A NON-HAMILTONIAN GRAPH

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(received June 30, 1959)

There is a remarkable cubic graph of 28 vertices, discovered by Prof. H.S.M. Coxeter, which has no circuit of fewer than 7 edges and in which all oriented arcs made up of three edges are equivalent under the automorphism group.

The structure of the graph is as follows. There are three disjoint heptagons $A(a_1a_2a_3a_4a_5a_6a_7a_1)$, $B(b_1b_5b_2b_6b_3b_7b_4b_1)$ and $C(c_1c_6c_4c_2c_7c_5c_3c_1)$. We denote the seven remaining vertices by $d_1, d_2, \ldots, d_7$. For each suffix i the three edges incident with d_i join it to a_i, b_i and c_i. In drawing a diagram of the graph it seems best to show only the heptagons A, B and C, leaving the rest of the figure to the imagination.

The purpose of this note is to establish another property of the graph, that it has no Hamiltonian circuit. A Hamiltonian circuit of a graph is a simple closed curve made up of edges and passing through all the vertices.

Now if the given graph G has a Hamiltonian circuit H passing through a_1a_2 and a_2a_3, then the circuit passes also through b_2d_2 and c_2d_2 (since it does not contain the edge a_2d_2). We abbreviate this proposition as

$$(a_1a_2,\ a_2a_3) \longrightarrow (b_2d_2,\ c_2d_2),$$

and we shall make extensive use of implications of this kind.

We note the existence of an automorphism U of G which maps x_i onto x_{i+1}, where x is a, b, c or d, $1 \leq i \leq 7$, and $x_8 = x_1$.

Assume G has a Hamiltonian circuit H. Clearly H does not include all the edges of A. Let s denote the maximum number of consecutive edges of A in H. Then $s \leq 6$. Moreover

Can. Math. Bull. vol. 3, no. 1, Jan. 1960.

$s \geqslant 2$ since each vertex of A is incident with two edges of H. We split up the proof into 5 cases corresponding to the values of s from 2 to 6.

Case I: $s = 6$.

Because of the automorphism U we can suppose H contains the edges a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6 and a_6a_7 of A. Since it does not contain a_1a_7 it must include also a_1d_1 and a_7d_7. We note the following implications.

$$(a_1a_2,\ a_2a_3) \longrightarrow (b_2d_2,\ c_2d_2)$$
$$(a_2a_3,\ a_3a_4) \longrightarrow (b_3d_3,\ c_3d_3)$$
$$(a_3a_4,\ a_4a_5) \longrightarrow (b_4d_4,\ c_4d_4)$$
$$(a_4a_5,\ a_5a_6) \longrightarrow (b_5d_5,\ c_5d_5)$$
$$(a_5a_6,\ a_6a_7) \longrightarrow (b_6d_6,\ c_6d_6)$$

The part of H so far determined is shown by thick lines in Figure 1. We note that it is invariant under an automorphism V of G mapping x_i onto x_{8-i}. But H must include one of the two edges b_1b_4 and b_4b_7. As these are equivalent under V we may suppose H includes b_1b_4. We can now extend our knowledge of H with the following implicatinns.

$$(b_1b_4,\ b_4d_4) \longrightarrow (b_3b_7,\ b_7d_7)$$
$$(b_3b_7,\ b_3d_3) \longrightarrow (b_2b_6,\ b_6d_6)$$
$$(b_2b_6,\ b_2d_2) \longrightarrow (b_1b_5,\ b_5d_5)$$
$$(b_1b_4,\ b_1b_5) \longrightarrow (a_1d_1,\ c_1d_1)$$
$$(a_7d_7,\ b_7d_7) \longrightarrow (c_2c_7,\ c_5c_7)$$
$$(c_5c_7,\ c_5d_5) \longrightarrow (c_1c_3,\ c_3d_3)$$
$$(c_1c_3,\ c_1d_1) \longrightarrow (c_4c_6,\ c_6d_6)$$

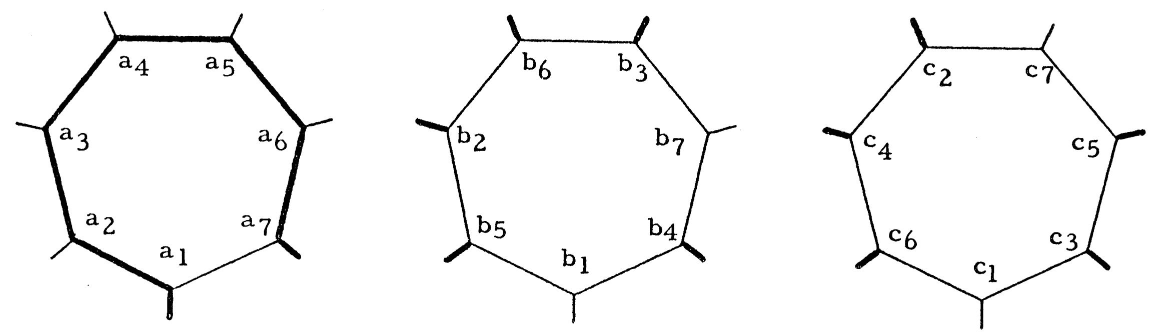
H is now completely determined. It consists of two separate circuits

$$H_1 = (a_1a_2a_3a_4a_5a_6a_7d_7b_7b_3d_3c_3c_1d_1a_1)$$

and

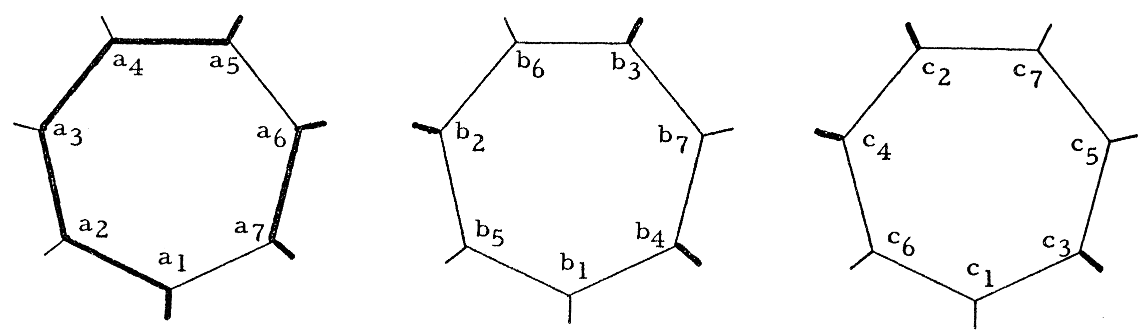
$$H_2 = (b_1b_5d_5c_5c_7c_2d_2b_2b_6d_6c_6c_4d_4b_4b_1).$$

So H is not Hamiltonian, contrary to assumption.



a4
a5
a3
a6
a2
a7
a1
b6
b3
b2
b7
b5
b4
b1
c2
c7
c4
c5
c6
c3
c1

Fig. 1



a4
a5
a3
a6
a2
a7
a1
b6
b3
b2
b7
b5
b4
b1
c2
c7
c4
c5
c6
c3
c1

Fig. 2

Case II: $s = 5$.

Because of the automorphism U we can assume H includes all the edges of A except a_7a_1 and a_1a_2. But H must include one of these two edges since it has two edges incident with a_1. The case $s = 5$ is thus impossible.

Case III: $s = 4$.

We may suppose H contains a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_1d_1 and a_5d_5. We have the following implications.

$$(a_1a_2,\ a_2a_3) \rightarrow (b_2d_2,\ c_2d_2)$$
$$(a_2a_3,\ a_3a_4) \rightarrow (b_3d_3,\ c_3d_3)$$
$$(a_3a_4,\ a_4a_5) \rightarrow (b_4d_4,\ c_4d_4)$$
$$(a_1a_2,\ a_1d_1) \rightarrow (a_6a_7,\ a_7d_7)$$
$$(a_4a_5,\ a_5d_5) \rightarrow (a_6a_7,\ a_6d_6)$$

The part of H so far determined is shown thickly in Figure 2. It is invariant under an automorphism W of G mapping x_i into x_{6-i}, where $x_0 = x_7$ and $x_{-1} = x_6$. But H must include one of the two edges b_3b_6 and b_3b_7, which are equivalent under W. We may therefore suppose H contains b_3b_7. We continue with the following implications.

$$(b_3b_7,\ b_3d_3) \rightarrow (b_2b_6,\ b_6d_6)$$
$$(b_2b_6,\ b_2d_2) \rightarrow (b_1b_5,\ b_5d_5)$$
$$(b_6d_6,\ a_6d_6) \rightarrow (c_1c_6,\ c_4c_6)$$
$$(b_5d_5,\ a_5d_5) \rightarrow (c_3c_5,\ c_5c_7)$$
$$(c_3c_5,\ c_3d_3) \rightarrow (c_1c_6,\ c_1d_1)$$
$$(a_1d_1,\ c_1d_1) \rightarrow (b_1b_4,\ b_1b_5)$$
$$(b_1b_4,\ b_4d_4) \rightarrow (b_3b_7,\ b_7d_7)$$
$$(c_4c_6,\ c_4d_4) \rightarrow (c_2c_7,\ c_2d_2)$$

H is now completely determined as the pair of circuits

$$(a_1a_2a_3a_4a_5d_5b_5b_1b_4d_4c_4c_6c_1d_1a_1)$$

and

$$(b_2d_2c_2c_7c_5c_3d_3b_3b_7d_7a_7a_6d_6b_6b_2),$$

contrary to its definition as Hamiltonian.

Before going on to the remaining cases we note the existence of an automorphism X defined by the following table.

X maps each element of the first row into the corresponding element of the second row.

$$\begin{Bmatrix} a_1a_2a_3a_4a_5a_6a_7b_1b_5b_2b_6b_3b_7b_4c_1c_6c_4c_2c_7c_5c_3d_1d_2d_3d_4d_5d_6d_7 \\ a_1a_2a_3d_3c_3c_1d_1d_7c_7c_2c_4d_4b_4b_7a_6d_6b_6b_2b_5d_5a_5a_7d_2a_4b_3c_5c_6b_1 \end{Bmatrix}$$

Case IV: $s = 2$.

We can suppose H contains the edges a_1a_2, a_2a_3, a_1d_1 and a_3d_3. But then H contains 4 consecutive edges of the heptagon $a_1a_2a_3d_3c_3c_1d_1a_1$, which is equivalent to A under X. So this case can be reduced to the preceding cases.

Case V: $s = 3$.

We can suppose H contains the edges a_4a_5, a_5a_6, a_6a_7, a_4d_4 and a_7d_7. We have the implications

$$(a_4a_5,\ a_4d_4) \longrightarrow (a_2a_3,\ a_3d_3),$$
$$(a_6a_7,\ a_7d_7) \longrightarrow (a_1a_2,\ a_1d_1).$$

So H contains a_1a_2, a_2a_3, a_1d_1 and a_3d_3. Accordingly Case V reduces to earlier cases in the same way as Case IV.

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