilde l(t)=l(t^{-1})^*$. In particular, if l is a continuous section of compact support of a Fell bundle $\mathcal{B}$, then $\tilde l$ also is.

Lemma 3.6. *The involution $*:\mathcal{A}\bigotimes_\alpha\mathcal{B}\to\mathcal{A}\bigotimes_\alpha\mathcal{B}$ is continuous.*

Proof. We know from [13, II-13.18] that a base for the topology defined in Proposition 3.5 is given by the sets

$$\mathcal{W}(l,U,\epsilon)=\{w\in\mathcal{A}\bigotimes_\alpha\mathcal{B}:\ p(w)\in U,\ \text{and}\ \alpha\big(l\big(p(w)\big)-w\big)<\epsilon\},$$

where $p:\mathcal{A}\bigotimes_\alpha\mathcal{B}\to G\times H$ is the projection, $U\subseteq G\times H$ is an open subset, $l=\sum_i f_i\oslash g_i$, with $f_i\in C_c(\mathcal{A})$, $g_i\in C_c(\mathcal{B})$, and $\epsilon>0$. In other words, $\mathcal{W}(l,U,\epsilon)=\bigcup_{t\in U}B(l(t),\epsilon)$, where $B(l(t),\epsilon)\subseteq(\mathcal{A}\bigotimes_\alpha\mathcal{B})_t$ is the open ϵ-ball with center $l(t)$. Then we have:

$$\begin{aligned}\mathcal{W}(l,U,\epsilon)^*&=\{w^*\in\mathcal{A}\bigotimes_\alpha\mathcal{B}:\ p(w)\in U,\ \text{and}\ \alpha\big(l\big(p(w)\big)-w\big)<\epsilon\}\\&=\{w^*\in\mathcal{A}\bigotimes_\alpha\mathcal{B}:\ p(w^*)\in U^{-1},\ \text{and}\ \alpha\big(l\big(p(w^*)^{-1}\big)^*-w^*\big)<\epsilon\}\\&=\{z\in\mathcal{A}\bigotimes_\alpha\mathcal{B}:\ p(z)\in U^{-1},\ \text{and}\ \alpha\big(\tilde l\big(p(z)\big)-z\big)<\epsilon\}\\&=\mathcal{W}(\tilde l,U^{-1},\epsilon),\end{aligned}$$

Thus $*$ is continuous. □

Proposition 3.7. *The product* $(\mathcal{A}\bigotimes_\alpha\mathcal{B})\times(\mathcal{A}\bigotimes_\alpha\mathcal{B})\to\mathcal{A}\bigotimes_\alpha\mathcal{B}$ *is continuous.*

Proof. We claim that if $a\to a_0$ in $\mathcal{A}$ and $b\to b_0$ in $\mathcal{B}$, then $a\otimes b\to a_0\otimes b_0$ in $\mathcal{A}\bigotimes_\alpha\mathcal{B}$. Let $W\subseteq\mathcal{A}\bigotimes_\alpha\mathcal{B}$ be an open set such that $a_0\otimes b_0\in W$, and let $f\in C_c(\mathcal{A})$, $g\in C_c(\mathcal{B})$ be such that $f(t_0)=a_0$ and $g(s_0)=b_0$. Then $(f\oslash g)(t_0,s_0)=a_0\otimes b_0$. Since $f\oslash g\in C_c(\mathcal{A}\bigotimes_\alpha\mathcal{B})$, and since the norm α is continuous, there exist $\epsilon>0$ and open sets $U\subseteq G$ and $V\subseteq H$ such that $(t_0,s_0)\in U\times V$ and $W\cap(\mathcal{A}\bigotimes_\alpha\mathcal{B})_{(t,s)}\supseteq B\big((f\oslash g)(t,s),\epsilon\big)$, $\forall(t,s)\in U\times V$. Consider now the open subsets $W(f,U,\epsilon^{1/2})$ and $W(g,V,\epsilon^{1/2})$ of $\mathcal{A}$ and $\mathcal{B}$ containing a_0 and b_0 respectively. We have $W(f,U,\epsilon^{1/2})\otimes W(g,V,\epsilon^{1/2})=\{a_t\otimes b_s\in\mathcal{A}\bigotimes_\alpha\mathcal{B}:\ (t,s)\in U\times V,\text{ and }\|f(t)-a_t\|<\epsilon^{1/2},\ \|g(s)-b_s\|<\epsilon^{1/2}\}\subseteq\{x_{(t,s)}\in\mathcal{A}\bigotimes_\alpha\mathcal{B}:\ (t,s)\in U\times V,\text{ and }\alpha\big((f\oslash g)(t,s)-x_{(t,s)}\big)<\epsilon\}=W(f\oslash g,U\times V,\epsilon)=\bigcup_{(t,s)\in U\times V}B\big((f\oslash g)(t,s),\epsilon\big)\subseteq W$, so it follows that $a\otimes b\to a_0\otimes b_0$ when $a\to a_0$, $b\to b_0$.

Note that $\forall f,f'\in C_c(\mathcal{A})$, $g,g'\in C_c(\mathcal{B})$, the map $\mu:(G\times H)\times(G\times H)\to\mathcal{A}\bigotimes_\alpha\mathcal{B}$ given by $\big((t,s),(t',s')\big)\mapsto(f\oslash g)(t,s)\otimes(f'\oslash g')(t',s')$ is continuous. Indeed the products on $\mathcal{A}$ and $\mathcal{B}$ are continuous, f,f',g,g' are continuous as well, and since $\mu(t,s,t',s')=f(t)f'(t')\otimes g(s)g'(s')$, the continuity of μ follows from the claim at the beginning of the proof.

Now pick elements $x_0\in(\mathcal{A}\bigotimes_\alpha\mathcal{B})_{(t_0,s_0)}$ and $x_0'\in(\mathcal{A}\bigotimes_\alpha\mathcal{B})_{(t_0',s_0')}$, and let $m\in L$, $1>\epsilon>0$ such that $x_0x_0'\in W(m,Z,\epsilon)$, where Z is some open subset of $G\times H$ containing (t_0t_0',s_0s_0'). Let $M>\epsilon+1+\alpha(x_0)+\alpha(x_0')$ and $l,l'\in L$ such that $\alpha(l(t_0,s_0)-x_0)<\epsilon/2M$, $\alpha(l'(t_0',s_0')-x_0')<\epsilon/2M$. Then we have $\alpha(l(t_0,s_0)l'(t_0',s_0')-x_0x_0')=:d<\epsilon$. Let $d<\epsilon'<\epsilon$. As seen above, the map $ll':(G\times H)\times(G\times H)\to\mathcal{A}\bigotimes_\alpha\mathcal{B}$ such that $\big((t,s),(t',s')\big)\mapsto l(t,s)l'(t',s')$ is continuous, so there exist open neighborhoods U, V, U' and V' of t_0, s_0, t_0' and s_0' respectively such that $ll'\big((U\times V)\times(U'\times V')\big)\subseteq W(m,Z,\epsilon')$. Let now $N>1+\|l\|_\infty+\|l'\|_\infty$, $0<\delta<(\epsilon-\epsilon')/4N$. We have $W(l,U\times V,\delta)W(l',U'\times V',\delta))\subseteq W(m,Z,\epsilon)$. In fact, if $x_{(t,s)}\in W(l,U\times V,\delta)$, $x'_{(t',s')}\in W(l,U'\times V',\delta)$

$$\begin{aligned}\alpha(x_{(t,s)}x'_{(t',s')}-m(tt',ss'))&\le\alpha(x_{(t,s)}x'_{(t',s')}-l(t,s)l'(t',s'))\\&\qquad+\alpha(l(t,s)l'(t',s')-m(tt',ss'))\\&\le\epsilon'+\alpha(x_{(t,s)}\big(x'_{(t',s')}-l'(t',s')\big))\\&\qquad+\alpha(\big(x_{(t,s)}-l(t,s)\big)l'(t',s'))\\&<\epsilon'+\frac{\epsilon-\epsilon'}{4}\left[\frac{1}{N}(\alpha(x_{(t,s)}-l(t,s))\,\alpha(l(t,s)))+1\right]\\&<\epsilon\end{aligned}$$

□

Definition 3.8. Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over the locally compact groups G and H, and let α be a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$. The tensor product

$\mathcal{A}\bigotimes_\alpha\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ with respect to α is the Fell bundle obtained by completing the algebraic tensor product $\mathcal{A}\bigodot\mathcal{B}$ with respect to the C^*-norm α, furnished with the topology provided by Proposition 3.5.

Proposition 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over the locally compact groups G and H. If $\alpha\geq\beta$ are C^*-norms on $\mathcal{A}\bigodot\mathcal{B}$, then there exists a unique homomorphism of Fell bundles $\sigma^\alpha_\beta:\mathcal{A}\bigotimes_\alpha\mathcal{B}\to\mathcal{A}\bigotimes_\beta\mathcal{B}$ such that $\sigma^\alpha_\beta(a\otimes b)=a\otimes b$, $\forall a\in\mathcal{A}$, $b\in\mathcal{B}$. This homomorphism is onto. Moreover, if $\alpha\geq\beta\geq\gamma$ are C^*-norms on $\mathcal{A}\bigodot\mathcal{B}$, we have $\sigma^\alpha_\gamma=\sigma^\alpha_\beta\sigma^\beta_\gamma$.*

Proof. Since $\alpha\geq\beta$ for each $(r,s)\in G\times H$ the identity map on $A_r\bigodot B_s$ has a (unique) continuous extension to a map $A_r\bigotimes_\alpha B_s\to A_r\bigotimes_\beta B_s$, which is surjective because its image is both dense and closed. The collection of all these maps is clearly a homomorphism σ^α_β from $\mathcal{A}_d\bigotimes_\alpha\mathcal{B}_d$ into $\mathcal{A}_d\bigotimes_\beta\mathcal{B}_d$. It is also continuous from $\mathcal{A}\bigotimes_\alpha\mathcal{B}$ into $\mathcal{A}\bigotimes_\beta\mathcal{B}$, because the vector space L of sections used to define the involved topologies is exactly the same, and the map σ^α_β is the identity on the set of such sections. Thus σ^α_β is continuous by [13, II-13.16]. The last assertion follows from the uniqueness of the maps σ^α_β. □

Summarizing the constructions and results obtained up to now we have:

Theorem 3.10. *Let $\mathcal{A}=(A_t)_{t\in G}$ and $\mathcal{B}=(B_s)_{s\in H}$ be Fell bundles over the locally compact groups G and H. Then $\mathcal{SN}_{\mathcal{A}\bigodot\mathcal{B}}(A_e\bigodot B_e)\cong\mathcal{SN}(\mathcal{A}\bigodot\mathcal{B})$ and $\mathcal{N}_{\mathcal{A}\bigodot\mathcal{B}}(A_e\bigodot B_e)\cong\mathcal{N}(\mathcal{A}\bigodot\mathcal{B})$ as a posets. Moreover $\mathcal{N}(\mathcal{A}\bigodot\mathcal{B})$ has a minimum and a maximum elements, namely the unique extensions of $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ on $A_e\bigodot B_e$ to C^*-norms on all of $\mathcal{A}\bigodot\mathcal{B}$.*

As a consequence we can extend Propositions 2.10 and 2.11 to the context of Fell bundles:

Proposition 3.11. *Let $\pi_1:\mathcal{A}_1\to\mathcal{B}_1$ and $\pi_2:\mathcal{A}_2\to\mathcal{B}_2$ be homomorphisms of Fell bundles. Then $\pi_1\odot\pi_2:\mathcal{A}_1\bigodot\mathcal{A}_2\to\mathcal{B}_1\bigodot\mathcal{B}_2$ is $\|\ \|_{min}$-continuous, so it extends to a homomorphism $\pi_1\bigotimes_{\min}\pi_2:\mathcal{A}_1\bigotimes_{\min}\mathcal{A}_2\to\mathcal{B}_1\bigotimes_{\min}\mathcal{B}_2$.*

Proposition 3.12. *Let $\pi_1:\mathcal{A}_1\to\mathcal{B}_1$ and $\pi_2:\mathcal{A}_2\to\mathcal{B}_2$ be homomorphisms of Fell bundles. Then $\pi_1\odot\pi_2:\mathcal{A}_1\bigodot\mathcal{A}_2\to\mathcal{B}_1\bigodot\mathcal{B}_2$ is $\|\ \|_{max}$-continuous, so it extends to a homomorphism $\pi_1\bigotimes_{\max}\pi_2:\mathcal{A}_1\bigotimes_{\max}\mathcal{A}_2\to\mathcal{B}_1\bigotimes_{\max}\mathcal{B}_2$.*

Consequently, as in the case of C*-algebras and of C*-trings, we see that the minimal and maximal tensor products of Fell bundles is a bifunctor $\mathsf{F}\times\mathsf{F}\to\mathsf{F}$, where F is the category of Fell bundles.

3.3. **Representations of tensor products.** We will study now the representations of tensor products of Fell bundles on Hilbert modules. The results obtained, similar to the case of C^*-algebras, will be useful in the next section. The first of them tells us how to obtain a representation of $\mathcal{A}\bigotimes_{\min}\mathcal{B}$ starting with representations of $\mathcal{A}$ and $\mathcal{B}$. The second one shows that there exists a bijective correspondence between non-degenerate representations of $\mathcal{A}\bigotimes_{\max}\mathcal{B}$ and non-degenerate commuting representations of $\mathcal{A}$ and $\mathcal{B}$.

Definition 3.13. Let $\mathcal{A}$ be a *-algebraic bundle over the discrete group G, and $\mathcal{H}$ a Hilbert module. A map $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is called a representation of $\mathcal{A}$ on $\mathcal{H}$ if $\pi(ab) = \pi(a)\pi(b)$, $\pi(a^*) = \pi(a)^*$ and $\pi\big|_{A_t}$ is linear, $\forall a, b \in \mathcal{A}, t \in G$. The representation π is said to be non-degenerate if $\overline{\mathrm{span}}\pi(\mathcal{A})\mathcal{H} = \mathcal{H}$. This is equivalent to the restriction $\pi\big|_{A_e}$ to be non-degenerate.

Definition 3.14. Let $\mathcal{A}$ be a Fell bundle over the locally compact group G. A representation of $\mathcal{A}$ on the Hilbert module $\mathcal{H}$ is a representation $\phi : \mathcal{A}_d \to \mathcal{L}(\mathcal{H})$ which is strongly continuous, that is, $\forall h \in \mathcal{H}$ the map $\mathcal{A} \to \mathcal{H}$ given by $a \mapsto \pi(a)h$ is continuous.

Note that for G discrete every representation of the Fell bundle $\mathcal{A}$ is automatically continuous, because $\|\pi(a)\| \leq \|a\|$, $\forall a \in \mathcal{A}$, as is easy to check.

If $\mathcal{A}$ is a Fell bundle (or just an *-algebraic bundle), and $\mathcal{H}$ is a Hilbert module, we will denote by $R(\mathcal{A}, \mathcal{H})$ the family of non-degenerate representations of $\mathcal{A}$ on $\mathcal{H}$. If $\mathcal{B}$ is another Fell bundle (or *-algebraic bundle), we set: $R(\mathcal{A}, \mathcal{B}, \mathcal{H}) := \{(\pi_1, \pi_2) \in R(\mathcal{A}, \mathcal{H}) \times R(\mathcal{B}, \mathcal{H}) : \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a), \forall a \in \mathcal{A}, b \in \mathcal{B}\}$. If A and B are *-algebras, we will also use the notations $R(A, \mathcal{H})$, $R(A, B, \mathcal{H})$, with the same meaning.

In what follows, given right Hilbert modules $\mathcal{H}$ and $\mathcal{K}$, over the C*-algebras C and D respectively, we will consider their *exterior tensor product* $\mathcal{H} \bigotimes_{\min} \mathcal{K}$, which is a right Hilbert module over $C \bigotimes_{\min} D$. The reader is referred to Subsection 2.2, as well as [15] or [6].

Proposition 3.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over the locally compact groups G and H respectively, and let $\pi_{\mathcal{A}} \in R(\mathcal{A}, \mathcal{H}_{\mathcal{A}})$, $\pi_{\mathcal{B}} \in R(\mathcal{B}, \mathcal{H}_{\mathcal{B}})$. Then there exists a unique representation $\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}} \in R(\mathcal{A} \bigotimes_{\min} \mathcal{B}, \mathcal{H}_{\mathcal{A}} \bigotimes_{\min} \mathcal{H}_{\mathcal{B}})$ such that $(\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}})(a \otimes b) = \pi_{\mathcal{A}}(a) \otimes \pi_{\mathcal{B}}(b)$, $\forall a \in \mathcal{A}$ and $\forall b \in \mathcal{B}$. If $\pi_{\mathcal{A}}\big|_{A_e}$ and $\pi_{\mathcal{B}}\big|_{B_e}$ are faithful, then $(\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}})\big|_{(\mathcal{A} \bigotimes_{\min} \mathcal{B})_e}$ also is faithful.*

Proof. According to [15, pages 36 and 37] (see also Corollary 2.13), we have an isometric embedding $\mathcal{L}(\mathcal{H}_{\mathcal{A}}) \bigotimes_{\min} \mathcal{L}(\mathcal{H}_{\mathcal{B}}) \hookrightarrow \mathcal{L}(\mathcal{H}_{\mathcal{A}} \bigotimes_{\min} \mathcal{H}_{\mathcal{B}})$, such that, $\forall T \in \mathcal{L}(\mathcal{H}_{\mathcal{A}}), S \in \mathcal{L}(\mathcal{H}_{\mathcal{B}})$, $h_{\mathcal{A}} \in \mathcal{H}_{\mathcal{A}}$, $h_{\mathcal{B}} \in \mathcal{H}_{\mathcal{B}}$: $(T \otimes S)(h_{\mathcal{A}} \otimes h_{\mathcal{B}}) = T(h_{\mathcal{A}}) \otimes S(h_{\mathcal{B}})$. Thus we may consider, for each $(t, s) \in G \times H$, the map $A_t \times B_s \to \mathcal{L}(\mathcal{H}_{\mathcal{A}} \bigotimes_{\min} \mathcal{H}_{\mathcal{B}})$ such that $(a_t, b_s) \mapsto \pi_{\mathcal{A}}(a_t) \otimes \pi_{\mathcal{B}}(b_s)$. This map is bilinear, so there exists a unique linear map $(\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}})_{(t,s)} : A_t \bigodot B_s \to \mathcal{L}(\mathcal{H}_{\mathcal{A}} \bigotimes_{\min} \mathcal{H}_{\mathcal{B}})$ such that $(\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}})_{(t,s)}(a_t \odot b_s) = \pi_{\mathcal{A}}(a_t) \otimes \pi_{\mathcal{B}}(b_s)$, $\forall a_t \in A_t$, $b_s \in B_s$. The collection of these linear maps is a representation $\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}}$ of the pre-Fell bundle $\mathcal{A} \bigodot \mathcal{B}$. Restricted to $A_e \bigodot B_e$ this map coincides with $\pi_{\mathcal{A}}\big|_{A_e} \otimes \pi_{\mathcal{B}}\big|_{B_e} : A_e \bigodot B_e \to \mathcal{L}(\mathcal{H}_{\mathcal{A}}) \bigotimes_{\min} \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ which is contractive with respect to $\| \, \|_{\min}$ on $A_e \bigodot B_e$ ([20, T.5.19]). It follows that $\pi_{\mathcal{A}} \otimes \pi_{\mathcal{B}}$ extends to a representation of the Fell bundle $\mathcal{A}_d \bigotimes_{\min} \mathcal{B}_d$. Moreover this representation is continuous in the topology of $\mathcal{A} \bigotimes_{\min} \mathcal{B}$ (recall Definition 3.8 and Proposition 3.5). Indeed, for $h_{\mathcal{A}} \in \mathcal{H}_{\mathcal{A}}$ and $h_{\mathcal{B}} \in \mathcal{H}_{\mathcal{B}}$ consider the Banach bundle $G \times H \times (\mathcal{H}_{\mathcal{A}} \bigotimes_{\min} \mathcal{H}_{\mathcal{B}})$ over $G \times H$ (with the product topology and the natural projection), and the map $\Phi : \mathcal{A}\bigotimes_{\min}\mathcal{B} \to G\times H\times (\mathcal{H}_\mathcal{A}\bigotimes_{\min}\mathcal{H}_\mathcal{B})$ given by $\Phi(c_{t,s}) = (t,s,(\pi_\mathcal{A}\otimes\pi_\mathcal{B})(c_{t,s})(h_\mathcal{A}\otimes h_\mathcal{B}))$, $\forall c_{t,s}\in A_t\bigotimes_{\min}B_s$. Let L be as in Proposition 3.5. To see that Φ is a continuous homomorphism of Banach bundles it is enough to show, according to [13, II-13.16], that for all $l\in L$ the map Φl is a continuous section of the bundle $G\times H\times(\mathcal{H}_\mathcal{A}\bigotimes_{\min}\mathcal{H}_\mathcal{B})$. Clearly it is sufficient to check this for sections of the form $f\oslash g$, with $f\in C_c(\mathcal{A})$, $g\in C_c(\mathcal{B})$. Thus assume that $(t,s)\to(t_0,s_0)$ in $G\times H$. We have to show that $\Phi(f(t)\otimes g(s))\to\Phi(f(t_0)\otimes g(s_0))$, which is equivalent to showing that $\pi_\mathcal{A}(f(t))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s))h_\mathcal{B}$ converges to $\pi_\mathcal{A}(f(t_0))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s_0))h_\mathcal{B}$. Now, if $\varepsilon(t,s) = \|\pi_\mathcal{A}(f(t))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s))h_\mathcal{B} - \pi_\mathcal{A}(f(t_0))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s_0))h_\mathcal{B}\|$, we have:

$$\begin{aligned}
\varepsilon(t,s) &\le \|\pi_\mathcal{A}(f(t))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s))h_\mathcal{B} - \pi_\mathcal{A}(f(t))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s_0))h_\mathcal{B}\|\\
&\qquad + \|\pi_\mathcal{A}(f(t))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s_0))h_\mathcal{B} - \pi_\mathcal{A}(f(t_0))h_\mathcal{A}\otimes\pi_\mathcal{B}(g(s_0))h_\mathcal{B}\|\\
&\le \|\pi_\mathcal{A}(f(t))\|\,\|h_\mathcal{A}\|\,\|\pi_\mathcal{B}(g(s))h_\mathcal{B} - \pi_\mathcal{B}(g(s_0))h_\mathcal{B}\|\\
&\qquad + \|\pi_\mathcal{A}(f(t))h_\mathcal{A} - \pi_\mathcal{A}(f(t_0))h_\mathcal{A}\|\,\|\pi_\mathcal{B}(g(s_0))h_\mathcal{B}\|\\
&\le \|f\|_\infty\,\|h_\mathcal{A}\|\,\|\pi_\mathcal{B}(g(s))h_\mathcal{B} - \pi_\mathcal{B}(g(s_0))h_\mathcal{B}\|\\
&\qquad + \|\pi_\mathcal{A}(f(t))h_\mathcal{A} - \pi_\mathcal{A}(f(t_0))h_\mathcal{A}\|\,\|g\|_\infty\,\|h_\mathcal{B}\|,
\end{aligned}$$

which converges to zero because $\pi_\mathcal{A}$ and $\pi_\mathcal{B}$ are continuous representations.

The fact that Φ is continuous implies that $\forall h_\mathcal{A}\in\mathcal{H}_\mathcal{A}$, $h_\mathcal{B}\in\mathcal{H}_\mathcal{B}$, the map $\mathcal{A}\bigotimes_{\min}\mathcal{B}\to\mathcal{H}_\mathcal{A}\bigotimes_{\min}\mathcal{H}_\mathcal{B}$ such that $c\mapsto(\pi_\mathcal{A}\otimes\pi_\mathcal{B})(c)(h_\mathcal{A}\otimes h_\mathcal{B})$ is continuous. Since $\|(\pi_\mathcal{A}\otimes\pi_\mathcal{B})(c)\|\le\|c\|$, $\forall c\in\mathcal{A}\bigotimes_{\min}\mathcal{B}$, we also have that $c\mapsto(\pi_\mathcal{A}\otimes\pi_\mathcal{B})(c)(h)$ is continuous, $\forall h\in\mathcal{H}_\mathcal{A}\bigotimes_{\min}\mathcal{H}_\mathcal{B}$. It follows that $\pi_\mathcal{A}\otimes\pi_\mathcal{B}$ is a representation.

If $\pi_\mathcal{A}$, $\pi_\mathcal{B}$ are non-degenerate, then so are $\pi_\mathcal{A}\big|_{A_e}$ and $\pi_\mathcal{B}\big|_{B_e}$. By Cohen-Hewitt, given $h_\mathcal{A}\in\mathcal{H}_\mathcal{A}$, $h_\mathcal{B}\in\mathcal{H}_\mathcal{B}$, there exist $a\in A_e$, $b\in B_e$, $h'_\mathcal{A}\in\mathcal{H}_\mathcal{A}$, $h'_\mathcal{B}\in\mathcal{H}_\mathcal{B}$ such that $\pi_\mathcal{A}(a)h'_\mathcal{A}=h_\mathcal{A}$ and $\pi_\mathcal{B}(b)h'_\mathcal{B}=h_\mathcal{B}$. Therefore $(\pi_\mathcal{A}\otimes\pi_\mathcal{B})(a\otimes b)(h'_\mathcal{A}\otimes h'_\mathcal{B}) = h_\mathcal{A}\otimes h_\mathcal{B}$. Consequently $(\pi_\mathcal{A}\otimes\pi_\mathcal{B}(\mathcal{A}\bigotimes_{\min}\mathcal{B}))(\mathcal{H}_\mathcal{A}\bigotimes_{\min}\mathcal{H}_\mathcal{B})$ is dense in $\mathcal{H}_\mathcal{A}\bigotimes_{\min}\mathcal{H}_\mathcal{B}$.

Finally, $(\pi_\mathcal{A}\otimes\pi_\mathcal{B})\big|_{(\mathcal{A}\bigotimes_{\min}\mathcal{B})_e} = \pi_\mathcal{A}\big|_{A_e}\otimes\pi_\mathcal{B}\big|_{B_e}$, and this one is injective if and only if $\pi_\mathcal{A}\big|_{A_e}$ and $\pi_\mathcal{B}\big|_{B_e}$ are injective ([20, T.5.19]). □

Suppose that $\mathcal{A}=(A_t)_{t\in G}$ is a Fell bundle and $L,R:\mathcal{A}\to\mathcal{A}$ are continuous maps such that there exists $t\in G$ for which $L(A_s)\subseteq A_{ts}$, $R(B_s)\subseteq B_{st}$, $\forall s\in G$, $L\big|_{A_s}: A_s\to A_{ts}$, $R\big|_{A_s}: A_s\to A_{st}$ are linear and bounded, and $\|L\| := \sup_s\|L\big|_{A_s}\| < \infty$, $\|R\| := \sup_s\|R\big|_{A_s}\| < \infty$. Then (L,R) is called a multiplier of order t of $\mathcal{A}$ ([13]) if $\forall a_1,a_2\in\mathcal{A}$ the following holds:

$$a_1L(a_2) = R(a_1)a_2 \qquad L(a_1a_2) = L(a_1)a_2 \qquad R(a_1a_2) = a_1R(a_2)$$

The set of multipliers of $\mathcal{A}$ of order t is denoted by $M_t(\mathcal{A})$, and $M(\mathcal{A}) = \bigcup_{t\in G}M_t(\mathcal{A})$ denotes the set of all multipliers of $\mathcal{A}$ (the notation differs from the one used in [13]). Every $M_t(\mathcal{A})$ is a Banach space with the obvious operations and the norm: $\|(L,R)\|_0 = \max\{\|L\|,\|R\|\}$. In fact we have

$\|L\| = \|R\|$. Moreover we have a product and an involution on $M(\mathcal{A})$:

$$(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1) \qquad (L, R)^* = (R^*, L^*)$$

where $L^*(a) = L(a^*)^*$ and $R^*(a) = R(a^*)^*$. With these operations and norm $M(\mathcal{A})$ is a Fell bundle over G_d. In addition $M(\mathcal{A})$ has a topology, in which $u_i = (L_i, R_i)$ converges to $u = (L, R)$ if $\forall a \in \mathcal{A}$ we have that $L_i(a) \to L(a)$ and $R_i(a) \to R(a)$. By analogy to the case of C^*-algebras, we call this topology *strict* (in [13, VIII-15.2] this topology is called *strong*). If $u = (L, R) \in M(\mathcal{A})$, we write ua and au instead of $L(a)$ and $R(a)$ respectively.

There is an isometric and continuous inclusion $\mathcal{A} \hookrightarrow M(\mathcal{A})$, given by $a \mapsto (L_a, R_a)$, where L_a is multiplication by a on the left, and R_a is multiplication by a on the right. In particular, the topology of $\mathcal{A}$ is stronger than the topology inherited from the strict topology of $M(\mathcal{A})$. If A_e is unital, these topologies agree. There is also an isomorphism $M(A_e) \cong M_e(\mathcal{A})$: since A_t is a Hilbert A_e-bimodule, then it is also a Hilbert $M(A_e)$-bimodule, and it can be shown that the actions of left and right multiplications by elements of $M(A_e)$ on $\mathcal{A}$ define multipliers of order e (see [13, VIII-3.8]). If $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a non-degenerate representation of $\mathcal{A}$, then there exists a unique extension ([13, VIII-15.3]) of π to a representation $\pi' : M(\mathcal{A}) \to \mathcal{L}(\mathcal{H})$ such that $\forall h \in \mathcal{H}$, the map $u \mapsto \pi'(u)h$ is strictly continuous on cylinders of $\mathcal{A}$ (the cylinder of radius r of $\mathcal{A}$ is $C_r := \{a \in \mathcal{A} : \|a\| \leq r\}$).

Lemma 3.16. *The maps $M_t(\mathcal{A}) \times \mathcal{A} \to \mathcal{A}$: $(u, a) \mapsto ua$ and $(u, a) \mapsto au$ are continuous, $\forall t \in G$.*

Proof. Recall that for any multiplier $u \in M_t(\mathcal{A})$, the maps $\mathcal{A} \to \mathcal{A}$: $a \mapsto ua$ $a \mapsto au$ are continuous. Suppose that $(u_i, a_i) \to (u, a)$ in $M_t(\mathcal{A}) \times \mathcal{A}$, with $a_i \in A_{s_i}, a \in A_s$. Since the norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}$ is continuous, there exist $M \geq 0$ and i_0 such that $\forall i \geq i_0$ we have $\|a_i\| \leq M$. Hence if $i \geq i_0$: $\max\{\|u_ia_i - ua_i\|, \|a_iu_i - a_iu\|\} \leq M\,\|u_i - u\| \to 0$, so we have $(u_ia_i - ua_i) \to 0_{ts}$ and $(a_iu_i - a_iu) \to 0_{st}$ when $i \to \infty$. On the other hand, we have that $ua_i \to ua$ and $a_iu \to au$. Thus $u_ia_i \to ua$ and $a_iu_i \to au$ if $i \to \infty$. □

The next result is analogous to [20, Lemma T.6.1.].

Lemma 3.17. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles and $\mathcal{A} \bigotimes \mathcal{B}$ a tensor product of $\mathcal{A}$ and $\mathcal{B}$. Then there exist unique inclusions $\iota_{\mathcal{A}} : M(\mathcal{A}) \to M(\mathcal{A} \bigotimes \mathcal{B})$ and $\iota_{\mathcal{B}} : M(\mathcal{B}) \to M(\mathcal{A} \bigotimes \mathcal{B})$ such that $\iota_{\mathcal{A}}(u)\iota_{\mathcal{B}}(v) = \iota_{\mathcal{B}}(v)\iota_{\mathcal{A}}(u)$, $\forall u \in M(\mathcal{A})$, $v \in M(\mathcal{B})$, and such that $\iota_{\mathcal{A}}(a)\iota_{\mathcal{B}}(b) = a \otimes b$, $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$. These inclusions are isometric and continuous in the strict topologies when restricted to cylinders.*

Proof. Let $u \in M_t(\mathcal{A})$. For $r \in G$, $s \in H$, the map $A_r \times B_s \to A_{tr} \bigotimes B_s$ such that $(a_r, b_s) \mapsto (ua_r, b_s)$ is bilinear, and therefore there exists a unique linear map $L_u : A_t \bigodot B_s \to A_{tr} \bigotimes B_s$ such that $a_r \otimes b_s \mapsto ua_r \otimes b_s$. Similarly, there exists $R_u : A_t \bigodot B_s \to A_{rt} \bigotimes B_s$ such that $R_u(a_r \otimes b_s) = a_ru \otimes b_s$. The collection of such maps define two applications $L_u, R_u : \mathcal{A} \bigodot \mathcal{B} \to \mathcal{A} \bigotimes \mathcal{B}$ such that $\forall x, y \in \mathcal{A} \bigodot \mathcal{B} \subseteq \mathcal{A} \bigotimes \mathcal{B}$ satisfy: $L_u(xy) = L_u(x)y$, $R_u(xy) =$

$xR_u(y)$, $xL_u(y) = R_u(x)y$. If we prove that L_u, R_u are bounded, then they extend by continuity on each fiber to continuous operators, which still satisfy the above algebraic relations. In other words, the pair formed by these extensions will be a multiplier of order (t,e) of $(\mathcal{A} \bigotimes \mathcal{B})_d$.

Let $x = \sum_{i=1}^n a_i \otimes b_i \in A_r \bigotimes B_s$. Then: $\|L_u x\|^2 = \|\sum_{i=1}^n ua_i \otimes b_i\|^2 = \|\sum_{i,j=1}^n a_i^* u^* u a_j \otimes b_i^* b_j\|$. Let $\mathfrak{u} = (u_{ij}) \in M_n(M(A_e))$, $\mathfrak{a} = (a_{ij}) \in M_n(A_r)$, given by: $u_{ij} = \begin{cases} \sqrt{u^*u} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$ and $a_{ij} = \begin{cases} a_j & \text{if } i = 1, \\ 0 & \text{otherwise} \end{cases}$. Since A_r is a right Hilbert $M(A_e)$-module, then $M_n(A_r)$ is a right Hilbert $M_n(M(A_e))$-module. Then we have $\langle \mathfrak{u}\mathfrak{a}, \mathfrak{u}\mathfrak{a}\rangle \leq \mathfrak{u}^*\mathfrak{u}\langle \mathfrak{a}, \mathfrak{a}\rangle \leq \|u\|^2 \langle \mathfrak{a}, \mathfrak{a}\rangle$. Thus $\mathfrak{c} := \|u\|^2\langle \mathfrak{a}, \mathfrak{a}\rangle_r - \langle \mathfrak{u}\mathfrak{a}, \mathfrak{u}\mathfrak{a}\rangle_r \geq 0$. An easy computation shows that if $\mathfrak{c} = (c_{ij})$, then $c_{ij} = a_i^*(\|u\|^2 - u^*u)a_j$. On the other hand, $M_n(B_s)$ is a Hilbert $M_n(M(B_e))$-module. In particular if $\mathfrak{b} = (b_{ij}) \in M_n(B_s)$ is given by $b_{ij} = \begin{cases} b_j & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$, the element $\mathfrak{b}^*\mathfrak{b} = (b_i^* b_j)$ is positive in $M_n(M(B_e))$. Now, Lemma 2.3 implies that $\mathfrak{c} \otimes \mathfrak{b}^*\mathfrak{b} = \big(a_i^*(\|u\|^2 - u^*u)a_j \otimes b_i^* b_j\big)$ is a positive element in any C^*-completion of $M_n(M(A_e)) \bigodot M_n(M(B_e))$, and $\sum_{i,j=1}^n a_i^*(\|u\|^2 - u^*u)a_j \otimes b_i^* b_j$ is a positive element in any C^*-completion of $M(A_e) \bigodot M(B_e)$ (alternatively, the positivity of $\mathfrak{c} \otimes \mathfrak{b}^*\mathfrak{b}$ can be deduced from the proof of [15, Lemma 4.3], which does not really use that the norm involved is $\|\ \|_{\min}$). Thus $\|\sum_{i,j=1}^n a_i^* u^* u a_j \otimes b_i^* b_j\| \leq \|u\|^2 \|\sum_{i,j=1}^n a_i^* a_j \otimes b_i^* b_j\|$, for any C^*-norm on $A_e \bigodot B_e$. This shows that $\|L_u x\|^2 \leq \|u\|^2 \|x\|^2$, so L_u is bounded. Similarly we see that $\|R_u x\|^2 \leq \|u\|^2 \|x\|^2$, and therefore (L_u, R_u) extends to a multiplier $\iota_{\mathcal{A}}(u)$ on $(\mathcal{A} \bigotimes \mathcal{B})_d$, and $\|\iota_{\mathcal{A}}(u)\| \leq \|u\|$. In fact $\|\iota_{\mathcal{A}}(u)\| = \|u\|$: if $a \in \mathcal{A}$, $b \in \mathcal{B}$ are such that $\|a\|, \|b\| \leq 1$, then $\|\iota_{\mathcal{A}}(u)\| \geq \|\iota_{\mathcal{A}}(u)(a \otimes b)\| = \|ua\|\,\|b\| = \|ua\|$, and therefore $\|\iota_{\mathcal{A}}(u)\| \geq \|u\|$. Then $\|\iota_{\mathcal{A}}(u)\| = \|u\|$, so $\iota_{\mathcal{A}}$ is an isometry.

To see that $\iota_{\mathcal{A}}(u) \in M(\mathcal{A} \bigotimes \mathcal{B})$, it remains to prove that it is continuous. To this end consider $f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$. Then the maps $G \times H \to \mathcal{A} \bigotimes \mathcal{B}$ such that $(t,s) \mapsto uf(t) \otimes g(s)$ and $(t,s) \mapsto f(t)u \otimes g(s)$ are continuous. Suppose that $x_i \to x$ in $\mathcal{A} \bigotimes \mathcal{B}$, and let $l = \sum_i f_i \oslash g_i$ be such that $\|l(t,s) - x\| < \epsilon$, where $x \in A_t \bigotimes B_s$, $x_i \in A_{t_i} \bigotimes B_{s_i}$. Since $x_i \to x$ and $l(t_i, s_i) \to l(t,s)$, there exists i_0 such that $\forall i \geq i_0$ we have $\|l(t_i,s_i) - x_i\| < \epsilon$. Now $\|L_u l(t_i,s_i) - L_u x_i\| \leq \epsilon\|u\|$, and $\|L_u l(t,s) - L_u x\| \leq \epsilon\|u\|$, and since $L_u l(t_i,s_i) \to L_u(t,s)$, we conclude that $L_u x_i \to L_u x$.

Let see now that $\iota_{\mathcal{A}}$ is strictly continuous on cylinders. If $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $(u_i) \subseteq \mathcal{A}$ is a net strictly convergent to $u \in \mathcal{A}$, with $\|u_i\|, \|u\| \leq C$, then: $\iota_{\mathcal{A}}(u_i)(a \otimes b) = u_i a \otimes b \to ua \otimes b = \iota_{\mathcal{A}}(u)(a \otimes b)$ and $(a \otimes b)\iota_{\mathcal{A}}(u_i) = au_i \otimes b \to au \otimes b = (a \otimes b)\iota_{\mathcal{A}}(u)$. Then $\iota_{\mathcal{A}}(u_i)x \to \iota_{\mathcal{A}}(u)x$ and $x\iota_{\mathcal{A}}(u_i) \to x\iota_{\mathcal{A}}(u)$, $\forall x \in \mathcal{A} \bigodot \mathcal{B}$. Since $\|\iota_{\mathcal{A}}(u_i)\|, \|\iota_{\mathcal{A}}(u)\| \leq C$, we conclude that $\iota_{\mathcal{A}}(u_i)x \to \iota_{\mathcal{A}}(u)x$ and $x\iota_{\mathcal{A}}(u_i) \to x\iota_{\mathcal{A}}(u)$, $\forall x \in \mathcal{A} \bigotimes \mathcal{B}$. Thus $\iota_{\mathcal{A}}(u_i)$ converges strictly to $\iota_{\mathcal{A}}(u)$.

Similarly we construct $\iota_{\mathcal{B}} : M(\mathcal{B}) \to M(\mathcal{A} \bigotimes \mathcal{B})$: if $v \in M(\mathcal{B})$ and $a \in \mathcal{A}$, $b \in \mathcal{B}$, then $\iota_{\mathcal{B}}(v)(a \otimes b) = a \otimes vb$, and $(a \otimes b)\iota_{\mathcal{B}}(v) = a \otimes bv$. It is clear that $\iota_{\mathcal{A}}(u)\iota_{\mathcal{B}}(v) = \iota_{\mathcal{B}}(v)\iota_{\mathcal{A}}(u)$, $\forall u \in M(\mathcal{A})$, $v \in M(\mathcal{B})$, and also that $\iota_{\mathcal{A}}(a)\iota_{\mathcal{B}}(b) = a \otimes b$, $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$.

This way we obtain a map $M(\mathcal{A}) \times M(\mathcal{B}) \to M(\mathcal{A} \bigotimes \mathcal{B})$, given by $(u, v) \mapsto \iota_{\mathcal{A}}(u)\iota_{\mathcal{B}}(v)$, which is bilinear on each $M_t(\mathcal{A}) \times M_s(\mathcal{B})$, and therefore we get a map $M(\mathcal{A}) \bigodot M(\mathcal{B}) \to M(\mathcal{A} \bigotimes \mathcal{B})$, which is linear on each $M_t(\mathcal{A}) \otimes M_s(\mathcal{B})$, and which is a homomorphism of Fell bundles because $\iota_{\mathcal{A}}(u)$ and $\iota_{\mathcal{B}}(v)$ commute, $\forall u \in M(\mathcal{A})$, $v \in M(\mathcal{B})$. □

Proposition 3.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over the locally compact groups G and H respectively, and let $\mathcal{H}$ be a Hilbert module. Then for each $(\pi_1, \pi_2) \in R(\mathcal{A}, \mathcal{B}, \mathcal{H})$, there exists a unique $\pi \in R(\mathcal{A} \bigotimes_{\max} \mathcal{B}, \mathcal{H})$ such that $\pi(a \otimes b) = \pi_1(a)\pi_2(b)$, $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$, and the map $(\pi_1, \pi_2) \mapsto \pi$ thus defined is a bijection between $R(\mathcal{A}, \mathcal{B}, \mathcal{H})$ and $R(\mathcal{A} \bigotimes_{\max} \mathcal{B}, \mathcal{H})$.*

Proof. Let $(\pi_1, \pi_2) \in R(\mathcal{A}, \mathcal{B}, \mathcal{H})$. The map $\mathcal{A} \times \mathcal{B} \to \mathcal{L}(\mathcal{H})$ such that $(a_t, b_s) \mapsto \pi_1(a_t)\pi_2(b_s)$ is bilinear on each $A_t \times B_s$, and therefore there exists a unique $\pi : \mathcal{A} \bigodot \mathcal{B} \to \mathcal{L}(\mathcal{H})$ such that $\pi(a_t \otimes b_s) = \pi_1(a_t)\pi_2(b_s)$. Since $\pi_1(a)$ and $\pi_2(b)$ commute, $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$, we have that $\pi : \mathcal{A} \bigodot \mathcal{B} \to \mathcal{L}(\mathcal{H})$ is a representation of the pre-Fell bundle $\mathcal{A} \bigodot \mathcal{B}$. Note that $x \mapsto \|x\|' := \max\{\|x\|_{\max}, \|\pi(x)\|\}$ is a C^*-norm on $A_t \bigodot B_s$, so $\|\cdot\|' = \|\cdot\|_{\max}$, and then $\|\pi(x)\| \leq \|x\|_{\max}$, $\forall x \in A_t \bigodot B_s$. Thus π has a unique extension to a representation $\pi : (\mathcal{A} \bigotimes_{\max} \mathcal{B})_d \to \mathcal{L}(\mathcal{H})$. It is easy to see that this representation is non-degenerate: since π_1 is non-degenerate, for every $h \in \mathcal{H}$ there exist $h' \in \mathcal{H}$ and $a \in A_e$ such that $\pi_2(a)h' = h$. Since π_2 is non degenerate, there exist $h'' \in \mathcal{H}$ and $b \in B_e$ such that $\pi_1(b)h'' = h'$. Therefore $\pi(a \otimes b)h'' = \pi_1(a)\pi_2(b)h'' = \pi_1(a)h' = h$.

To see that π is continuous is sufficient, by [13, II-13.16], to prove that $\forall f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$, and $h \in \mathcal{H}$, the map $F : G \times H \to \mathcal{H}$ such that $F(t, s) = \pi\left((f \oslash g)(t, s)\right) h$ is continuous. Now

$$\begin{aligned}
\|F(t,s) - F(t_0,s_0)\| &= \|\pi_1\left(f(t)\right)\pi_2\left(g(s)\right)h - \pi_1\left(f(t_0)\right)\pi_2\left(g(s_0)\right)h\| \\
&\leq \|\pi_1(f(t))\left(\pi_2(g(s)) - \pi_2(g(s_0))\right)h\| \\
&\quad + \|\left(\pi_1(f(t)) - \pi_1(f(t_0))\right)\pi_2(g(s_0))h\| \\
&\leq \|f\|_\infty \|(\pi_2(g(s))h - \pi_2(g(s_0))h\| \\
&\quad + \|\pi_1(f(t))\pi_2(g(s_0))h - \pi_1(f(t_0))\pi_2((s_0))h\|
\end{aligned}$$

Then $F(t, s) \to F(t_0, s_0)$ if $(t, s) \to (t_0, s_0)$. Therefore $\pi \in R(\mathcal{A} \bigotimes_{\max} \mathcal{B}, \mathcal{H})$.

Conversely suppose that $\pi \in R(\mathcal{A} \bigotimes_{\max} \mathcal{B}, \mathcal{H})$. By [13, VIII-15.3], π can be uniquely extended to a representation π' of $M(\mathcal{A} \bigotimes_{\max} \mathcal{B})$ such that $x \mapsto \pi'(x)h$ is strictly continuous on cylinders, $\forall h \in \mathcal{H}$. Let $\pi_1 = \pi'\iota_{\mathcal{A}}\big|_{\mathcal{A}} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, and $\pi_2 = \pi'\iota_{\mathcal{B}}\big|_{\mathcal{B}} : \mathcal{B} \to \mathcal{L}(\mathcal{H})$, where $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ are the inclusions provided by Lemma 3.17. Since π', $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ are continuous on cylinders, immediately follows that $\forall h \in \mathcal{H}$ the maps $\mathcal{A} \to \mathcal{H}$ and $\mathcal{B} \to \mathcal{H}$ given by $a \mapsto \pi_1(a)h$ and $b \mapsto \pi_2(b)h$ respectively are continuous, from where it follows that π_1 and π_2 are continuous representations. Since $\iota_{\mathcal{A}}(a)$ and $\iota_{\mathcal{B}}(b)$ commute, $\forall a \in \mathcal{A}$ and $b \in \mathcal{B}$, then $\pi_1(a)$ and $\pi_2(b)$ commute as well. Finally, the representations π_1 and π_2 are non-degenerate. To see this is enough to show that $\pi(a \otimes b)h$ is in the image of π_1 and the image of π_2, $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$ and $h \in \mathcal{H}$. By Cohen-Hewitt a and b can be factorized as $a = a_1a_2$, $b = b_1b_2$, and therefore $\pi_1(a_1)\pi(a_2 \otimes b)h = \pi(a_1a_2 \otimes b)h = \pi(a \otimes b)h$ and $\pi_2(b_1)\pi(a \otimes b_2)h = \pi(a \otimes b_1b_2)h = \pi(a \otimes b)h$.

In conclusion we constructed two correspondences $(\pi_1, \pi_2) \mapsto \pi$ and $\pi \mapsto (\pi\iota_{\mathcal{A}}\big|_{\mathcal{A}}, \pi\iota_{\mathcal{B}}\big|_{\mathcal{B}})$, which clearly are mutually inverse. □

4. C*-algebras of Tensor Products of Fell Bundles

The first goal of this section is to compare tensor products of the cross-sectional algebras of the Fell bundles $\mathcal{A}$ and $\mathcal{B}$ with the cross-sectional algebras of tensor products of $\mathcal{A}$ and $\mathcal{B}$. This is accomplished in Propositions 4.2 and 4.3, and in Theorem 4.7. The second objective is to give some applications.

Let $\mathcal{B}$ be a Fell bundle over a locally compact group G. Then there are two important cross-sectional C^*-algebras associated with $\mathcal{B}$: the full cross-sectional algebra $C^*(\mathcal{B})$, and the reduced cross-sectional algebra $C^*_r(\mathcal{B})$. We recall next their definitions.

Suppose that G is a locally compact group with Haar measure λ and modular function Δ. Let $\mathcal{B}$ be a Fell bundle over G and let $L^1(\mathcal{B}) := \{f : G \to \mathcal{A} :\ f(t) \in B_t, \forall t \in G,$ and $(t \mapsto \|f(t)\|) \in L^1(G,\lambda)\}$. Then $C_c(\mathcal{B})$ and $L^1(\mathcal{B})$ are *-algebras with the operations: $f * g(t) = \int_G f(r)g(r^{-1}t)$, $f^*(t) = \Delta(t)^{-1}f(t^{-1})^*$. Moreover, $L^1(\mathcal{B})$ is a Banach *-algebra with the norm: $\|f\|_1 = \int_G \|f(t)\|$. The enveloping C^*-algebra of $L^1(\mathcal{B})$ is called the cross-sectional algebra of $\mathcal{B}$, and it is denoted by $C^*(\mathcal{B})$.

Suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of Fell bundles. If $f \in L^1(\mathcal{A})$ we have that $\phi^1(f) : G \to \mathcal{B}$, given by $\phi^1(f)(t) = \phi\big(f(t)\big)$, belongs to $L^1(\mathcal{B})$, and $\|\phi^1(f)\|_1 \leq \|f\|_1$. Moreover ϕ^1 is a homomorphism of *-algebras, so it uniquely extends to a homomorphism $C^*(\phi) : C^*(\mathcal{A}) \to C^*(\mathcal{B})$. This way we obtain a functor from the category of Fell bundles over G to the category of C^*-algebras. In fact this functor is the compostion of the functor $A \mapsto C^*(A)$ from the category of Banach *-algebras with approximate unit and contractive homomorphisms to the category of C^*-algebras, with the functor: $\mathcal{A} \mapsto L^1(\mathcal{A})$, $\big(\mathcal{A} \xrightarrow{\phi} \mathcal{B}\big) \mapsto \big(L^1(\mathcal{A}) \xrightarrow{\phi^1} L^1(\mathcal{B})\big)$.

There is a bijection between non-degenerate representations of the Fell bundle $\mathcal{B}$ and non-degenerate representations of the C^*-algebra $C^*(\mathcal{B})$. In one direction this correspondence consists of passing from a representation $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ to its integrated form $\int_G \pi : C^*(\mathcal{B}) \to \mathcal{L}(\mathcal{H})$, characterized by $\langle \int_G \pi(f)\xi, \eta\rangle = \int_G \langle \pi(f(t))\xi, \eta\rangle dt$, $\forall f \in C_c(\mathcal{B})$, $\xi, \eta \in \mathcal{H}$ (see [13, VIII-13.2]).

Among the representations of $\mathcal{B}$ there is one of particular importance: the *(left) regular representation*, which we describe below. Note that $C_c(\mathcal{B})$ is a right B_e-module with the action given by pointwise multiplication. Moreover the map $\langle\cdot,\cdot\rangle : C_c(\mathcal{B}) \times C_c(\mathcal{B}) \to B_e$ such that $\langle \xi, \eta\rangle = \int_G \xi(s)^*\eta(s)ds$ is a pre-inner product. Completing $C_c(\mathcal{B})$ with respect to the norm defined by $\langle\cdot,\cdot\rangle$ we obtain a full right Hilbert B_e-module, which is denoted by $L^2(\mathcal{B})$. Again, it is not dificult to check that $\mathcal{B} \mapsto L^2(\mathcal{B})$ is a functor. There exists a unique representation $\Lambda^{\mathcal{B}} : \mathcal{B} \to \mathcal{L}(L^2(\mathcal{B}))$ such that $\Lambda^{\mathcal{B}}_{b_t}\xi(s) = b_t\xi(t^{-1}s)$ $\forall s, t \in G$, $b_t \in B_t$ and $\xi \in C_c(\mathcal{B})$ (if no confusion can arise we write just Λ instead of $\Lambda^{\mathcal{B}}$). This is called the regular representation of $\mathcal{B}$ on $L^2(\mathcal{B})$. Its integrated form is also called the regular representation, and it satisfies $\Lambda_f(\xi) = f * \xi$, $\forall f \in C_c(\mathcal{B}) \subseteq C^*(\mathcal{B})$ and $\forall \xi \in C_c(\mathcal{B}) \subseteq L^2(\mathcal{B})$, where the *convolution* $f * \xi$ is defined as: $f * \xi(t) = \int_G f(s)\xi(s^{-1}t)ds$. The reduced cross-sectional algebra is then defined as: $C^*_r(\mathcal{B}) := \Lambda^{\mathcal{B}}(C^*(\mathcal{B})) \subseteq \mathcal{L}(L^2(\mathcal{B}))$.

When we look at the regular representation as a homomorphism $\Lambda^{\mathcal{B}} : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$, then it is clear that $\Lambda^{\mathcal{B}}$ is onto. In the case that $\Lambda^{\mathcal{B}}$ is also injective, thus an isomorphism, we say that the Fell bundle $\mathcal{B}$ is *amenable*. The reader is referred to [12], [9] and [3] for further information on the reduced cross-sectional algebra.

It can be shown that $\mathcal{B} \mapsto C^*_r(\mathcal{B})$ also is a functor, and in fact Λ is a natural transformation from C^* to C^*_r ([5, page 277]).

4.1. **Cross-sectional algebras.**

Lemma 4.1. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles and suppose that $\mathcal{A} \bigotimes \mathcal{B}$ is a tensor product of $\mathcal{A}$ and $\mathcal{B}$. Then there exists a unique homomorphism of algebras $j_c : C_c(\mathcal{A}) \bigodot C_c(\mathcal{B}) \to C_c(\mathcal{A} \bigotimes \mathcal{B})$, such that $j_c(f \odot g) = f \oslash g$, that is: $j_c(f \odot g)(t, s) = f(t) \otimes g(s)$, $\forall f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$, $t \in G$, and $s \in H$. Moreover j_c is injective and $j_c(C_c(\mathcal{A}) \bigodot C_c(\mathcal{B}))$ is dense in $C_c(\mathcal{A} \bigotimes \mathcal{B})$ in the inductive limit topology.*

Proof. The existence and uniqueness of the linear map j_c follows from the universal property of tensor products. It is clear that j_c is a homomorphism of *-algebras. To see that it is injective suppose that $l = \sum_{i=1}^n f_i \odot g_i \in \ker j_c$. Then $0 = \langle j_c(l), j_c(l)\rangle = \int_{G\times H} \sum_{i,j=1}^n f_i(t)^* f_j(t) \otimes g_i(s)^* g_j(s) d(t, s)$. On the other hand we have $\int_{G\times H} \sum_{i,j=1}^n f_i(t)^* f_j(t) \otimes g_i(s)^* g_j(s) d(t, s) = \big(\int_G \sum_{i,j=1}^n f_i(t)^* f_j(t)dt\big) \otimes \big(\int_H \sum_{i,j=1}^n g_i(s)^* g_j(s)ds\big)$. Therefore, if we think of l as an element of $L^2(\mathcal{A}) \bigodot L^2(\mathcal{B})$ we have that $\langle j_c(l), j_c(l)\rangle = \langle l, l\rangle$, where the latter is the pre-inner product of $L^2(\mathcal{A}) \bigodot L^2(\mathcal{B})$ computed in l. Since $\langle l, l\rangle = 0$, it follows that $l = 0$.

Let see that $j_c(C_c(\mathcal{A}) \bigodot C_c(\mathcal{B}))$ is dense in $C_c(\mathcal{A} \bigotimes \mathcal{B})$ in the inductive limit topology. It is clear that $j_c(C_c(\mathcal{A}) \bigodot C_c(\mathcal{B})))(t, s)$ is dense in $(\mathcal{A} \bigotimes \mathcal{B})_{(t,s)}$, $\forall (t, s) \in G \times H$. On the other hand, if $\Theta = C_c(G) \bigodot C_c(H)$, let $\theta \in \Theta$ and $l \in C_c(\mathcal{A}) \bigodot C_c(\mathcal{B})$, say $\theta = \sum_i \phi_i \odot \psi_i$ and $l = \sum_j f_j \odot g_j$, then: $\theta j_c(l)(t, s) = \big(\sum_i \phi_i(t)\psi_i(s)\big)\big(\sum_j f_j(t) \otimes g_j(s)\big) = \sum_i \sum_j (\phi_i f_j)(t) \otimes (\psi_i g_j)(s) = j_c(l')(t, s)$, where $l' = \sum_i \sum_j \phi_i f_j \odot \psi_i g_j \in C_c(\mathcal{A}) \bigodot C_c(\mathcal{B})$. Thus $j_c(C_c(\mathcal{A}) \bigodot C_c(\mathcal{B}))$ is dense in $C_c(\mathcal{A} \bigotimes \mathcal{B})$ by [3, Lemma 5.1]. ☐

Proposition 4.2. *Let $\mathcal{A}=(A_t)_{t\in G}$ and $\mathcal{B}=(B_s)_{s\in H}$ be Fell bundles. Then there exists a unique isomorphism $j: C^*(\mathcal{A})\bigotimes_{\max} C^*(\mathcal{B})\to C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})$, such that $j(f\otimes g)(t,s)=f(t)\otimes g(s)$, $\forall f\in C_c(\mathcal{A})$, $g\in C_c(\mathcal{B})$, and $(t,s)\in G\times H$.*

Proof. Recall that if $\mathcal{H}$ is a Hilbert space and $\mathcal{C}=(C_t)_{t\in G}$ is a Fell bundle, then there is a bijection between $R(\mathcal{C},\mathcal{H})$ and $R\big(C^*(\mathcal{C}),\mathcal{H}\big)$ such that to each $\pi\in R(\mathcal{C},\mathcal{H})$ corresponds the integrated representation $\int_G\pi$ of $C^*(\mathcal{C})$, which is determined by its values on elements of $C_c(\mathcal{C})$: if $f\in C_c(\mathcal{C})$ and $h\in\mathcal{H}$, then $(\int_G\pi)f\big|_h=\int_G\pi\big(f(t)\big)hdt$. Note as well that if $\mathcal{C}'=(C'_s)_{s\in H}$ is another Fell bundle, then the map $R(\mathcal{C},\mathcal{C}',\mathcal{H})\to R\big(C^*(\mathcal{C}),C^*(\mathcal{C}'),\mathcal{H}\big)$ such that $(\pi,\pi')\mapsto(\int_G\pi,\int_H\pi')$ is also a bijection, because the corresponding integrands commute. On the other hand, by Proposition 3.18 we have a bijection between $R(\mathcal{C},\mathcal{C}',\mathcal{H})$ and $R(\mathcal{C}\bigotimes_{\max}\mathcal{C}',\mathcal{H})$, given by $(\pi_1,\pi_2)\mapsto\pi_1\times\pi_2$, where $(\pi_1\times\pi_2)(a\otimes b)=\pi_1(a)\pi_2(b)$.

Let $j_c: C_c(\mathcal{A})\bigodot C_c(\mathcal{B})\to C_c(\mathcal{A}\bigotimes_{\max}\mathcal{B})$ be the map provided by Lemma 4.1. The comments above imply that $C^*(\mathcal{A})\bigotimes_{\max}C^*(\mathcal{B})$ and $C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})$ are respectively the completions of $C_c(\mathcal{A})\bigodot C_c(\mathcal{B})$ and $j_c(C_c(\mathcal{A})\bigodot C_c(\mathcal{B}))$ with respect to the norms:

$$\|\sum_i f_i\odot g_i\|=\sup\{\|\sum_i\int_G\pi_1(f_i)\int_H\pi_2(g_i)\|:\,(\pi_1,\pi_2)\in R(\mathcal{A},\mathcal{B},\mathcal{H})\},$$

$$\|j_c(\sum_i f_i\odot g_i)\|=\sup\{\|\int_{G\times H}(\pi_1\times\pi_2)(\sum_i f_i\otimes g_i)\|:(\pi_1,\pi_2)\in R(\mathcal{A},\mathcal{B},\mathcal{H})\}.$$

Now, if $h\in\mathcal{H}$:

$$\begin{aligned}\left(\int_{G\times H}(\pi_1\times\pi_2)\big(\sum_i f_i\otimes g_i\big)\right)h&=\int_{G\times H}\sum_i\pi_1\big(f_i(t)\big)\pi_2\big(g_i(s)\big)hd(t,s)\\&=\sum_i\int_G\int_H\pi_1\big(f_i(t)\big)\pi_2\big(g_i(s)\big)hdsdt\\&=\sum_i\int_G\pi_1\big(f_i(t)\big)\int_H\pi_2\big(g_i(s)\big)hdsdt\\&=\sum_i\int_G\pi_1\big(f_i(t)\big)\left(\int_H\pi_2(g_i)\right)hdt\\&=\sum_i\left(\int_G\pi_1(f_i)\int_H\pi_2(g_i)\right)h.\end{aligned}$$

Thus $j_c: C_c(\mathcal{A})\bigodot C_c(\mathcal{B})\to j_c(C_c(\mathcal{A})\bigodot C_c(\mathcal{B}))$ is an isometry with these norms so it extends uniquely to an isomorphism between $C^*(\mathcal{A})\bigotimes_{\max}C^*(\mathcal{B})$ and the C*-algebra $\overline{j_c(C_c(\mathcal{A})\bigodot C_c(\mathcal{B})}$. Since by Lemma 4.1 $j_c(C_c(\mathcal{A})\bigodot C_c(\mathcal{B})$ is dense in $C_c(\mathcal{A}\bigotimes_{\max}\mathcal{B})$ in the inductive limit topology, then it is also dense in $C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B})$. ∎

Proposition 4.3. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles, and suppose that $\alpha \geq \beta$ are C^*-norms on $\mathcal{A}\bigodot\mathcal{B}$. Then there exist a unique homomorphism $\sigma^\alpha_\beta : C^*(\mathcal{A}\bigotimes_\alpha\mathcal{B}) \to C^*(\mathcal{A}\bigotimes_\beta\mathcal{B})$ such that $\sigma^\alpha_\beta(f \oslash g) = f \oslash g$, $\forall f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$. Moreover σ^α_β is surjective.*

Proof. By Proposition 3.9 there exists a surjective homomorphism of Fell bundles $\sigma^\alpha_\beta : \mathcal{A}\bigotimes_\alpha\mathcal{B} \to \mathcal{A}\bigotimes_\beta\mathcal{B}$. Then there is an induced homomorphism $\sigma^\alpha_\beta : C^*(\mathcal{A}\bigotimes_\alpha\mathcal{B}) \to C^*(\mathcal{A}\bigotimes_\beta\mathcal{B})$, which we still denote by σ^α_β. If $f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$, $\sigma^\alpha_\beta(f\oslash g)(t,s) = \sigma^\alpha_\beta\big(f(t)\otimes g(s)\big) = f(t)\otimes g(s) = f\oslash g(t,s)$, from where it follows that $\sigma^\alpha_\beta(f \oslash g) = f \oslash g$. Since $\operatorname{span}\{f \oslash g : f \in C_c(\mathcal{A}), g \in C_c(\mathcal{B})\}$ is dense in $C^*(\mathcal{A}\bigotimes_\alpha\mathcal{B})$, we conclude that σ^α_β is surjective. □

Consider two Fell bundles $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$. Then $L^2(\mathcal{A})$ and $L^2(\mathcal{B})$ are full right Hilbert modules over A_e and B_e respectively. If α is a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$, then $\alpha|_{(A_e\bigodot B_e)} \in \mathcal{N}(A_e\bigodot B_e)$. Since $L^2(\mathcal{A})^r = A_e$ and $L^2(\mathcal{B})^r = B_e$, $\alpha|_{(A_e\bigodot B_e)}$ defines a C*-norm $\tilde\alpha$ on $L^2(\mathcal{A})\bigodot L^2(\mathcal{B})$, given by (2), that is $\tilde\alpha(\mu) := \sqrt{\alpha(\langle\mu,\mu\rangle)}$, $\forall\mu \in L^2(\mathcal{A})\bigodot L^2(\mathcal{B})$. The completion $L^2(\mathcal{A})\bigotimes_{\tilde\alpha} L^2(\mathcal{B})$ of $L^2(\mathcal{A})\bigodot L^2(\mathcal{B})$ with respect to $\tilde\alpha$ is a full right Hilbert module over $A_e\bigotimes_{\alpha|_{A_e\otimes B_e}} B_e$, so we have $(L^2(\mathcal{A})\bigotimes_{\tilde\alpha} L^2(\mathcal{B}))^r = A_e\bigotimes_{\alpha|_{A_e\otimes B_e}} B_e$, whose its corresponding inner product is determined by $\langle\xi_1\otimes\eta_1, \xi_2\otimes\eta_2\rangle = \langle\xi_1,\xi_2\rangle\otimes\langle\eta_1,\eta_2\rangle$, $\forall\xi_1,\xi_2 \in L^2(\mathcal{A})$, $\eta_1,\eta_2 \in L^2(\mathcal{B})$ (see Theorem 2.7).

Lemma 4.4. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles, and let α be a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$. If $\tilde\alpha$ is as above, then there exists a unique isomorphism $j_2 : L^2(\mathcal{A})\bigotimes_{\tilde\alpha} L^2(\mathcal{B}) \to L^2(\mathcal{A}\bigotimes_\alpha\mathcal{B})$, such that $j_2(\xi\otimes\eta) = \xi\oslash\eta$, $\forall\xi \in C_c(\mathcal{A}) \subseteq L^2(\mathcal{A})$, $\eta \in C_c(\mathcal{B}) \subseteq L^2(\mathcal{B})$. In particular we have $L^2(\mathcal{A})\bigotimes_{\min} L^2(\mathcal{B}) \cong L^2(\mathcal{A}\bigotimes_{\min}\mathcal{B})$ and $L^2(\mathcal{A})\bigotimes_{\max} L^2(\mathcal{B}) \cong L^2(\mathcal{A}\bigotimes_{\max}\mathcal{B})$.*

Proof. Let j_c be the map defined in Lemma 4.1. If $\xi_1,\xi_2 \in C_c(\mathcal{A})$, $\eta_1,\eta_2 \in C_c(\mathcal{B})$, then $j_c(\xi_1\otimes\eta_1)$, $j_c(\xi_2\otimes\eta_2) \in C_c(\mathcal{A}\bigotimes_\alpha\mathcal{B}) \subseteq L^2(\mathcal{A}\bigotimes_\alpha\mathcal{B})$ and we have

$$\langle j_c(\xi_1\otimes\eta_1), j_c(\xi_2\otimes\eta_2)\rangle = \int_{G\times H}(\xi_1\otimes\eta_1)(t,s)^*(\xi_2\otimes\eta_2)(t,s)d(t,s)$$
$$= \int_G\int_H \xi_1(t)^*\xi_2(t)\otimes\eta_1(s)^*\eta_2(s)dsdt = \langle\xi_1,\xi_2\rangle\otimes\langle\eta_1,\eta_2\rangle$$

On the other hand, if $a \in A_e$, $b \in B_e$, $\xi \in C_c(\mathcal{A})$, and $\eta \in C_c(\mathcal{B})$, we have $\big(j_c(\xi\otimes\eta)\big)(a\otimes b)(t,s) = (\xi(t)\otimes\eta(s))(a\otimes b) = \xi(t)a\otimes\eta(s)b = j_c\big((\xi\otimes\eta)(a\otimes b)\big)(t,s)$. Thus j_c is a homomorphism of pre-Hilbert modules over $A_e\bigotimes_\alpha B_e$ which is injective by Lemma 4.1 and has dense image in $L^2(\mathcal{A}\bigotimes_\alpha\mathcal{B})$: by Lemma 4.1, the image of j_c is dense in the $C_c(\mathcal{A}\bigotimes_\alpha\mathcal{B})$ in the inductive limit topology, and therefore is dense in $L^2(\mathcal{A}\bigotimes_\alpha\mathcal{B})$. Thus j_c extends uniquely to an isomorphism $j_2 : L^2(\mathcal{A})\bigotimes_{\tilde\alpha} L^2(\mathcal{B}) \to L^2(\mathcal{A}\bigotimes_\alpha\mathcal{B})$. The last two statements follow from the fact that if $\alpha = \|\ \|_{\min}$, then also $\tilde\alpha = \|\ \|_{\min}$ and, similarly, if $\alpha = \| \ \|_{\max}$, then also $\tilde\alpha = \| \ \|_{\max}$ (because $\alpha \mapsto \alpha_{A_e \bigodot B_e} \mapsto \tilde\alpha$ are isomorphisms of posets). □

With the notation as above, we have inclusions $C_r^*(\mathcal{A}) \subseteq \mathcal{L}\big(L^2(\mathcal{A})\big)$ and $C_r^*(\mathcal{B}) \subseteq \mathcal{L}\big(L^2(\mathcal{B})\big)$, so $C_r^*(\mathcal{A}) \bigodot C_r^*(\mathcal{B})$ is included in $\mathcal{L}\big(L^2(\mathcal{A})\big) \bigodot \mathcal{L}\big(L^2(\mathcal{B})\big)$, which in turn is included in $\mathcal{L}\big(L^2(\mathcal{A}) \bigotimes_{\tilde\alpha} L^2(\mathcal{B})\big)$according to Corollary 2.13. Therefore we have an inclusion $C_r^*(\mathcal{A}) \bigodot C_r^*(\mathcal{B}) \hookrightarrow \mathcal{L}\big(L^2(\mathcal{A}) \bigotimes_{\tilde\alpha} L^2(\mathcal{B})\big)$.

Definition 4.5. If α is a C^*-norm on $\mathcal{A} \bigodot \mathcal{B}$, we define $C_r^*(\mathcal{A}) \bigotimes_{\overline{\alpha}} C_r^*(\mathcal{B})$ to be the closure of $C_r^*(\mathcal{A}) \bigodot C_r^*(\mathcal{B})$ in $\mathcal{L}\big(L^2(\mathcal{A}) \bigotimes_{\tilde\alpha} L^2(\mathcal{B})\big)$ (that is: we call $\overline{\alpha}$ the norm on $C_r^*(\mathcal{A}) \bigodot C_r^*(\mathcal{B})$ inherited by the inclusion above). Recall that, in particular, if $\alpha = \| \ \|_{\min}$, then we also have $\overline{\alpha} = \| \ \|_{\min}$.

Suppose that $u : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary operator between the Hilbert modules $\mathcal{H}_1$ and $\mathcal{H}_2$. Then u induces an isomorphism $Ad_u : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$, given by $Ad_u(T) = uTu^*$, $\forall T \in \mathcal{L}(\mathcal{H}_1)$.

Proposition 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over the locally compact groups G and H respectively, $j_2 : L^2(\mathcal{A}) \bigotimes_{\tilde\alpha} L^2(\mathcal{B}) \to L^2(\mathcal{A} \bigotimes_\alpha \mathcal{B})$ the isomorphism given by Lemma 4.4, and $\overline{\alpha}$ the C^*-norm given by Definition 4.5. Then $Ad_{j_2}\big(C_r^*(\mathcal{A}) \bigotimes_{\overline{\alpha}} C_r^*(\mathcal{B})\big) = C_r^*(\mathcal{A} \bigotimes_\alpha \mathcal{B})$, and there is a unique isomorphism $j_r : C_r^*(\mathcal{A}) \bigotimes_{\overline{\alpha}} C_r^*(\mathcal{B}) \to C_r^*(\mathcal{A} \bigotimes_\alpha \mathcal{B})$ such that $j_r\big(\Lambda_f^{\mathcal{A}} \otimes \Lambda_g^{\mathcal{B}}\big) = \Lambda_{(f\otimes g)}^{\mathcal{A} \bigotimes_\alpha \mathcal{B}}$, $\forall f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$.*
In particular $C_r^*(\mathcal{A} \bigotimes_{\min} \mathcal{B}) \cong C_r^*(\mathcal{A}) \bigotimes_{\min} C_r^*(\mathcal{B})$.

Proof. As usual, by the universal property of tensor products we see that there exists a unique map $\Lambda^{\mathcal{A}} \odot \Lambda^{\mathcal{B}} : \mathcal{A} \bigodot \mathcal{B} \to \mathcal{L}\big(L^2(\mathcal{A}) \bigotimes_\alpha L^2(\mathcal{B})\big)$ such that $(\Lambda^{\mathcal{A}} \odot \Lambda^{\mathcal{B}})(a \odot b) = \Lambda_a^{\mathcal{A}} \otimes \Lambda_b^{\mathcal{B}}$, $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$. Writing just Λ instead of $\Lambda^{\mathcal{A} \bigotimes_\alpha \mathcal{B}}$ we have

$$\begin{aligned}
\Lambda_{(a_t \otimes b_s)}\big(j_2(\xi \otimes \eta)\big)(t_0, s_0) &= (a_t \otimes b_s)(\xi \otimes \eta)\big((t,s)^{-1}(t_0, s_0)\big) \\
&= a_t \xi(t^{-1}t_0) \otimes b_s \eta(s^{-1}s_0) \\
&= \big(\Lambda_{a_t}^{\mathcal{A}} \xi\big)(t_0) \otimes \big(\Lambda_{b_s}^{\mathcal{B}} \eta\big)(s_0) \\
&= \big(\Lambda_{a_t}^{\mathcal{A}} \xi \otimes \Lambda_{b_s}^{\mathcal{B}} \eta\big)(t_0, s_0) \\
&= j_2\big(\Lambda_{a_t}^{\mathcal{A}} \xi \otimes \Lambda_{b_s}^{\mathcal{B}} \eta\big)(t_0, s_0) \\
&= j_2\big((\Lambda^{\mathcal{A}} \odot \Lambda^{\mathcal{B}})(a_t \odot b_s)(\xi \otimes \eta)\big)(t_0, s_0),
\end{aligned}$$

It follows that $\Lambda_x = j_2(\Lambda^{\mathcal{A}} \odot \Lambda^{\mathcal{B}})(x) j_2^*$, $\forall x \in \mathcal{A} \bigodot \mathcal{B}$, so $\Lambda^{\mathcal{A}} \odot \Lambda^{\mathcal{B}}$ extends uniquely to a representation $\Lambda^{\mathcal{A}} \otimes \Lambda^{\mathcal{B}} : \mathcal{A} \bigotimes_\alpha \mathcal{B} \to \mathcal{L}(L^2(\mathcal{A}) \bigotimes_\alpha L^2(\mathcal{B}))$ such that $\Lambda^{\mathcal{A}} \otimes \Lambda^{\mathcal{B}}(x) = j_2^* \Lambda_x j_2$, $\forall x \in \mathcal{A} \bigotimes_\alpha \mathcal{B}$. Taking the corresponding integrated representations, we have that $\Lambda_{(f \otimes g)} = j_2(\Lambda^{\mathcal{A}} \otimes \Lambda^{\mathcal{B}})(f \otimes g) j_2^*$, $\forall f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$. If j_c is the map given by Lemma 4.1, then $j_c(C_c(\mathcal{A}) \bigodot C_c(\mathcal{B}))$ is dense in $C_r^*(\mathcal{A} \bigotimes_\alpha \mathcal{B})$. Therefore we conclude that $C_r^*(\mathcal{A} \bigotimes_\alpha \mathcal{B}) = j_2\big(C_r^*(\mathcal{A}) \bigotimes_{\overline{\alpha}} C_r^*(\mathcal{B})\big) j_2^*$, as we wanted to prove.

In particular, $j_r : C_r^*(\mathcal{A}) \bigotimes_{\overline{\alpha}} C_r^*(\mathcal{B}) \to C_r^*(\mathcal{A} \bigotimes_\alpha \mathcal{B})$ given by $x \mapsto j_2 x j_2^*$ is an isomorphism satisfying $j_r\big(\Lambda_f^{\mathcal{A}} \otimes \Lambda_g^{\mathcal{B}}\big) = \Lambda_{(f\otimes g)}$, $\forall f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$. The uniqueness of such an isomorphism is clear. As for the last statement just recall that $\overline{\alpha} = \| \ \|_{\min}$ if $\alpha = \| \ \|_{\min}$ (Corollary 2.13). $\square$

In functorial language, Proposition 4.2 and the last statement of Proposition 4.6 can be stated as follows. Let $\mathcal{B} \mapsto C^*(\mathcal{B})$ and $\mathcal{B} \mapsto C_r^*(\mathcal{B})$ be the functors sending a Fell bundle $\mathcal{B}$ to its cross-sectional and reduced croseed sectional algebras respectively, and let $\otimes_{\max}$ and $\otimes_{\min}$ be the bifunctors of taking maximal and minimal tensor products respectively (of Fell bundles or of C*-algebras). Then we have:

$$C^* \circ \otimes_{\max} = \otimes_{\max} \circ (C^* \times C^*) : \mathsf{F} \times \mathsf{F} \to \mathsf{C}^*$$
$$C_r^* \circ \otimes_{\min} = \otimes_{\min} \circ (C_r^* \times C_r^*) : \mathsf{F} \times \mathsf{F} \to \mathsf{C}^*,$$

where F is the category of Fell bundles and C^* the category of C*-algebras. That is: taking full (reduced) cross-sectional algebras commute with taking maximal (respectively: minimal) tensor products.

Theorem 4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over the locally compact groups G and H respectively. Then for every C^*-norm α on $\mathcal{A} \bigodot \mathcal{B}$ we have the following commutative diagram D_α:*

$$\begin{array}{ccccc}
C^*(\mathcal{A} \bigotimes_{\max} \mathcal{B}) & \xrightarrow{\sigma_\alpha^{\max}} & C^*(\mathcal{A} \bigotimes_\alpha \mathcal{B}) & \xrightarrow{\Lambda} & C_r^*(\mathcal{A} \bigotimes_\alpha \mathcal{B}) \\
{\scriptstyle j\cong}\uparrow & & & & \uparrow{\scriptstyle \cong j_r} \\
C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B}) & \xrightarrow[\Lambda^{\mathcal{A}} \otimes_{\max} \Lambda^{\mathcal{B}}]{} & C_r^*(\mathcal{A}) \bigotimes_{\max} C_r^*(\mathcal{B}) & \xrightarrow[\tilde{\sigma}_{\overline{\alpha}}^{\max}]{} & C_r^*(\mathcal{A}) \bigotimes_{\overline{\alpha}} C_r^*(\mathcal{B})
\end{array}$$

where $\Lambda = \Lambda^{\mathcal{A} \bigotimes_\alpha \mathcal{B}}$, the map $\sigma_\alpha^{\max}$ is provided by Proposition 4.3, j is given by Proposition 4.2, j_r by Proposition 4.6, and $\Lambda^{\mathcal{A}} \otimes_{\max} \Lambda^{\mathcal{B}}$ is the tensor product of the regular representations of $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$ respectively. Finally, the existence and the surjectivity of $\tilde{\sigma}_{\tilde{\alpha}}^{\max}$ is obvious.

Proof. Let $f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$. Then, by Proposition 4.2 and Lemma 4.1 we have $\Lambda \sigma_\alpha^{\max} j(f \otimes g) = \Lambda \sigma_\alpha^{\max}(f \oslash g) = \Lambda_{(f \oslash g)}$. On the other hand Lemma 4.1 and Proposition 4.6 imply that $j_r \tilde{\sigma}_\alpha^{\max}(\Lambda^{\mathcal{A}} \otimes_{\max} \Lambda^{\mathcal{B}})(f \otimes g) = j_r \tilde{\sigma}_\alpha^{\max}\big(\Lambda_f^{\mathcal{A}} \otimes \Lambda_g^{\mathcal{B}}\big) = j_r\big(\Lambda_f^{\mathcal{A}} \otimes \Lambda_g^{\mathcal{B}}\big) = \Lambda_{(f \oslash g)}$. Since $C_c(\mathcal{A}) \bigodot C_c(\mathcal{B})$ is dense in $C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B})$, we conclude that $\Lambda_{\mathcal{A} \bigotimes_\alpha \mathcal{B}} \sigma_\alpha^{\max} j(x) = j_r \tilde{\sigma}_\alpha^{\max}(\Lambda^{\mathcal{A}} \otimes_{\max} \Lambda^{\mathcal{B}})(x)$, $\forall x \in C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B})$, and therefore the diagram commutes. $\square$

Corollary 4.8. *The Fell bundle $\mathcal{A} \bigotimes_{\max} \mathcal{B}$ is amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are amenable and $C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B}) = C^*(\mathcal{A}) \bigotimes_{\overline{\max}} C^*(\mathcal{B})$.*

Proof. For $\alpha = \max$, the diagram $D_{\max}$ becomes:

$$\begin{array}{ccc} C^*(\mathcal{A}\bigotimes_{\max}\mathcal{B}) & \xrightarrow{\quad\Lambda\quad} & C^*_r(\mathcal{A}\bigotimes_{\max}\mathcal{B}) \\ {\scriptstyle j\cong}\uparrow & & \uparrow{\scriptstyle \cong j_r} \\ C^*(\mathcal{A})\bigotimes_{\max}C^*(\mathcal{B}) & \xrightarrow[\tilde{\sigma}^{\max}_{\overline{\max}}\circ(\Lambda^{\mathcal{A}}\otimes_{\max}\Lambda^{\mathcal{B}})]{} & C^*_r(\mathcal{A})\bigotimes_{\overline{\max}}C^*_r(\mathcal{B}) \end{array}$$

If Λ is an isomorphism, then so is $\tilde{\sigma}^{\max}_{\overline{\max}}\circ(\Lambda^{\mathcal{A}}\otimes\Lambda^{\mathcal{B}})$, and therefore also $\tilde{\sigma}^{\max}_{\overline{\max}}$, because $\Lambda^{\mathcal{A}}\otimes\Lambda^{\mathcal{B}}$ is surjective. Moreover the injectivity Λ implies that of $\Lambda^{\mathcal{A}}\otimes\Lambda^{\mathcal{B}}$, and therefore that of $\Lambda^{\mathcal{A}}$ and $\Lambda^{\mathcal{B}}$, and also that $\|\cdot\|_{\max} = \|\cdot\|_{\overline{\max}}$. In other words, the amenability of $\mathcal{A}\bigotimes_{\max}\mathcal{B}$ implies the amenability of $\mathcal{A}$ and of $\mathcal{B}$, and also that $\|\cdot\|_{\max} = \|\cdot\|_{\overline{\max}}$. The converse is clear. □

4.2. **Some applications.** Suppose that $\mathcal{B} = (B_t)_{t\in G}$ is a Fell bundle over the locally compact group G, and for $\phi,\psi \in C_c\big(G, M(B_e)\big)$ and $b \in B_t$ let $\phi\cdot b\cdot\psi := \int_G \phi_i(s)^* b\psi_i(t^{-1}s)ds$. Since every $x \in M(B_e)$ defines a multiplier of $\mathcal{B}$ of order e, then we have that $\phi\cdot b\cdot\psi \in B_t$, $\forall b \in B_t$. So we have a map $\Phi_{\phi,\psi} : \mathcal{B}\to\mathcal{B}$ defined by $b \mapsto \phi\cdot b\cdot\psi$. For $b_t \in B_t$ we have

$$\phi\cdot b_t\cdot\psi = \int_{(\operatorname{supp}\phi)\cap(t\operatorname{supp}\psi)} \phi(s)^* b_t\psi(t^{-1}s)ds, \text{ so if } m \text{ is Haar measure:}$$

$$\|\phi\cdot b_t\cdot\psi\| \le m\big((\operatorname{supp}\phi)\cap(t\operatorname{supp}\psi)\big)\|\phi\|_\infty\|\psi\|_\infty\|b_t\|.$$

Besides, if $f \in C_c(\mathcal{B})$, we have $\phi\cdot f\cdot\psi \in C_c(\mathcal{B})$, with $\operatorname{supp}(\phi\cdot f\cdot\psi) \subseteq \operatorname{supp}(f)$ and $\|\phi\cdot f\cdot\psi\|_\infty \le m\big((\operatorname{supp}\phi)\cap\operatorname{supp}(f)(\operatorname{supp}\psi)\big)\|\phi\|_\infty\|\psi\|_\infty\|f\|_\infty$. By Lemma 3.16 the map $(s,t)\mapsto \phi(s)^*f(t)\psi(t^{-1}s)$ is continuous. Then [13, II-15.19] implies that $\phi\cdot f\cdot\psi \in C_c(\mathcal{B})$. It follows that the map $\mathcal{B}\to\mathcal{B}$ such that $b\mapsto\phi\cdot b\cdot\psi$ is a continuous map on the bundle $\mathcal{B}$, and that $\Phi_{\phi,\psi} : C_c(\mathcal{B})\to C_c(\mathcal{B})$ given by $f\mapsto\phi\cdot f\cdot\psi$ is continuous in the inductive limit topology. In fact, in [9, Lemma 3.2] is shown that

$$\|\Phi_{\phi,\psi}(b)\| \le \|\phi\|\,\|\psi\|\|b\|, \tag{6}$$

where $\|\phi\|$ and $\|\psi\|$ are the norms of ϕ and ψ as elements of $L^2(G, M(B_e))$. Hence we also have $\|\Phi_{\phi,\psi}(f)\|_\infty \le \|\phi\|\,\|\psi\|\|f\|_\infty$ and $\|\Phi_{\phi,\psi}(f)\|_1 \le \|\phi\|\,\|\psi\|\|f\|_1$ $\forall f \in C_c(\mathcal{B})$, and therefore $\Phi_{\phi,\psi}$ extends to a bounded map on $L^1(\mathcal{B})$.

Definition 4.9. (cf. [9, Definition 3.6]) Let $\mathcal{B}$ be a Fell bundle over the locally compact group G, and $M \ge 0$.

(1) We say that $\mathcal{B}$ has the pointwise M-approximation property if there exist nets $(\phi_i)_{i\in I}, (\psi_i)_{i\in I} \subseteq C_c\big(G, M(B_e)\big)$ such that:
(i) $\sup_{i\in I}\{\|\phi_i\|\,\|\psi_i\|\} \le M$ (as elements of $L^2(G)\bigotimes M(B_e)$), and
(ii) $\phi_i\cdot b\cdot\psi_i$ converges to b, $\forall b \in \mathcal{B}$.
If $I = \mathbb{N}$ we say that $\mathcal{B}$ has the countable M-pointwise approximation property. We say that $\mathcal{B}$ has the (countable) pointwise approximation property if $\mathcal{B}$ has the (countable) M-pointwise approximation property for some $M > 0$.

(2) $\mathcal{B}$ is said to have the M-approximation property if there are nets $(\phi_i), (\psi_i)$ as in (1) such that $\phi_i \cdot f \cdot \psi_i$ converges uniformly to f, $\forall f \in C_c(\mathcal{B})$. It is said to have the *approximation property* if it has the M-approximation property for some $M \geq 0$.
(3) We say that $\mathcal{B}$ has the L^1-approximation property if there are nets $(\phi_i), (\psi_i)$ as in (1) such that $\phi_i \cdot f \cdot \psi_i$ converges to f in $L^1(\mathcal{B})$, $\forall f \in C_c(\mathcal{B})$.

In all the cases above we say that $\mathcal{B}$ has the *positive* corresponding approximation property if we can choose $\phi_i = \psi_i$, $\forall i$.

The fact that we allow the approximating nets $(\phi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ to take values on the multiplier algebra $M(B_e)$ rather than in B_e is not an essential change in relation to the original definition of approximation property, but it allows some more flexibility (it is enough to multiply the approroximating nets by an approximate unit of B_e to obtain nets as in [9, Definition 3.6]).

It was proved in [9] that if G is an amenable group then the Fell bundle has the positive 1-approximation property.

For a Fell bundle $\mathcal{B}$ over a discrete group it is currently customary to say that $\mathcal{B}$ has the approximation property when it has the positive 1-approximation property. The corresponding net is called a Cesaro net for $\mathcal{B}$ by Exel in [8, Definition 20.4].

Since $L^2(G)$ is a Hilbert space, it is a nuclear C^*-tring, so there is a unique tensor product $L^2(G) \bigotimes M(B_e)$. On the other hand $L^2(G) \bigotimes M(B_e)$ is naturally identified with $L^2\big(G, M(B_e)\big)$, the completion of $C_c\big(G, M(B_e)\big)$ with respect to the inner product: $\langle f, g\rangle = \int_G f(t)^* g(t)dt$. Thus we also have that $L^2\big(G, M(B_e)\big) = L^2\big(G \times M(B_e)\big)$, where $G \times M(B_e)$ is the Fell bundle over G with the product topology and pointwise defined operations.

Proposition 4.10. *Let $\mathcal{B}$ be a Fell bundle over the locally compact group G. We have:*

(1) If G is discrete, the three next statements are equivalent to each other: $\mathcal{B}$ has the M-pointwise approximation property; $\mathcal{B}$ has the M-approximation property; $\mathcal{B}$ has the M- L^1-approximation property.
(2) If $\mathcal{B}$ has the approximation property then it also has the L^1-approximation property.
(3) If $\mathcal{B}$ has the countable pointwise approximation property, then $\mathcal{B}$ has the L^1-approximation property.

Proof. The first statement easily follows by observing that, if G is discrete and $f \in C_c(\mathcal{B})$, then $\operatorname{supp}(f)$ is finite. Suppose now that (ϕ_i), $(\psi_i) \subseteq C_c(G, M(B_e))$ are nets such that $\phi_i \cdot f \cdot \psi_i$ converges uniformly to f, $\forall f \in C_c(\mathcal{B})$, with $\sup_i \|\phi_i\| \, \|\psi_i\| \leq M < \infty$. Thus $\phi_i \cdot f \cdot \psi_i$ converges to f in the inductive limit topology, because $\operatorname{supp}(\phi \cdot f \cdot \psi) \subseteq \operatorname{supp}(f)$. Therefore the net $\phi_i \cdot f \cdot \psi_i$ converges to f in $L^1(\mathcal{B})$.

Suppose now that $\mathcal{B}$ has the countable pointwise approximation property: there exist sequences $(\phi_n), (\psi_n) \subseteq C_c(G, M(B_e))$ with $\sup_{n\in\mathbb{N}}\{\|\phi_n\|\,\|\psi_n\|\} = M < \infty$ and $\phi_n\cdot b\cdot\psi_n \to b$, $\forall b \in \mathcal{B}$. Let $\Phi_n := \Phi_{\phi_n,\psi_n} : C_c(\mathcal{B}) \to C_c(\mathcal{B})$ be the corresponding induced map. Then, since $\|\Phi_n(f) - f\|_\infty \leq (M+1)\|f\|_\infty$, we have that $\|(\Phi_n(f) - f)\|_1 = \int_{\mathrm{supp}(f)} \|\Phi_n(f)(t) - f(t)\|dt \to 0$ by the dominated convergence theorem. □

The following theorem is a direct generalization of the corresponding result [12, Theorem 4.6] for discrete groups, so we omit the proof here, although for the convenience of the reader we have provided its details in Appendix 5.

Theorem 4.11. *If $\mathcal{B}$ is a Fell bundle with the L^1-approximation property, then $\mathcal{B}$ is amenable. In particular if $\mathcal{B}$ has the approximation property, then $\mathcal{B}$ is amenable.*

Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles, and α a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$. Let $\overline{\alpha}$ be the C^*-norm on $M(A_e)\bigodot M(B_e)$ as a subalgebra of $M(A_e\bigotimes_\alpha B_e)$ (see [20, T.6.3], or alternatively use Corollary 2.13 and [15, Theorem 2.4] for the right Hilbert modules A_e and B_e over themselves), and $M(A_e)\bigotimes_{\overline{\alpha}} M(B_e)$ the corresponding tensor product. If $\phi \in C_c(G, M(A_e))$ and $\phi' \in C_c(H, M(B_e))$, we have a section $\phi\otimes\phi' \in C_c\big(G\times H, M(A_e)\bigotimes_{\overline{\alpha}} M(B_e)\big) \subseteq L^2(G\times H, M(A_e\bigotimes_\alpha B_e))$ such that $\phi\otimes\phi'(t,s) = \phi(t)\otimes\phi'(s)$, $\forall (t,s) \in G\times H$. Moreover: $\|\phi\otimes\phi'\| = \|\phi\|\,\|\phi'\|$ by Remark 2.9.

Proposition 4.12. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_s)_{s\in H}$ be Fell bundles, and α a C^*-norm on $\mathcal{A}\bigodot\mathcal{B}$. Suppose that $(\phi_i)_{i\in I}, (\psi_i)_{i\in I} \subseteq C_c(G, M(A_e))$ and $(\phi'_j)_{j\in J}, (\psi'_j)_{j\in J} \subseteq C_c(H, M(B_e))$. Consider $(\phi_i\otimes\phi'_j)_{(i,j)\in I\times J}$, $(\psi_i\otimes\psi'_j)_{(i,j)\in I\times J} \subseteq C_c\big(G\times H, M(A_e\bigotimes_\alpha B_e)\big)$. Then:*

1. *If $\phi_i\cdot a\cdot\psi_i \to a$, $\forall a \in \mathcal{A}$ and $\phi'_j\cdot b\cdot\psi'_j \to b$ and $\sup_{i\in I}\{\|\phi_i\|\,\|\psi_i\|\} \leq M < \infty$, $\sup_{j\in J}\{\|\phi'_j\|\,\|\psi'_j\|\} \leq N < \infty$, then $(\phi_i\otimes\phi'_j)\cdot x\cdot(\phi_i\otimes\phi'_j) \to x$, $\forall x \in \mathcal{A}\bigotimes_\alpha\mathcal{B}$, and $\sup_{(i,j)\in I\times J}\{\|\phi_i\otimes\phi'_j\|\,\|\psi_i\otimes\psi'_j\|\} \leq MN < \infty$.*
2. *If $\mathcal{A}$ and $\mathcal{B}$ have the (positive, countable) pointwise approximation property, then $\mathcal{A}\bigotimes_\alpha\mathcal{B}$ also has the (respectively: positive, countable) pointwise approximation property.*
3. *If $\mathcal{A}$ and $\mathcal{B}$ have the L^1-approximation property then $\mathcal{A}\bigotimes_\alpha\mathcal{B}$ also has the L^1-approximation property, and therefore it is amenable.*

Proof. Note first that 2) follows from 1). To prove 1), let $\Phi_i : \mathcal{A} \to \mathcal{A}$ and $\Phi'_j : \mathcal{B} \to \mathcal{B}$ be the maps induced by the pairs (ϕ_i, ψ_i) and (ϕ'_j, ψ'_j), $\forall (i,j) \in I\times J$. Let $a_{t_0} \in \mathcal{A}$ and $b_{s_0} \in \mathcal{B}$. Since Φ_i and Φ'_j converge pointwise to the identity maps respectively on $\mathcal{A}$ and $\mathcal{B}$, there exist $i_0 \in I$, $j_0 \in J$ such that $\forall i \geq i_0$, $j \geq j_0$ we have $\|\Phi_i(a_{t_0}) - a_{t_0}\| < \epsilon/N(1 + \|a_{t_0}\| + \|b_{s_0}\|)$ and $\|\Phi'_j(b_{s_0}) - b_{s_0}\| < \epsilon/(1 + \|a_{t_0}\| + \|b_{s_0}\|)$. Consider $\Phi_{i,j} : \mathcal{A}\bigotimes_\alpha\mathcal{B} \to \mathcal{A}\bigotimes_\alpha\mathcal{B}$ such that

$$\Phi_{i,j}(x_{(t,s)}) = \int_{G\times H} (\phi_i\otimes\phi'_j)(t',s')x_{(t,s)}(\psi_i\otimes\psi'_j)(t^{-1}t', s^{-1}s')d(t',s').$$

Then for $(i,j) \geq (i_0, j_0)$ we have:

$$\begin{aligned}\|\Phi_{i,j}(a_{t_0} \otimes b_{s_0}) - (a_{t_0} \otimes b_{s_0})\| &= \|\Phi_i(a_{t_0}) \otimes \Phi'_j(b_{s_0}) - (a_{t_0} \otimes b_{s_0})\| \\ &\leq \|\big(\Phi_i(a_{t_0}) - a_{t_0}\big) \otimes \Phi'_j(b_{s_0})\| + \|a_{t_0} \otimes \big(\Phi'_j(b_{s_0}) - b_{s_0}\big)\| \\ &\leq \|\Phi_i(a_{t_0}) - a_{t_0}\| \, \|\Phi'_j(b_{s_0})\| + \|a_{t_0}\| \, \|\Phi'_j(b_{s_0}) - b_{s_0}\| \\ &\leq \frac{\epsilon}{N(1 + \|a_{t_0}\| + \|b_{s_0}\|)} N\|b_{s_0}\| + \frac{\epsilon}{(1 + \|a_{t_0}\| + \|b_{s_0}\|)}\|a_{t_0}\| < \epsilon.\end{aligned}$$

By (6) we have $\|\Phi_{i,j}(x)\| \leq MN\|x\|$, $\forall x \in \mathcal{A} \bigotimes_\alpha \mathcal{B}$, and consequently $\Phi_{i,j}(x) \to x$, $\forall x \in \mathcal{A} \bigotimes_\alpha \mathcal{B}$. This proves 1) and therefore also 2).

To see that 3) holds, suppose now that for the maps Φ_i and Φ'_j above and every $f \in C_c(\mathcal{A})$, $g \in C_c(\mathcal{B})$ we have that $\|\Phi_i(f) - f\|_1 \to 0$ and $\|\Phi'_j(g) - g\|_1 \to 0$. Note that if $f \in C_c(\mathcal{A})$ and $g \in C_c(\mathcal{B})$, then $\Phi_{i,j}(f \oslash g) = \Phi_i(f) \oslash \Phi'_j(g)$, and therefore

$$\begin{aligned}\|\Phi_{i,j}(f \oslash g) - f \oslash g\|_1 &= \|\Phi_i(f) \oslash \Phi'_j(g) - f \oslash g\|_1 \\ &\leq \|\Phi_i(f) \oslash \big(\Phi'_j(g) - g\big)\|_1 + \|\big(\Phi_i(f) - f\big) \oslash g\|_1 \\ &\leq M \, \|f\|_1 \, \|\Phi'_j(g) - g\|_1 + \|\Phi_i(f) - f\|_1 \, \|g\|_1 \\ &\to 0 \qquad \text{when } i,j \to \infty\end{aligned}$$

It follows that $\Phi_{i,j}(l) \to l$ in $L^1(\mathcal{A} \bigotimes_\alpha \mathcal{B})$, $\forall l \in L = \{\sum_k f_k \oslash g_k\}$. Since L is dense in $C_c(\mathcal{A} \bigotimes_\alpha \mathcal{B})$ in the inductive limit topology, it is also dense in $L^1(\mathcal{A} \bigotimes_\alpha \mathcal{B})$. Since $\|\Phi_{i,j}\| \leq MN$, $\forall i \in I, j \in J$, then $\|\Phi_{i,j}(h) - h\|_1 \to 0$, $\forall h \in L^1(\mathcal{A} \bigotimes_\alpha \mathcal{B})$. □

The last statement of the next result was first proved by the author in the case of discrete groups in the previous preprint version of the present paper mentioned at the end of the introduction, and was later proved for arbitrary locally compact groups in [2] and in [9].

Corollary 4.13. *If $\mathcal{A} = (A_t)_{t \in G}$ and $\mathcal{B} = (B_s)_{s \in H}$ are Fell bundles with the L^1-approximation property, then $A_e \bigodot B_e$ admits exactly one C^*-norm if and only if $C^*(\mathcal{A}) \bigodot C^*(\mathcal{B})$ admits exactly one C^*-norm. In particular, if $\mathcal{A}$ is a Fell bundle with the L^1-approximation property (this is automatically true if G is amenable) and nuclear unit fiber A_e, then $C^*(\mathcal{A})$ is also nuclear.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are Fell bundles with the L^1-approximation property, then $\mathcal{A} \bigotimes_{\min} \mathcal{B}$ also has the L^1-approximation property by Proposition 4.12, so the diagram $D_{\min}$ becomes:

$$\begin{array}{ccc} C^*(\mathcal{A} \bigotimes_{\max} \mathcal{B}) & \xrightarrow{\sigma^{\max}_{\min}} & C^*(\mathcal{A} \bigotimes_{\min} \mathcal{B}) \\ {\scriptstyle j \cong} \uparrow & & \uparrow {\scriptstyle \cong j_r} \\ C^*(\mathcal{A}) \bigotimes_{\max} C^*(\mathcal{B}) & \xrightarrow[\tilde{\sigma}^{\max}_{\min}]{} & C^*(\mathcal{A}) \bigotimes_{\min} C^*(\mathcal{B}) \end{array}$$

If $A_e \bigodot B_e$ admits just one C^*-norm, then $\mathcal{A} \bigotimes_{\max} \mathcal{B} = \mathcal{A} \bigotimes_{\min} \mathcal{B}$ which implies $\sigma^{\max}_{\min} = id$, and therefore $\tilde{\sigma}^{\max}_{\min} = id$, from where it follows that

$C^*(\mathcal{A})\bigotimes_{\max} C^*(\mathcal{B}) = C^*(\mathcal{A})\bigotimes_{\min} C^*(\mathcal{B})$. Conversely, suppose now that $C^*(\mathcal{A})\bigodot C^*(\mathcal{B})$ admits just one C^*-norm. Then $\tilde{\sigma}_{\min}^{\max} = id$, and therefore $\sigma_{\min}^{\max} = id$. Thus $\mathcal{A}\bigotimes_{\max}\mathcal{B} = \mathcal{A}\bigotimes_{\min}\mathcal{B}$, so $A_e\bigotimes_{\max} B_e = A_e\bigotimes_{\min} B_e$.

As for the last assertion, notice that every C^*-algebra B may be considered as a Fell bundle over the trivial group, and it is clear that this Fell bundle has the (positive, countable) L^1-approximation property: it is enough to take $\phi : G \to M(B)$ such that $\phi(t) = 1$, $\forall t \in G$. Consequently, by the first part of this Corollary we have $C^*(\mathcal{A})\bigotimes_{\max} B = C^*(\mathcal{A})\bigotimes_{\min} B$, that is, $C^*(\mathcal{A})$ is a nuclear C^*-algebra. □

Corollary 4.14. *If $\mathcal{A} = (A_t)_{t\in G}$ is a Fell bundle with the L^1-approximation property and nuclear unit fiber, and if $\mathcal{B} = (B_s)_{s\in H}$ is an amenable Fell bundle, then $\mathcal{A}\bigotimes\mathcal{B}$ also is amenable.*

Proof. By Corollary 4.13, our assumptions on $\mathcal{A}$ imply that $C^*(\mathcal{A})$ is nuclear. Therefore we have $C^*(\mathcal{A})\bigotimes_{\max} C^*(\mathcal{B}) = C^*(\mathcal{A})\bigotimes_{\overline{\max}} C^*(\mathcal{B})$, and then the result follows from Corollary 4.8. □

Corollary 4.15. *Any twisted partial crossed product of a nuclear C^*-algebra by an amenable group is nuclear. In particular, the partial C^*-algebra $C_p^*(G)$ of an amenable discrete group G is nuclear.*

Definition 4.16. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Fell bundles over the locally compact group G. We say that a sequence $0\longrightarrow\mathcal{A}\xrightarrow{\phi}\mathcal{B}\xrightarrow{\psi}\mathcal{C}\longrightarrow 0$ is exact if ϕ is injective, ψ is surjective, and $\ker\psi = \mathrm{Im}\phi$, where $\ker\psi := \{b \in \mathcal{B} : \psi(b)$ is a zero element$\}$.

Proposition 4.17. *The functors $\mathcal{A} \mapsto L^1(\mathcal{A})$ and $\mathcal{A} \mapsto C^*(\mathcal{A})$ are exact. That is, if $0\longrightarrow\mathcal{A}\xrightarrow{\phi}\mathcal{B}\xrightarrow{\psi}\mathcal{C}\longrightarrow 0$ is an exact sequence of Fell bundles over the locally compact group G, then:*

(1) $0\longrightarrow L^1(\mathcal{A})\xrightarrow{\phi^1} L^1(\mathcal{B})\xrightarrow{\psi^1} L^1(\mathcal{C})\longrightarrow 0$ *is exact, and*

(2) $0\longrightarrow C^*(\mathcal{A})\xrightarrow{C^*(\phi)} C^*(\mathcal{B})\xrightarrow{C^*(\psi)} C^*(\mathcal{C})\longrightarrow 0$ *also is exact.*

Proof. Since every non-degenerate representation of $L^1(\mathcal{A})$ has a unique extension to a representation of $L^1(\mathcal{B})$, we have that $C^*(\mathcal{A})$ is the closure of $L^1(\mathcal{A})$ in $C^*(\mathcal{B})$, so it is enough to prove 1), because then 2) follows from [21, 2.29] and the fact that $L^1(\mathcal{F})$ has an approximate unit, for every Fell bundle $\mathcal{F}$.

Since $\|\phi(a)\| = \|a\|$, $\forall a \in \mathcal{A}$, it follows that ϕ^1 is an isometry.

Let see that $\ker\psi^1 = \mathrm{Im}\phi^1$. The inclusion $\mathrm{Im}\phi^1 \subseteq \ker\psi^1$ is clear. In order to see the converse inclusion let $g \in \ker\psi^1$. Then $\|\psi^1(g)\|_1 = 0$, so $\psi\big(g(t)\big) = 0$ almost everywhere in G. Without loss of generality we may suppose that $\psi\big(g(t)\big) = 0$, $\forall t$ in G. Thus, $g(t) \in \ker\psi = \mathrm{Im}\phi$, $\forall t \in G$, and therefore there exists a unique $f(t) \in A_t$ such that $\phi\big(f(t)\big) = g(t)$, $\forall t \in G$. Since ϕ is a continuous and isometric isomorphism between $\mathcal{A}$ and $\phi(\mathcal{A})$ (by [13, II-13.17]), and we have $f = (\phi^1)^{-1}(g)$, then $f \in L^1(\mathcal{A})$. Thus $g \in \mathrm{Im}\phi^1$.

Finally, we show that ψ^1 is surjective. We will suppose, as we can, that $\mathcal{A} \subseteq \mathcal{B}$. Note that $L^1(\mathcal{A})$ is a closed *-ideal of $L^1(\mathcal{B})$. Thus there exists an isomorphism of *-algebras $\frac{L^1(\mathcal{B})}{L^1(\mathcal{A})} \xrightarrow{\bar{\psi}} \psi^1\big(L^1(\mathcal{B})\big) \subseteq L^1(\mathcal{C})$. The image of $\bar{\psi}$ contains $\psi^1\big(C_c(\mathcal{B})\big)$, which is dense in $C_c(\mathcal{C})$ in the inductive limit topology: since ψ is surjective then we may apply [3, 5.1] to $\psi^1\big(C_c(\mathcal{B})\big)$, thus concluding that $\mathrm{Im}\bar{\psi}$ is dense in $L^1(\mathcal{C})$. Then it is sufficient to prove that $\bar{\psi}$ is an isometry, where $\frac{L^1(\mathcal{B})}{L^1(\mathcal{A})}$ is endowed with the quotient norm. Let $f \in C_c(\mathcal{B})$ and $\bar{f}$ its projection into the quotient space. Then $\|\bar{\psi}\| = \|\psi^1\|$, and therefore $\|\bar{\psi}(\bar{f})\| \leq \|\psi^1\|\,\|\bar{f}\| \leq \|\bar{f}\|$. To prove the converse inequality consider an arbitrary $\epsilon > 0$, and let M be the measure of a compact neighborhood V of $\mathrm{supp}(f)$. For each $s \in V$, there exists $g_s \in C_c(\mathcal{A})$ such that $\|f(s) - g_s(s)\| < \|\overline{f(s)}\| + \epsilon/M$, and we may suppose that $\mathrm{supp}(g_s) \subseteq V$. Since f, g_s, and $t \mapsto \|\overline{f(t)}\| = \|\psi(f(t))\|$ are continuous, for every $s \in \mathrm{supp}(f)$ must exist an open neighborhood V_s of s, which we may suppose to be contained in V, such that $\|f(t) - g_s(t)\| < \|\overline{f(t)}\| + \epsilon/M$, $\forall t \in V_s$. Now, $\{V_s : \ s \in \mathrm{sup}(f)\}$ is an open covering of the compact set $\mathrm{supp}(f)$. Let $V_{s_1}, \ldots, V_{s_n}$ be a finite subcovering. Let $G_\star$ be the one point compactification of G, and define $s_{n+1} := \star$, $V_{s_{n+1}} = G_\star \setminus \mathrm{supp}(f)$ and $g_{s_{n+1}} = 0$, where $\star$ represents the adjoined point at infinity. Then $\{V_{s_i}\}_{i=1}^{n+1}$ is an open covering of $G_\star$. Let $(\phi_i)_{i=1}^{n+1}$ be a partition of the unit of $G_\star$, subordinated to $\{V_{s_i}\}_{i=1}^{n+1}$, and define $g(t) = \sum_{i=1}^{n+1} \phi_i(t) g_{s_i}(t)$, $\forall t \in G$. Then $g \in C_c(\mathcal{A})$, $\mathrm{supp}(g) \subseteq V$, and

$$
\begin{aligned}
\|\bar{f}\| &\leq \int_G \|f(t) - g(t)\| dt = \int_V \|\sum_{i=1}^{n+1} \big(\phi_i(t) f(t) - \phi_i(t) g_{s_i}(t)\big)\| dt \\
&\leq \sum_{i=1}^{n+1} \int_{V_{s_i}} \phi_i(t) \|f(t) - g_{s_i}(t)\| dt \leq \int_V \sum_{i=1}^{n+1} \phi_i(t) \big(\|\overline{f(t)}\| + \epsilon/M\big) dt \\
&\leq \int_G \|\psi\big(f(t)\big)\| dt + \epsilon = \|\psi^1(f)\|_1 + \epsilon.
\end{aligned}
$$

Since ϵ was arbitrary, we conclude that $\|\bar{f}\| \leq \|\psi^1(f)\|_1$, and therefore $\|\bar{f}\| = \|\bar{\psi}(f)\|_1$. Moreover $\bar{\psi}$ has dense image in $L^1(\mathcal{C})$, so $\mathrm{Im}(\bar{\psi}) = L^1(\mathcal{C})$ and, since $\mathrm{Im}(\psi^1) = \mathrm{Im}(\bar{\psi})$, we conclude that ψ^1 is surjective. □

For our next result, recall from [6, 5.3] that the definition of exact C*-algebra extends to C*-trings, and that a C*-tring E is exact if and only if E^r is exact. In particular, if $\mathcal{A} = (A_t)_{t \in G}$ is a Fell bundle, its unit fiber is an exact C*-algebra if and only if each fiber A_t is an exact C*-tring.

Theorem 4.18. *Let $\mathcal{A} = (A_t)_{t \in G}$ be a Fell bundle with exact unit fiber and the L^1-approximation property. Then $C^*(\mathcal{A})$ is an exact C^*-algebra.*

Proof. Let B be a C^*-algebra and $I \lhd B$. Since A_t is exact $\forall t \in G$, the sequence of Fell bundles $0 \longrightarrow \mathcal{A} \bigotimes I \longrightarrow \mathcal{A} \bigotimes B \longrightarrow \mathcal{A} \bigotimes (B/I) \longrightarrow 0$ is exact, and each one of the bundles in the sequence has the L^1-approximation property by Proposition 4.12 (here $\bigotimes = \bigotimes_{\min}$). Since C^* is an exact functor from the category of Banach *-algebras with approximate unit to the category of C^*-algebras, and in this case we have $C^* = C^*_r$, the sequence of C^*-algebras

$$0 \longrightarrow C^*_r\big(\mathcal{A} \bigotimes I\big) \longrightarrow C^*_r\big(\mathcal{A} \bigotimes B\big) \longrightarrow C^*_r\big(\mathcal{A} \bigotimes (B/I)\big) \longrightarrow 0$$

also is exact.

Now Proposition 4.6 provides a natural isomorphism between $C^*_r\big(\mathcal{A} \bigotimes C\big)$ and $C^*_r(\mathcal{A}) \bigotimes C$, for every C^*-algebra C. Thus we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*_r(\mathcal{A} \bigotimes I) & \longrightarrow & C^*_r(\mathcal{A} \bigotimes B) & \longrightarrow & C^*_r(\mathcal{A} \bigotimes (B/I)) & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & C^*_r(\mathcal{A}) \bigotimes I & \longrightarrow & C^*_r(\mathcal{A}) \bigotimes B & \longrightarrow & C^*_r(\mathcal{A}) \bigotimes (B/I) & \longrightarrow & 0 \end{array}$$

Since the first row is exact and the diagram is commutative, then the second row also is exact. Hence it follows that $C^*(\mathcal{A})$, which is equal to $C^*_r(\mathcal{A})$, is an exact C^*-algebra. □

Since any Fell bundle over an amenable locally compact group has the approximation property, from Theorem 4.18 we obtain the following generalization of [14, Proposition 7.1] (see also [7, Proposition 7.5]):

Corollary 4.19. *Any twisted partial crossed product of an exact C^*-algebra by an amenable group G is also exact.*

5. Appendix

Proof of Theorem 4.11. Let $\pi : \mathcal{B} \to B(H)$ be a non-degenerate representation such that $\pi\big|_{B_e}$ is faithful. We also call π the integrated representation of π, and to its unique extension to $M(\mathcal{B})$ as well. Since $\pi\big|_{B_e}$ is faithful, then so is $\pi\big|_{M(B_e)}$. Given $\phi \in C_c\big(G, M(B_e)\big) \subseteq L^2(G) \bigotimes M(B_e)$, consider the operator $V_\phi : H \to L^2(G) \bigotimes H$ such that $V_\phi h\big|_t = \pi\big(\phi(t)\big)h$. We have:

$$\begin{aligned} \|V_\phi\|^2 &= \sup_{\|h\|=1} \int_G \langle \pi\big(\phi(t)\big)h, \pi\big(\phi(t)\big)h\rangle dt = \sup_{\|h\|=1} \int_G \langle \pi\big(\phi(t)^*\phi(t)\big)h, h\rangle dt \\ &= \sup_{\|h\|=1} \langle \int_G \pi\big(\phi(t)^*\phi(t)\big) dt\, h, h\rangle = \sup_{\|h\|=1} \langle \pi(\langle \phi, \phi\rangle)h, h\rangle \\ &= \|\pi(\langle \phi, \phi\rangle)\| = \|\langle \phi, \phi\rangle\|. \end{aligned}$$

where we used that $\pi(\langle \phi, \phi\rangle)$ is positive and $\pi\big|_{M(B_e)}$ is faithful to obtain the last two equalities. We compute V^*_ϕ: if $h \in H$, $\xi \in L^2(G)$, then

$$\langle V_\phi h, \xi\rangle = \int_G \langle \pi\big(\phi(t)\big)h, \xi(t)\rangle dt = \int_G \langle h, \pi\big(\phi(t)^*\big)\xi(t)\rangle dt = \langle h, \int_G \pi\big(\phi(t)^*\big)\xi(t) dt\rangle,$$

so $V_\phi^*(\xi)=\int_G \pi\big(\phi(t)^*\big)\xi(t)dt$. Note that if $\psi\in C_c\big(G,M(B_e)\big)$, then

$$V_\phi^*V_\psi h=\int_G \pi\big(\phi(t)^*\big)V_\psi h\big|_t dt=\int_G \pi\big(\phi(t)^*\big)\pi\big(\psi(t)\big)hdt=\pi(\langle\phi,\psi\rangle)h.$$

Moreover, if ϕ_1, ϕ_2, $\phi_3\in C_c\big(G,M(B_e)\big)$, $h\in H$, we have:

$$V_{\phi_1\langle\phi_2,\phi_3\rangle}h\big|_t=\pi(\phi_1(t)\langle\phi_2,\phi_3\rangle)h=\pi(\phi_1(t))\pi(\langle\phi_2,\phi_3\rangle)h=V_{\phi_1}V_{\phi_2}^*V_{\phi_3}h\big|_t$$

Therefore, since $\phi\mapsto V_\phi$ is an isometry on the dense subspace $C_c\big(G,M(B_e)\big)$ of the C*-tring $L^2(G,M(B_e))$, it extends to a homomorphism of positive C*-trings $\pi_2:L^2(G)\bigotimes M(B_e)\to\pi_2\big(L^2(G)\bigotimes M(B_e)\big)\subseteq B\big(H,L^2(G)\bigotimes H\big)$, which is consequently an isomorphism of C*-trings.

Consider now the representation $\pi_\lambda:\mathcal{B}\to B\big(L^2(G)\bigotimes H\big)$, such that $\pi_\lambda(b_t)=\lambda_t\otimes\pi(b_t)$ and its integrated representation, which we continue to call $\pi_\lambda:C^*(\mathcal{B})\to B(L^2(G)\bigotimes H)$ (here λ is the left regular representation of G). Define, for ϕ, $\psi\in L^2(G)\bigotimes M(B_e)$, the completely bounded map $\Psi:\pi_\lambda\big(C^*(\mathcal{B})\big)\to B(H)$, given by $\Psi(x)=V_\phi^*xV_\psi$, $\forall x\in\pi_\lambda\big(C^*(\mathcal{B})\big)$. We have $\|\Psi(x)\|\le\|\phi\|\,\|\psi\|\,\|x\|$, so $\|\Psi\|\le\|\phi\|\,\|\psi\|$. Consider also, for $f\in C_c(\mathcal{B})$, the function $\Phi(f):G\to\mathcal{B}$ such that $\Phi(f)\big|_t=\int_G\phi(s)^*f(t)\psi(t^{-1}s)ds$. Let $F(t,s)=\phi(s)^*f(t)\psi(t^{-1}s)$. By 3.16, $F:G\times G\to\mathcal{B}$ is continuous and of compact support, and such that $F(t,s)\in B_t$, $\forall t\in G$. Then by [13, II-15.19], the function $t\mapsto\int_G F(t,s)ds$ is a compactly supported continuous section of $\mathcal{B}$. In other words, $\Phi(f)\in C_c(\mathcal{B})$. In fact, it is clear that supp $\big(\Phi(f)\big)\subseteq$ supp(f). Besides, we have $\pi\big(\Phi(f)\big)=\Psi\big(\pi_\lambda(f)\big)$, for if $h,k\in H$:

$$\begin{aligned}\langle\pi\big(\Phi(f)\big)h,k\rangle&=\langle\int_G\pi\big(\Phi(f)\big|_t\big)h,k\rangle=\int_G\langle\pi\big[\int_G\phi(s)^*f(t)\psi(t^{-1}s)ds\big]h,k\rangle\\&=\int_G\int_G\langle\pi\big(\phi(s)^*f(t)\psi(t^{-1}s)\big)h,k\rangle dsdt\\&=\int_G\int_G\langle\pi\big(\phi(s)^*\big)\pi\big(f(t)\big)\pi\big(\psi(t^{-1}s)\big)h,k\rangle dsdt\\&=\int_G\langle\pi\big(\phi(s)^*\big)\int_G\pi\big(f(t)\big)V_\psi h\big|_{t^{-1}s}dt,k\rangle ds\\&=\int_G\langle\pi\big(\phi(s)^*\big)\big[\int_G(\lambda_t\otimes\pi)\big(f(t)\big)(V_\psi h)dt\big]\big|_s,k\rangle ds\\&=\int_G\langle\pi\big(\phi(s)^*\big)\pi_\lambda(f)(V_\psi h)(s)ds,k\rangle=\langle\int_G\pi\big(\phi(s)^*\big)\big(\pi_\lambda(f)V_\psi h(s)\big)ds,k\rangle\\&=\langle(V_\phi^*\pi_\lambda(f)V_\psi)h,k\rangle=\langle\Psi\big(\pi_\lambda(f)\big)h,k\rangle,\end{aligned}$$

whence $\pi\big(\Phi(f)\big)=\Psi\big(\pi_\lambda(f)\big)$. In particular we have $\Psi\big(\pi_\lambda\big(C_c(\mathcal{B})\big)\big)\subseteq\pi\big(C_c(\mathcal{B})\big)\subseteq\pi\big(C^*(\mathcal{B})\big)$, which is closed, and therefore $\Psi\big(\pi_\lambda\big(C^*(\mathcal{B})\big)\big)\subseteq\pi\big(C^*(\mathcal{B})\big)$. Then we have $\Psi:\pi_\lambda\big(C^*(\mathcal{B})\big)\to\pi\big(C^*(\mathcal{B})\big)$. Suppose now that (ϕ_i), (ψ_i) are approximating nets as in (3) of Definition 4.9, with $\|\phi_i\|\,\|\psi_i\|\le M$, $\forall i$, so we have $\Phi_i:C_c(\mathcal{B})\to C_c(\mathcal{B})$ and $\Phi_i(f)$ converges to f in $L^1(\mathcal{B})$, for all $f\in C_c(\mathcal{B})$. Let $\Psi_i:\pi_\lambda\big(C^*(\mathcal{B})\big)\to\pi\big(C^*(\mathcal{B})\big)$ be the correspondending completely bounded maps, that is: $\Psi_i(x) = V_{\phi_i}^* x V_{\psi_i}$, $\forall x \in \pi_\lambda\big(C^*(\mathcal{B})\big)$. Since $\Phi_i(f) \to f$ in $L^1(\mathcal{B})$, then $\Phi_i(f) \to f$ also in $C^*(\mathcal{B})$, thus $\pi\big(\Phi_i(f)\big) \to \pi(f)$ in $\pi\big(C^*(\mathcal{B})\big)$. Consequently, $\|\pi\big(\Phi_i(f)\big)\| \to \|\pi(f)\|$. On the other hand, $\pi\big(\Phi_i(f)\big) = \Psi_i\big(\pi_\lambda(f)\big)$, whence

$$\begin{aligned}\|\pi(f)\| &= \lim_i \|\pi\big(\Phi_i(f)\big)\| = \lim_i \|\Psi_i\big(\pi_\lambda(f)\big)\| \\ &\leq \limsup_i \|\Psi_i\| \, \|\pi_\lambda(f)\| \leq M\|\pi_\lambda(f)\|.\end{aligned}$$

Since $C_c(\mathcal{B})$ is dense in $L^1(\mathcal{B})$, it follows that $\|\pi(y)\| \leq M\|\pi_\lambda(y)\|$, $\forall y \in C^*(\mathcal{B})$. In particular, if π is a faithful representation of $C^*(\mathcal{B})$, we conclude that π_λ is also faithful. On the other hand, it is proved in [10, Proposition 2.3] that the representation $\Lambda \otimes id : \mathcal{B} \to B(L^2(\mathcal{B}) \otimes_{B_e} H)$, given by $(\Lambda \otimes id)_b(\xi \otimes h) := \Lambda_b \xi \otimes h$, is equivalent to a subrepresentation of π_λ, so it is faithful as well. This implies that Λ is faithful, which is to say that $C^*(\mathcal{B}) = C_r^*(\mathcal{B})$. □

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