Elementary Proofs on Optimal Stopping [1]

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Keywords: Optimal stopping, Lévy processes, American options, Derivative pricing, Black-Scholes model.
JEL: G12, MSC (1991): 60G40

Abstract

Elementary proofs of classical theorems on pricing perpetual call and put options in the standard Black-Scholes model are given. The method presented does not rely on stochastic calculus and is also applied to give prices and optimal stopping rules for perpetual call options when the stock is driven by a Lévy process with no positive jumps, and for perpetual put options for stocks driven by a Lévy process with no negative jumps

[1] This work was partially written at the Laboratoire de Statistique et Probabilités de l'Université Paul Sabatier, Toulouse, and benefited from helpful discussion with Walter Moreira.

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December 29, 2000

1 Introduction

1.1 Consider a model of a financial market with two assets, a savings account $B = \{B_t\}_{t\geq 0}$ and a stock $S = \{S_t\}_{t\geq 0}$. The evolution of B is deterministic, with

$$B_t = B_0 e^{rt}, \quad B_0 = 1, \quad r > 0,$$

and the stock is random, and evolves according to the formula

$$S_t = S_0 e^{X_t}, \quad S_0 > 0,$$

where $X = \{X_t\}_{t\geq 0}$, the driving process, is an adapted stochastic process defined in a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions. We always assume that the process

$$\left\{\frac{S_t}{B_t}\right\}_{t\geq 0} \text{ is a martingale under } \mathbf{P}. \tag{1.1}$$

$$\mathcal{M} = \{\tau\colon \Omega \to [0, +\infty], \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$

denotes the class of all $\mathbf{F}$-stopping times.

1.2 If we are interested in pricing perpetual call and put options, possible discounted at a constant rate $\delta \geq 0$, we are led to solve an optimal stopping problem consisting in finding a cost function $V = V(S_0)$ and an optimal stopping rule τ^* that satisfy

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$$V(S_0) = \sup_{\tau \in \mathcal{M}} \mathbf{E} e^{-(r+\delta)\tau} G(S_\tau) = \mathbf{E} e^{-(r+\delta)\tau^*} G(S_{\tau^*}). \tag{1.2}$$

where the function $G = G(S)$ is given by $G(S) = (S - K)^+$ for call options and $G(S) = (K - S)^+$ for put options. As usual, we assume that $e^{-(r+\delta)\tau} G(S_\tau) \mathbf{1}_{\{\tau=\infty\}} = \limsup_{t\to\infty} e^{-(r+\delta)t} G(S_t)$.

Given a function $\bar{V}$ and a stopping time $\bar{\tau} \in \mathcal{M}$, if we are able to verify

(A) $\bar{V}(S_0) = \mathbf{E} e^{-(r+\delta)\bar{\tau}} G(S_{\bar{\tau}})$,

(B) $\bar{V}(S_0) \geq \mathbf{E} e^{-(r+\delta)\tau} G(S_\tau)$, for all $\tau \in \mathcal{M}$

we then prove that the pair $\bar{V}, \bar{\tau}$ is the solution to the problem (1.2).

The aim of this paper is twofold. First we give elementary proofs of classical results on perpetual calls and puts in the Black-Scholes model. The proofs consist on the verification of the martingale and supermartingale property of certain auxiliary processes, in order to see (A) and (B) above, and are based on independence of increments and Jensen's inequality respectively, not requiring Itô calculus. In this sense the proofs are elementary, and this task is carried in section 2. In section 3 we observe that the presented arguments can be extended to a more general setting, obtaining admissible prices of perpetual call options for upper semi-continuous Lévy processes, that is processes with no positive jumps, and admissible prices of perpetual put options for lower semi-continuous Lévy processes, that is, processes with no negative jumps. By admissible prices we mean that, as the market is incomplete, the martingale measure $\mathbf{P}$ chosen is such that X is a Lévy process under $\mathbf{P}$, and prices are computed according to this measure. More general results can be obtained with different techniques as in Mordecki (2000). Section 4 summarizes the results, and concludes with a discussion of related works and references.

2 Black-Scholes model

In this section we assume that $X = \{X_t\}_{t\geq 0}$ in **1.1** is given by

$$X_t = \sigma W_t + (r - \frac{\sigma^2}{2})t$$

where $W = \{W_t\}_{t\geq 0}$ is a Wiener process under $\mathbf{P}$, and $\sigma > 0$. This corresponds to the classical Black and Scholes (1973) model.

2.1 It is well known, that the solution to (1.2) with $G(S) = (S-K)^+$ and $\delta > 0$ i.e. the pricing of a perpetual call option with discount has cost function (see Merton, (1973)),

$$C(S_0) = \begin{cases} AS_0^\gamma & 0 \le S_0 < S_c^*, \\ \\ S_0 - K & S_c^* \le S_0, \end{cases} \tag{2.1}$$

where

$$\gamma = \frac{1}{2} - \frac{r}{\sigma^2} + \sqrt{\Big(\frac{1}{2} + \frac{r}{\sigma^2}\Big)^2 + \frac{2\delta}{\sigma^2}},$$

$$S_c^* = K\frac{\gamma}{\gamma - 1}, \quad A = \frac{1}{\gamma^\gamma}\Big(\frac{\gamma-1}{K}\Big)^{\gamma-1},$$

and optimal stopping rule

$$\tau_c^* = \inf\{t \ge 0\colon S_t \ge S_c^*\}.$$

See also Karatzas and Shreve (1998) or Shiryaev (1999).

Let us prove this result in an elementary way. To see (A) observe that the process $\{e^{-(r+\delta)t}S_t^\gamma\}_{t\ge 0}$ is a martingale. In fact, for $h > 0$

$$\mathbf{E}(e^{-(r+\delta)(t+h)}S_{t+h}^\gamma - e^{-(r+\delta)t}S_t^\gamma|\mathcal{F}_t)$$

$$= e^{-(r+\delta)t}S_t^\gamma \mathbf{E}e^{-(r+\delta)h + \gamma\sigma W_h + \gamma(r-\frac{\sigma^2}{2})h} - 1$$

$$= e^{-(r+\delta)t}S_t^\gamma \Big(e^{-[r+\delta+\gamma^2\frac{\sigma^2}{2} + \gamma(r-\frac{\sigma^2}{2})]h} - 1\Big) = 0 \tag{2.2}$$

by the election of γ. We claim that

$$\lim_{t\to\infty} \gamma X_t - (r+\delta)t = \lim_{t\to\infty} X_t - (r+\delta)t = -\infty. \tag{2.3}$$

This follows from

$$\mathbf{E}(\gamma X_1 - (r+\delta)) < \mathbf{E}e^{\gamma X_1 - (r+\delta)} = 0,$$

and

$$\mathbf{E}(X_1 - (r+\delta)) < \mathbf{E}e^{X_1 - (r+\delta)} = e^{-\delta} - 1 < 0$$

respectively. Now, if $S_0 < S_c^*$

$$\mathbf{E}e^{-(r+\delta)\tau_c^*}(S_{\tau_c^*} - K)^+ = \mathbf{E}e^{-(r+\delta)\tau_c^*}(S_{\tau_c^*} - K)^+\mathbf{1}_{\{\tau_c^*<\infty\}} =$$

$$A\mathbf{E}e^{-(r+\delta)\tau_c^*}S_{\tau_c^*}^{\gamma}\mathbf{1}_{\{\tau_c^*<\infty\}} = \lim_{t\to\infty} A\mathbf{E}e^{-(r+\delta)(\tau_c^*\wedge t)}S_{(\tau_c^*\wedge t)}^{\gamma} = AS_0^{\gamma}. \tag{2.4}$$

where we used (2.3). If $S_c^* \leq S_0$ then $\tau_c^* = 0$ and

$$\mathbf{E}e^{-(r+\delta)\tau_c^*}(S_{\tau_c^*} - K)^+ = S_0 - K.$$

In this way we proved (A).

Let us see (B). Consider for $y \geq 0$ the functions $v = v(y)$, and $\Phi = \Phi(y)$ given respectively by $v(y) = Ay^{\gamma}$, and

$$\Phi(y) = \begin{cases} y, & 0 \leq y \leq S_c^* - K, \\ \left(\frac{y}{A}\right)^{\frac{1}{\gamma}}, & S_c^* - K < y. \end{cases}$$

It is easy to verify the following three properties

- $\Phi(v(y)) = C(y)$ with $C(y)$ given in (2.1),
- Φ is concave, because $\gamma > 1$,
- $\Phi(\alpha y) \leq \alpha\Phi(y)$ for $\alpha \geq 1$.

The process $\{e^{-(r+\delta)t}C(S_t)\}_{t\geq 0}$ is a supermartingale. In fact, for $h > 0$

$$\mathbf{E}(e^{-(r+\delta)(t+h)}C(S_{t+h})|\mathcal{F}_t) = e^{-(r+\delta)(t+h)}\mathbf{E}(\Phi(v(S_{t+h}))|\mathcal{F}_t)$$

$$\leq e^{-(r+\delta)(t+h)}\Phi(\mathbf{E}(v(S_{t+h})|\mathcal{F}_t)) = e^{-(r+\delta)(t+h)}\Phi(e^{(r+\delta)h}v(S_t)) \leq$$

$$e^{-(r+\delta)t}\Phi(v(S_t)) = e^{-(r+\delta)t}C(S_t). \tag{2.5}$$

Take now $\sigma \in \mathcal{M}$.

$$\mathbf{E}e^{-(r+\delta)\sigma}(S_\sigma - K)^+ = \mathbf{E}\lim_{t\to\infty} e^{-(r+\delta)(\sigma\wedge t)}(S_{\sigma\wedge t} - K)^+$$

$$\leq \liminf_{t\to\infty} \mathbf{E}e^{-(r+\delta)(\sigma\wedge t)}(S_{\sigma\wedge t} - K)^+ \leq \liminf_{t\to\infty} \mathbf{E}e^{-(r+\delta)(\sigma\wedge t)}C(S_{\sigma\wedge t}) \leq C(S_0)$$

where we used (2.3). This proves (B) and concludes the proof.

2.2 We consider now the put case. For $\delta \geq 0$ the solution to (1.2) with $G(S) = (K - S)^+$ was given in Mc. Kean (1965), see also Merton (1973), and has cost function

$$P(S_0) = \begin{cases} K - S_0, & 0 \leq S_0 \leq S_p^*, \\ \\ BS_0^{-\beta}, & S_p^* < S_0, \end{cases} \tag{2.6}$$

where

$$\beta = \sqrt{\Big(\frac{1}{2} + \frac{r}{\sigma^2}\Big)^2 + \frac{2\delta}{\sigma^2}} - \Big(\frac{1}{2} - \frac{r}{\sigma^2}\Big) > 0,$$

$$S_p^* = K\frac{\beta}{\beta+1}, \quad B = \beta^\beta\Big(\frac{K}{\beta+1}\Big)^{\beta+1},$$

and optimal stopping rule

$$\tau_p^* = \inf\{t \geq 0 \colon S_t \leq S_p^*\}.$$

See also Karatzas and Shreve (1998) or Shiryaev (1999). Let us verify (A) and (B) in order to prove this result. Denote by $w = w(y), y \geq 0$ the function given by $w(y) = By^{-\beta}$. As in **2.1** the election of β allows to show that $\{e^{-(r+\delta)t}w(S_t)\}$ is a martingale. Also we know that

$$\lim_{t\to\infty} -\beta X_t - (r+\delta)t = \lim_{t\to\infty} X_t - (r+\delta)t = -\infty, \tag{2.7}$$

because

$$\mathbf{E}(-\beta X_1 - (r+\delta)) < \mathbf{E}e^{-\beta X_1 - (r+\delta)} = 0,$$

and

$$\mathbf{E}(X_1 - (r+\delta)) < \mathbf{E}e^{X_1 - (r+\delta)} = e^{-\delta} - 1 < 0,$$

respectively. Also as before, when $S_0 > S_p^*$

$$\mathbf{E}e^{-(r+\delta)\tau_p^*}(K - S_{\tau_p^*})^+ = \mathbf{E}e^{-(r+\delta)\tau_p^*}(K - S_{\tau_p^*})^+\mathbf{1}_{\{\tau_p^*<\infty\}} =$$

$$B\mathbf{E}e^{-(r+\delta)\tau_p^*}S_{\tau_p^*}^{-\beta}\mathbf{1}_{\{\tau_p^*<\infty\}} = \lim_{t\to\infty}\mathbf{E}e^{-(r+\delta)(\tau_p^*\wedge t)}v(S_{(\tau_p^*\wedge t)}) = BS_0^{-\beta}. \tag{2.8}$$

by (2.7). If $S_0 < S_p^*$, $\tau_p^* = 0$ and $\mathbf{E}e^{-(r+\delta)\tau_p^*}(K - S_{\tau_p^*})^+ = K - S_0$, concluding (A). In order to see (B) consider the function $\phi = \phi(y), y \geq 0$ given

$$\phi(y) = \begin{cases} y, & 0 \leq y \leq S_p^* - K, \\ \\ K - \left(\frac{y}{B}\right)^{-\frac{1}{\beta}}, & S_p^* - K < y. \end{cases}$$

The following properties hold

- $\phi(w(y)) = P(y)$ with $P(y)$ given in (2.6),
- ϕ is concave, because $\beta > 0$,
- $\phi(\alpha y) \leq \alpha\phi(y)$ for $\alpha \geq 1$,

and allow to see, as in **2.1** that $\{e^{-(r+\delta)t}P(S_t)\}_{t\geq 0}$ is a supermartingale. From this, for $\sigma \in \mathcal{M}$ we have

$$\mathbf{E}e^{-(r+\delta)\sigma}(K - S_\sigma)^+ = \mathbf{E}\lim_{t\to\infty} e^{-(r+\delta)(\sigma\wedge t}(K - S_{\sigma\wedge t})^+$$

$$\leq \liminf_{t\to\infty} \mathbf{E}e^{-(r+\delta)(\sigma\wedge t)}(K - S_{\sigma\wedge t})^+$$

$$\leq \liminf_{t\to\infty} \mathbf{E}e^{-(r+\delta)(\sigma\wedge t)}P(S_{\sigma\wedge t}) \leq P(S_0)$$

by (2.7), proving (B) and completing the proof.

3 Perpetual options for semi-continuous Lévy process

We now see that the proofs above remain in force in a more general setting. For a call option we consider a stock driven by a Lévy process without positive jumps, i.e. an upper semi-continuous Lévy process. In the put case the driving process has no negative jumps, i.e. it is lower semi-continuous. For general reference on Lévy processes see Jacod and Shiryaev (1987), Skorokhod (1991), Bertoin (1996) or Sato (1999). Lévy-Khinchine formula states, for a Lévy process $X = \{X_t\}_{t\geq 0}$

$$\mathbf{E}e^{i\mu X_t} = \exp\Big\{t\Big[i\mu a - \frac{1}{2}\sigma^2\mu^2 + \int_{\mathbf{R}}(e^{i\mu y} - 1 - i\mu y\mathbf{1}_{\{|y|<1\}})\Pi(dy)\Big]\Big\}, \quad (3.1)$$

where a and $\sigma \geq 0$ are real constants, and Π is a positive measure on $\mathbf{R} - \{0\}$ such that $\int(1 \wedge y^2)\Pi(dy) < +\infty$, called the Lévy measure. The triplet of parameters (a, σ^2, Π) completely determine the law of the process.

3.1 Call perpetual options for upper semi-continuous Lévy processes

Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process with no positive jumps. This is equivalent to say that the measure Π in (3.1) is supported on the set $(-\infty, 0)$. We exclude from consideration the case of the negative of a subordinator, i.e. an a.s. non-increasing process. If $\lambda \geq 0$ we introduce the Laplace exponent $\kappa = \kappa(\lambda)$ of X

$$\kappa(\lambda) = a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{0} (e^{\lambda y} - 1 - \lambda y \mathbf{1}_{\{-1<y<0\}})\Pi(dy), \tag{3.2}$$

that satisfies $\mathbf{E}e^{\lambda X_t} = e^{t\kappa(\lambda)}$. It is known that $\kappa(0) = 0$, κ is convex, and $\lim_{\lambda\to\infty} \kappa(\lambda) = \infty$. The martingale condition (1.1) reads $\kappa(1) = r$, so for any $\delta > 0$ there exists $\gamma > 1$ such that $\kappa(\gamma) = r + \delta$.

Theorem 3.1 *Consider the model of a financial market of* **1.1** *with driving process X, a Lévy process with no positive jumps, triplet (a, σ^2, Π) and exponent (3.2). Then, the solution to the problem (1.2) with $G(S) = (S-K)^+$ and $\delta > 0$ has cost function*

$$C(S_0) = \begin{cases} AS_0^\gamma & 0 \leq S_0 < S_c^*, \\ \\ S_0 - K & S_c^* \leq S_0, \end{cases}$$

where $\gamma > 1$ is such that $\kappa(\gamma) = r + \delta$, $S_c^* = K\frac{\gamma}{\gamma-1}$, $A = \frac{1}{\gamma^\gamma}\left(\frac{\gamma-1}{K}\right)^{\gamma-1}$, and optimal stopping rule

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\}.$$

Proof. We follow the proof in **2.1**. $\{e^{-(r+\delta)t}S_t^\gamma\}_{t\geq 0}$ is a martingale, as

$$E(e^{-(r+\delta)(t+h)}S_{t+h}^\gamma - e^{-(r+\delta)t}S_t^\gamma | \mathcal{F}_t) =$$

$$e^{-(r+\delta)t}S_t^\gamma(e^{(-r-\delta+\kappa(\gamma))h}) - 1) = 0.$$

The rest follows exactly as before, taking into account that on $\{\tau_c^* < \infty\}$, we have

$$(S_{\tau_c^*} - K)^+ = AS_{\tau_c^*}^\gamma,$$

because the process has no positive jumps.

3.2 Put perpetual options for lower semi-continuous Lévy processes

Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process with no negative jumps. The measure Π in (3.1) is supported now on the set $(0,\infty)$. We exclude the case of a subordinator, i.e. an a.s. increasing process. The Laplace exponent is now defined for $\lambda \leq 0$ by

$$\kappa(\lambda) = a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{\lambda y} - 1 - \lambda y \mathbf{1}_{\{0<y<1\}})\Pi(dy), \tag{3.3}$$

and satisfies $\mathbf{E}e^{\lambda X_t} = e^{t\kappa(\lambda)}$. Observe that the martingale condition (1.1) requires in this case an extra assumption on the jumps, i.e. $\mathbf{E}e^{X_1} = e^r < \infty$. The finiteness of the exponential moment is ensured by the condition $\int_0^\infty e^y \Pi(dy) < \infty$. As $\kappa(0) = 0$, κ is convex, and $\lim_{\lambda\to-\infty}\kappa(\lambda) = \infty$ we obtain that for any $\delta \geq 0$ there exists $\beta > 0$ such that $\kappa(-\beta) = r + \delta$.

Theorem 3.2 *Consider the model of a financial market of* **1.1** *with driving process X, a Lévy process with no negative jumps, triplet (a, σ^2, Π) and exponent (3.3). Then, the solution to the problem (1.2) with $G(S) = (K - S)^+$ and $\delta \geq 0$ has cost function*

$$P(S_0) = \begin{cases} K - S_0 & 0 \leq S_0 \leq S_p^*, \\ \\ BS_0^{-\beta} & S_p^* < S_0, \end{cases}$$

where $\beta > 0$ is such that $\kappa(-\beta) = r + \delta$, $S_p^* = K\frac{\beta}{\beta+1}$, $B = \beta^\beta\left(\frac{K}{\beta+1}\right)^{\beta+1}$, and optimal stopping rule

$$\tau_p^* = \inf\{t \geq 0 \colon S_t \leq S_p^*\}.$$

Proof. We follow the proof in **2.2**. $\{e^{-(r+\delta)t}w(S_t)\}$ is a martingale in view of $\kappa(-\beta) = r + \delta$. (2.8) follows because on the set $\{\tau_p^* < \infty\}$ we have $(K - S_{\tau_p^*})^+ = BS_{\tau_c^*}^{-\beta}$, as the process has no negative jumps. The rest follows exactly.

4 Conclusion

This article presents new proofs of classical theorems on pricing call and put perpetual American options in the Black-Scholes model, with a savings account $B = \{B_t\}_{t\geq 0}$ and a stock $S = \{S_t\}_{t\geq 0}$. The proofs consist in the verification of the martingale and supermartingale properties, for process of the type $\{f(S_t)\}_{t\geq 0}$ for certain auxiliary functions f. The martingale property is established elementary, by independence of increments. The supermartingale property is verified through Jensen's inequality. No stochastic calculus is involved. By observing that the overshot when the process reaches a fixed level is null for semi-continuous Lévy processes, upper for call options, and lower for put options, the results are extended to these cases. More general results based on different techniques can be found in Mordecki (2000) and the references therein.

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