

Constructing cospectral graphs

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Abstract. Some new constructions for families of cospectral graphs are derived, and some old ones are considerably generalized. One of our new constructions is sufficiently powerful to produce an estimated 72% of the 51039 graphs on 9 vertices which do not have unique spectrum. In fact, the number of graphs of order n without unique spectrum is believed to be at least $\alpha n^3 g_{n-1}$ for some $\alpha > 0$, where g_n is the number of graphs of order n and $n \geq 7$.

1. Introduction

1.1. We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G is the $n \times n$ matrix with (i, j) th entry equal to 1 if vertices i and j are adjacent and equal to 0 otherwise. The adjacency matrix of G will also be denoted by the symbol G . The *characteristic polynomial* of G is the polynomial $\phi(G) = \phi(G, x) = \det(xI_n - G)$, where I_n is the $n \times n$ identity matrix.

Two graphs G and H are *cospectral* if $\phi(G) = \phi(H)$. We say that G is *characterized by its spectrum* if every graph cospectral to G is isomorphic to G . It was proved by Schwenk [14] that the proportion of trees on n vertices which are characterized by their spectra converges to zero as n increases. The corresponding asymptotic question for graphs in general remains one of the outstanding unsolved problems in the theory of graph spectra.

Schwenk's proof depends on a construction which provides pairs of cospectral trees. Thus, if we wish to settle the question for graphs in general, it is natural to

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look for constructions for pairs of cospectral graphs. In this paper we present some new and powerful constructions for pairs of cospectral graphs and considerably generalize some old ones. One of our new methods is sufficiently powerful to generate an estimated 72% of the 51039 graphs on 9 vertices which are not characterized by their spectra.

1.2. TERMINOLOGY. We will use J_m to denote the $m \times n$ matrix with each entry one and I_n to denote the identity matrix of order n . In each case the subscripts will be deleted if the order is clear from the context. The column vector $J_{m,1}$ will also be denoted by j_m .

The concept of *switching* was introduced by Seidel [13]. Let S be a subset of $V(G)$. Then the graph H formed from G by *switching about* S has

$$V(H) = V(G),$$

and

$$E(H) = \{xy \in E(G) \mid x, y \in S \text{ or } x, y \notin S\} \cup \{xy \notin E(G) \mid x \in S \text{ and } y \notin S\}.$$

We say that G and H are *switching equivalent*.

2.1. CONSTRUCTION. Let G be a graph and let $\pi = (C_1, C_2, \dots, C_k, D)$ be a partition of $V(G)$. Suppose that, whenever $1 \leq i, j \leq k$ and $v \in D$, we have

- (a) any two vertices in C_i have the same number of neighbours in C_j , and
- (b) v has either 0, $n_i/2$ or n_i neighbours in C_i , where $n_i = |C_i|$.

The graph $G^{(\pi)}$ formed by *local switching in* G with respect to π is obtained from G as follows. For each $v \in D$ and $1 \leq i \leq k$ such that v has $n_i/2$ neighbours in C_i , delete these $n_i/2$ edges and join v instead to the other $n_i/2$ vertices in C_i .

For our purposes the most important property of our construction is provided by the next theorem.

2.2. THEOREM. *Let G be a graph and let π be a partition of $V(G)$ which satisfies properties (a) and (b) above. Then $G^{(\pi)}$ and G are cospectral, with cospectral complements.*

Proof. The most direct way of showing that two graphs are cospectral is to show that their adjacency matrices are similar. We now proceed to do this.

For any positive integer m , define $Q_m = 2J_m/m - I_m$. The following claims can be verified easily.

- (a) $Q_m^2 = I_m$.
- (b) If X is an $m \times n$ matrix with constant row sums and constant column sums, then $Q_m X Q_n = X$.
- (c) If x is a vector with $2m$ entries, m of which are zero and m of which are one, then $Q_{2m} x = j_m - x$.

If the vertices of G are labelled in an order consistent with π , the adjacency matrix of G has the form

$$G = \begin{bmatrix} C_1 & C_{12} & \cdots & C_{1k} & D_1 \\ C_{12}^T & C_2 & \cdots & C_{2k} & D_2 \\ \vdots & \vdots & & \vdots & \vdots \\ C_{1k}^T & C_{2k}^T & \cdots & C_k & D_k \\ D_1^T & D_2^T & \cdots & D_k^T & D \end{bmatrix}.$$

The required properties of π ensure that each C_i and each C_{ij} has constant row sums and constant column sums, and that each column of each D_i has either 0, $n_i/2$ or n_i ones. Therefore $Q G Q$ is the adjacency matrix $G^{(\pi)}$, where Q is the block-diagonal matrix $\text{diag}(Q_{n_1}, Q_{n_2}, \dots, Q_{n_k}, I_{|D|})$. Since $Q^2 = I$, this proves G and $G^{(\pi)}$ to be cospectral. G and $G^{(\pi)}$ have cospectral complements by the same argument, since $(\bar{G})^{(\pi)}$ is the complement of $G^{(\pi)}$. □

2.3. EXAMPLES. We consider some cases of local switching that are of particular interest.

(a) Form G by taking a regular graph H with an even number of vertices and adjoining a new vertex v adjacent to exactly half the vertices of H . Then $G^{(\pi)}$ for $\pi = (V(H), \{v\})$ is formed by joining v instead to the other vertices of H . An example with $H = C_8$ is shown in Figure 1.

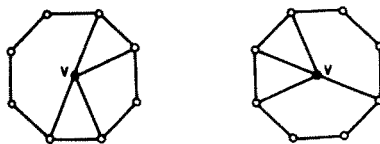


Figure 1

If H has $2m$ vertices and a trivial automorphism group, it is easy to show that all of the $\binom{2m}{m}$ possibilities for G are nonisomorphic. Hence, for example, if we choose H to be a cubic graph on 12 vertices with no nontrivial automorphisms we can construct $\binom{12}{6}/2 = 462$ pairs of cospectral nonisomorphic graphs on 13 vertices.

(b) Let G be regular with degree k and let S be a subset of $V(G)$ such that the graph H obtained from G by switching about S is also regular with degree k . Then

H and G are cospectral because $H = G^{(\pi)}$, where $\pi = (V(G) \setminus S, S)$. This construction is well known.

In examples (a) and (b), $G^{(\pi)}$ and G are switching equivalent. However this is not always the case, as in the next example.

(c) This example is most easily explained by reference to Figure 2, where $\pi = (\{a, b, c, d\}, H)$.

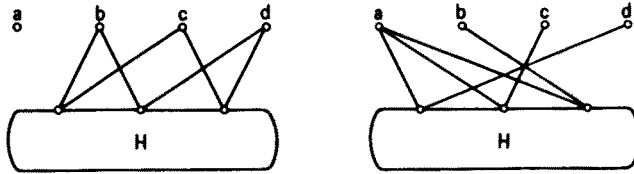


Figure 2

A similar example is shown in Figure 3, where again $\pi = (\{a, b, c, d\}, H)$. If H is chosen to have vertices of degree three, except that vertices 1 through 6 have degree one, then G and $G^{(\pi)}$ are both cubic. This construction provides 2 of the 3 pairs of cospectral cubic graphs on 14 vertices. There are no cospectral cubic graphs on less than 14 vertices [2].

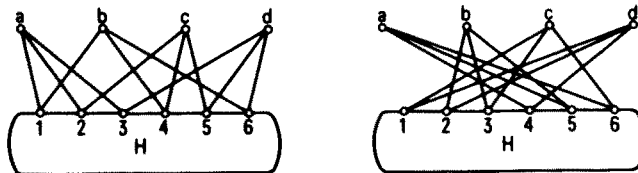


Figure 3

2.4. STATISTICS. We now demonstrate the efficiency of local switching as a means of producing pairs of cospectral graphs. We will concentrate on the subcase which appears to produce the most examples, namely that when $\pi = (C_1, D)$ and $|C_1| = 4$.

Define l_n to be the proportion of (unlabelled) graphs G on n vertices such that there exists $\pi = (C_1, D)$ with $|C_1| = 4$ satisfying conditions (a) and (b) of Section 2.1 and $G^{(\pi)} \not\cong G$. Also define c_n to be the proportion of all graphs on n vertices which are not characterized by their spectra.

It is easy to show that $l_n = 0$ for $n \leq 6$. For each value of n in the range $7 \leq n \leq 16$ a large number N_n of random labelled graphs with n vertices was generated. For each graph the order of the automorphism group was found, and then, for each π of the required type, $G^{(\pi)}$ was tested for isomorphism with G . An estimate of l_n was then obtained by weighting each graph according to the order of

its automorphism group, to get an unbiased estimator for *unlabelled* graphs. The results are shown in Table 1. The values of c_n are taken from [10] for $n \leq 9$ and are unknown for $n > 9$. The estimates of l_n are given with approximate 95% confidence limits. The latter were computed under the untested assumption that the distribution of the estimates of l_n obtained from samples of size $N_n/10$ is normal, and should only be taken as a rough guide to the accuracy of l_n . There are convincing reasons to suspect that $l_n \sim \binom{n}{4}2^{1-n}$, which would imply the claim made in the last sentence of the abstract.

Table 1

n	N_n	c_n	l_n	l_n/c_n
5	—	0.059	0	0.00
6	—	0.064	0	0.00
7	50000	0.105	0.037 ± 0.003	0.35
8	50000	0.139	0.084 ± 0.004	0.60
9	50000	0.186	0.135 ± 0.003	0.73
10	30000		0.164 ± 0.008	
11	20000		0.165 ± 0.008	
12	10000		0.145 ± 0.011	
13	10000		0.127 ± 0.010	
14	6000		0.095 ± 0.010	
15	5000		0.074 ± 0.008	
16	3000		0.042 ± 0.009	

It is seen that for $n \leq 9$ the ratio l_n/c_n is steadily increasing and probably exceeds 70% for $n = 9$. It is interesting to note that $\liminf_{n \rightarrow \infty} l_n/c_n > 0$ would imply $c_n \rightarrow 0$, since $l_n \rightarrow 0$. However, we feel that our data provide only a small amount of evidence that $c_n \rightarrow 0$, and that the behaviour of c_n for small n may not be typical.

2.5. In the special case where $\pi = (C_1, D)$ and there are no edges within C_1 , the construction above can be considerably generalized. We will call two $m \times n$ matrices A and B *congruent* if $A^T A = B^T B$. If we view the columns of A and B as points in \mathbb{R}^m then it is clear that A and B are congruent if and only if the corresponding sets of points in \mathbb{R}^m are congruent in the geometric sense, i.e., there is an $m \times m$ orthogonal matrix which maps the columns of A onto the corresponding columns of B .

2.6. CONSTRUCTION. Let H be a graph on n vertices and let A be an $m \times n$ 0-1 matrix. Then $H(A)$ is the graph with adjacency matrix

$$\begin{pmatrix} 0 & A \\ A^T & H \end{pmatrix}.$$

2.7. THEOREM. *Let H be a graph on n vertices and let A and B be two congruent $m \times n$ 0-1 matrices. Then the graphs $H(A)$ and $H(B)$ are cospectral.*

Proof. Let $Q_1 = \text{diag}(Q, I_n)$, where $QA = B$ and Q is orthogonal. Then Q_1 is orthogonal and $Q_1 H(A) Q_1^T = H(B)$. □

One source of congruent matrices is Construction 2.1 with $k = 1$. Another example is provided when A and B are the transposed incidence matrices of two BIBDs with the same parameters. If the graph H has no edges in this case then Construction 2.7 reduces to a known result. If H is not empty, then $H(A)$ may be not isomorphic to $H(B)$ even if the two corresponding designs are isomorphic. As an example, let A be the transposed incidence matrix of a Steiner Triple System with $v = 15$ and trivial automorphism group, and let H be any 15 vertex graph with trivial automorphism group. Then $H(A) \not\cong H(B)$ for any column permutation B of A . Thus we get a family of $15! = 130767438000$ nonisomorphic graphs on 50 vertices, all of which are cospectral and have cospectral complements. More generally, there are $v^{\binom{v}{3} + O(v)}$ labelled Steiner triple systems with v points (see [1]), and $2^{\binom{v}{2}}/v!(1 + O(1/v))$ graphs of order v with trivial automorphism groups. Thus we can construct $2^{1+\binom{v}{2}}/v!(1 + O(1/v))$ families of graphs of order $v(v + 5)/6$, each containing $v^{\binom{v}{3} + O(v)}$ cospectral graphs. Although high, the number of graphs involved here is miniscule compared to the number produced by Construction 2.1.

Construction 2.6 has also been investigated (without proof) by Davidson [6], who gives many examples of congruent matrices.

3. Tensor products

3.1. In this section we describe a very general procedure which uses the matrix tensor product to construct families of cospectral graphs. Although many special cases of this construction have appeared before, the general case is new.

All the matrices in this section are real, but otherwise not restricted. The necessary conditions for the matrices so constructed to be adjacency matrices of graphs will be obvious in every case.

The tensor (direct) product of matrices A and B will be denoted by $A \otimes B$. For the most elementary properties of this operation we refer the reader to [11] or [7].

3.2. Consider a sequence of matrices

$$A = (A_1^{(1)}, A_2^{(1)}, \dots, A_k^{(1)}; A_1^{(2)}, A_2^{(2)}, \dots, A_k^{(2)}),$$

where $A_i^{(1)}$ has order $n_1 \times n_1$ and $A_i^{(2)}$ has order $n_2 \times n_2$, for $1 \leq i \leq k$. The next lemma is just an elementary property of the tensor product.

3.3. LEMMA. *For any monomial f in k noncommuting variables*

$$\begin{aligned} \text{tr } f(A_1^{(1)} \otimes A_1^{(2)}, A_2^{(1)} \otimes A_2^{(2)}, \dots, A_k^{(1)} \otimes A_k^{(2)}) \\ = \text{tr } f(A_1^{(1)}, A_2^{(1)}, \dots, A_k^{(1)}) \text{tr } f(A_1^{(2)}, A_2^{(2)}, \dots, A_k^{(2)}). \end{aligned} \quad \square$$

Let \mathbf{B} be a sequence of matrices with the same orders as those of \mathbf{A} . Define $T(\mathbf{A}) = \sum_{i=1}^k (A_i^{(1)} \otimes A_i^{(2)})$ and $T(\mathbf{B}) = \sum_{i=1}^k (B_i^{(1)} \otimes B_i^{(2)})$. The general construction we are considering is based on the following theorem, which follows immediately from the multinomial theorem and Lemma 3.3.

3.4. THEOREM. *Suppose that*

$$\text{tr } f(A_1^{(j)}, A_2^{(j)}, \dots, A_k^{(j)}) = \text{tr } f(B_1^{(j)}, B_2^{(j)}, \dots, B_k^{(j)}),$$

for any monomial f and $j \in \{1, 2\}$. Then $T(\mathbf{A})$ and $T(\mathbf{B})$ are cospectral. □

The cases where $A_i^{(j)} \in \{I, G_j, \bar{G}_j\}$ for $1 \leq j \leq 2$ and $1 \leq i \leq k$ have been investigated in depth by Cvetković and others ([3], [5]). In this connection, we note that, although we are using tensor products of only two factors, the class of graphs constructed is not thereby reduced, since the case of more than two factors can be obtained by repeated application.

A simple family of applications of Theorem 3.4 can be obtained with the help of the following lemma.

3.5. LEMMA. *If G_1 and G_2 are cospectral, then $\text{tr } G_1^r = \text{tr } G_2^r$ for any $r \geq 0$. If also \bar{G}_1 and \bar{G}_2 are cospectral, then $\text{tr } f(G_1, \bar{G}_1, J) = \text{tr } f(G_2, \bar{G}_2, J)$ for any monomial f .*

Proof. The first claim is obvious, while the second follows from Lemma 2.1 of McKay [12]. □

3.6. CONSTRUCTION. Let G_1 and G_2 be cospectral, and let X and H be square matrices of the same order. Then $H \otimes I + X \otimes G_1$ is cospectral to $H \otimes I + X \otimes G_2$.

The choices $X = I$, $X = J$ and $X = I + H$ give the cartesian product $H \times G_i$, the lexicographic product $G_i[H]$ and the strong product $H * G_i$, respectively.

3.7. CONSTRUCTION. Let G_1 and G_2 be cospectral, with \bar{G}_1 and \bar{G}_2 also cospectral. Let C, D, E and F be square matrices of the same order. Then $C \otimes I + D \otimes J + E \otimes G_1 + F \otimes \bar{G}_1$ is cospectral to $C \otimes I + D \otimes J + E \otimes G_2 + F \otimes \bar{G}_2$.

If $C = F = 0$ and $E = I$ we have the lexicographic product $D[G_i]$. Another interesting subcase comes from taking $C = D = 0, E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which yields the *switching graphs* $\text{Sw}(G_1)$ and $\text{Sw}(G_2)$. These are regular even if G_1 and G_2 are not, and are known [8] to be isomorphic if and only if G_1 and G_2 are switching equivalent. Thus, for example, if G_1 and G_2 are cospectral nonisomorphic trees then $\text{Sw}(G_1)$ and $\text{Sw}(G_2)$ are cospectral nonisomorphic regular graphs. This follows from the easily proved fact that switching equivalent trees are isomorphic.

Constructions 3.6 and 3.7 have the property that the graphs constructed will be isomorphic if G_1 and G_2 are isomorphic. This is not a necessary characteristic of $T(A)$ and $T(B)$, as our next example demonstrates.

Let G_1 and G_2 be bipartite graphs with parts of cardinality m and n . For $i = 1, 2$, label G_i so that

$$G_i = \begin{pmatrix} 0 & B_i \\ B_i^T & 0 \end{pmatrix},$$

where the partition is $m + n$. Define

$$E = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \quad E^* = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad C_i = \begin{pmatrix} 0 & B_i \\ 0 & 0 \end{pmatrix}.$$

3.8. LEMMA. *If G_1 and G_2 are cospectral then, for any monomial f in four noncommuting variables, $\text{tr } f(E, E^*, C_1, C_1^T) = \text{tr } f(E, E^*, C_2, C_2^T)$. If also $m = n$ then $\text{tr } f(E, E^*, C_1, C_1^T) = \text{tr } f(E^*, E, C_2^T, C_2)$.*

Proof. The only monomials f for which $\text{tr } f(E, E^*, C_i, C_i^T)$ is possibly nonzero are those for which $f(E, E^*, C_i, C_i^T)$ equals $(C_i C_i^T)^r$ or $(C_i^T C_i)^r$ for some $r \geq 0$. The first claim now follows from Lemma 3.5, together with the observation that $\text{tr}(C_i C_i^T)^r = \text{tr}(C_i^T C_i)^r = \frac{1}{2} \text{tr } G_i^{2r}$ if $r \geq 1$. The second claim can be proved by noting, in addition, that $\text{tr}(C_i C_i^T)^0 = m$ and $\text{tr}(C_i^T C_i)^0 = n$. □

Lemma 3.8 can be used to produce many diverse pairs of cospectral graphs. For example, if G_1 and G_2 are cospectral as above, and H_1 and H_2 are arbitrary cospectral graphs, then the four “half-cartesian-products” $G_i \otimes I + E^* \otimes H_j$ ($i, j = 1, 2$) are cospectral. However, the most interesting application is perhaps the *partitioned tensor product*, first defined in [9].

3.9. THEOREM [9]. Let G_1 and G_2 be cospectral bipartite graphs with parts of cardinality m and n . Define E, E^*, C_1 and C_2 as above. Let G and H be arbitrary graphs and let Q be an $r \times s$ matrix, where r and s are the orders of G and H , respectively. Define

$$P = E \otimes G + E^* \otimes H + C_1 \otimes Q + C_2 \otimes Q^T,$$

and

$$P^* = E^* \otimes G + E \otimes H + C_2^T \otimes Q + C_1^T \otimes Q^T.$$

Then

$$\phi(P)\phi(H)^{m-n} = \phi(P^*)\phi(G)^{m-n}.$$

Consequently, $\phi(P) = \phi(P^*)$ if either $m = n$ or $\phi(H) = \phi(G)$.

Proof. Without loss of generality, assume that $m \geq n$. Add $m - n$ isolated vertices to the second parts of G_1 and G_2 . The effect is to add $m - n$ isolated copies of H to P and $m - n$ isolated copies of G to P^* . The claim is now immediate from 3.4 and 3.8. □

Informally, P is obtained as follows: Replace each vertex in the first part of G_1 by a copy of G , and each vertex in the second part by a copy of H . Then, for each edge of G_1 , join the corresponding copies of G and H according to the entries of Q . An example is shown in Figure 4.

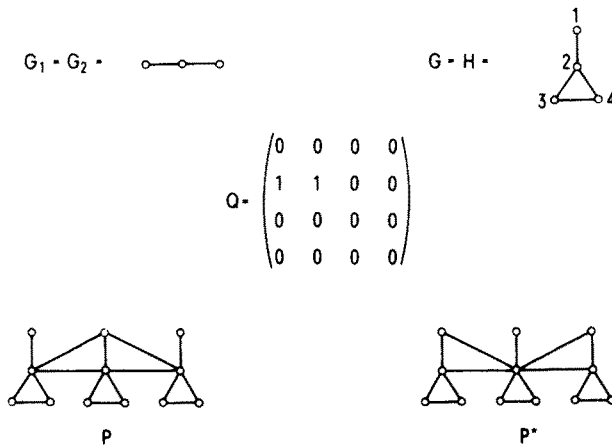


Figure 4

In [9] it is shown that the smallest pair of cospectral graphs, the smallest pair of cospectral forests and the smallest pair of cospectral connected graphs can each be obtained using the partitioned tensor product.

Of course, there is no guarantee that P and P^* are nonisomorphic in general, although some sufficient conditions are known (see [9]). A special case of Theorem 3.9 has been rediscovered by Schwenk, Herndon and Ellzey [16].

4. Cospectral points

4.1. If G is a rooted graph with root v , then G_v denotes the graph formed by deleting v from G . Given two rooted graphs G and H with roots v and w , respectively, we can define the following composite graphs:

- (i) $G \cdot H$ is formed from G and H by identifying v and w .
 - (ii) $G \dashv H$ is formed from disjoint copies of G and H by adding one edge joining v and w .
 - (iii) $G \equiv H$ is formed from disjoint copies of G_v and H_w by joining every vertex in G_v which is adjacent to v in G to every vertex in H_w which is adjacent to w in H .
- Examples are given in Figure 5.

The spectrum of $G \cdot H$ was first determined by Schwenk [14]. That of $G \dashv H$ is just a special case. The spectrum of $G \equiv H$ has not been previously determined.

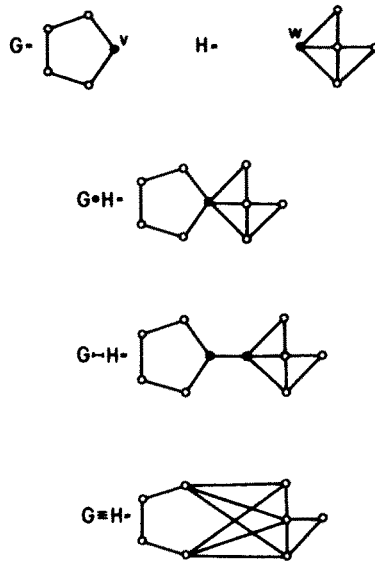


Figure 5

4.2. THEOREM.

- (i) $\phi(G \cdot H) = \phi(G)\phi(H_w) + \phi(G_v)\phi(H) - x\phi(G_v)\phi(H_w)$;
- (ii) $\phi(G \dashv H) = \phi(G)\phi(H) - \phi(G_v)\phi(H_w)$;
- (iii) $\phi(G \equiv H) = \phi(G_v)\phi(H_w) - (x\phi(G_v) - \phi(G))(x\phi(H_w) - \phi(H))$.

Proof. For part (i), and thus (ii), see [14] or [4]. To prove part (iii) we need some additional notation. Let M and N be square matrices of order m and n , respectively. Let \mathbf{a} and \mathbf{b} be (column) vectors of length m and n , respectively. The notation $M \mid \mathbf{a}$ represents the matrix $\begin{pmatrix} M & \mathbf{a} \\ \mathbf{0} & 0 \end{pmatrix}$. The two claims following can be proved by applying elementary row and column operations. For any α ,

$$\phi(M + \alpha \mathbf{a} \mathbf{a}^T) = \alpha \phi(M \mid \mathbf{a}) + (1 - \alpha x) \phi(M). \tag{1}$$

$$\phi \begin{pmatrix} 0 & \mathbf{a}^T & \mathbf{b}^T \\ \mathbf{a} & M & 0 \\ \mathbf{b} & 0 & N \end{pmatrix} = \phi(M) \phi(N \mid \mathbf{b}) + \phi(N) \phi(M \mid \mathbf{a}) - x \phi(M) \phi(N). \tag{2}$$

Now suppose $G = G_v \mid \mathbf{g}$ and $H = H_w \mid \mathbf{h}$. Then

$$\phi(G \equiv H) = \phi \begin{pmatrix} G_v & \mathbf{g} \mathbf{h}^T \\ \mathbf{h} \mathbf{g}^T & H_w \end{pmatrix} = \phi \left[\begin{pmatrix} G_v - \mathbf{g} \mathbf{g}^T & 0 \\ 0 & H_w - \mathbf{h} \mathbf{h}^T \end{pmatrix} + \mathbf{k} \mathbf{k}^T \right],$$

where $\mathbf{k}^T = (\mathbf{g}^T \mathbf{h}^T)$. Application of (1) and (2) produces

$$\begin{aligned} \phi(G \equiv H) &= \phi(G_v) \phi(H_w - \mathbf{h} \mathbf{h}^T) + \phi(G_w) \phi(G_v - \mathbf{g} \mathbf{g}^T) \\ &\quad - \phi(H_w - \mathbf{h} \mathbf{h}^T) \phi(G_v - \mathbf{g} \mathbf{g}^T). \end{aligned}$$

Further application of (2) gives the desired form. □

4.3. COROLLARY. $\phi(G \dashv H) + \phi(G \equiv H) = x \phi(G \cdot H)$. □

4.4. COROLLARY. For $i = 1, 2$, let $G^{(i)}$ and $H^{(i)}$ be rooted graphs with roots $v^{(i)}$ and $w^{(i)}$, respectively. Suppose that $\phi(G^{(i)})$, $\phi(G_v^{(i)})$, $\phi(H^{(i)})$ and $\phi(H_w^{(i)})$ are independent of i . Then $\phi(G^{(i)} \cdot H^{(i)})$, $\phi(G^{(i)} \dashv H^{(i)})$ and $\phi(G^{(i)} \equiv H^{(i)})$ are independent of i . □

The case $\phi(G^{(i)} \cdot H^{(i)})$ was used by Schwenk [14] to prove that almost no tree is uniquely identified by its spectrum. Stronger results of similar form appeared in [10] and especially in [12]. A generalization (for the “ \cdot ” operation) to graphs rooted at more than one point has been given by Schwenk [15]. Construction 2.1 can also be described in this manner.

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