NECESSARY CONDITIONS FOR STABLE CONVERGENCE OF SEMIMARTINGALES

E. Mordecki

Steklov Mathematical Institute, Vavilov 42, 117966 Moscow GSP–1, Russia
Universidad de la República, Montevideo, Uruguay

Abstract: We prove an inverse to a theorem on stable convergence of semimartingales due to (Feigin, 1985). As a consequence, it can be stated (under some control in the jumps) that a sequence of martingales X^n converges stably to a continuous martingale X with conditionally independent increments if and only if the quadratic variations of X^n converge in probability to the quadratic variation of X for each $t \in \mathbf{R}^+$.

Keywords: Stable convergence, semimartingales.

1.Introduction and Results.

The subject of the present paper is the characterization of the behaviour of a sequence of semimartingales when they converge stably to a continuous semimartingale with conditionally independent increments. In view of the results due to (Lipster and Shiryaev, 1983), and more recently to (Feigin, 1985), it remains only to prove some inverse theorems, and this is done, based on the extension of existing results for weak convergence, and a convenient characterization of stable convergence. The results presented here were announced in (Mordecki, 1993).

For general concepts about semimartingales and their characteristics we refer to (Jacod and Shiryaev, 1987, ch. I and II).

Let us now define the sequence of semimartingales and the limit.

Let h denote a truncation function, and all the characterictics will be considered with respect to h.

(S) *Consider for each integer $n \in \mathbf{N}$ a d-dimensional semimartingale X^n defined on a stochastic basis $\mathcal{B}^n = (\Omega, \mathcal{F}, \mathbb{F}^n, \mathbf{P})$, with $\mathbb{F}^n = (\mathcal{F}^n_t : t \geq 0)$. Denote by $(B^n, \tilde{C}^n, \nu^n)$ the triplet of modified characteristics of X^n under $\mathbf{P}$, and by $[X^n, X^n]$ the $\mathbf{R}^d \otimes \mathbf{R}^d$ - valued processes whose components are the (co-)quadratic variation $[X^{n,j}, X^{n,k}]$. Set for simplicity $X^n_0 = 0$.*

As for the limit consider a transition probability $Q(\omega, d\gamma)$ from $(\Omega, \mathcal{F})$ into the Skorokhod space

$$(\mathbf{D}(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d), (\mathcal{D}_t : t \geq 0)),$$

such that for each $\omega \in \Omega$ the canonical process on $\mathbf{D}(\mathbf{R}^d)$ defined by $X_t(\gamma) = \gamma_t$ is a semimartingale with independent increments, and characteristics $(B(\omega), C(\omega), 0)$

Now we set

$$\bar{\Omega} = \Omega \times \mathbf{D}(\mathbf{R}^d), \qquad \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{D}(\mathbf{R}^d),$$

$$\bar{\mathbf{P}}(d\omega, d\gamma) = \mathbf{P}(d\omega) Q(\omega, d\gamma), \qquad \bar{\mathbb{F}} = (\cap_{s>t}(\mathcal{F} \otimes \mathcal{D}_s) : t \geq 0)$$

and every process, variable or σ-field defined on $(\Omega, \mathcal{F})$ or $(\mathbf{D}(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d))$ is naturally extended to $(\bar{\Omega}, \bar{\mathcal{F}})$ with the same symbol, e.g., $X(\omega, \gamma) = X(\gamma)$, $X^n(\omega, \gamma) = X^n(\omega)$, etc.

(L) *Let then X be a process defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{P})$ with $(B, C, 0)$ as characteristics, that is, a process with $\mathcal{F}$-conditionally independent increments. Denote by $[X, X]$ the quadratic variation of X (defined as above), and observe that $\tilde{C} = C$.*

(N) *Nesting condition: Assume there exists a sequence of real numbers (α_n) decreasing to 0, such that*

(1) $\mathcal{F}^n_{\alpha_n} \subset \mathcal{F}^{n+1}_{\alpha_{n+1}}$,

(2) $\mathcal{F} = \bigvee_n \mathcal{F}^n_{\alpha_n}$.

Then consider the conditions:

$[\gamma - \mathbf{R}^+]: \quad \tilde{C}^n_t \xrightarrow{\mathbf{P}} C_t \quad \forall t \in D$

$[\delta - \mathbf{R}^+]: \quad \mathbb{I}_{\{|x|>\varepsilon\}} * \nu^n_t \xrightarrow{\mathbf{P}} 0 \quad \forall t \in D \quad \forall \varepsilon > 0$

Denote $Var(B^n)_t = \sum_{i=1}^d Var(B^{n,i})_t$, where $Var(A)_t$ stands for the total variation of a function A on $[0, t]$.

We will say that the sequence X^n converges $\mathcal{F}$-stably to X in $\mathbf{D}(\mathbf{R}^d)$ (denoted by $X^n \xrightarrow{d} X \quad (\mathcal{F}\text{-stably})$), if

$$\mathbf{E}(Y f(X^n)) \to \mathbf{E}(Y f(X)) \quad \text{as} \quad n \to +\infty$$

for all bounded and $\mathcal{F}$-measurable random variables Y and continuous and bounded functions $f : \mathbf{D}(\mathbf{R}^d) \to \mathbf{R}$.

Theorem 1. *Assume conditions* **(S)** *and* **(L)** *are fulfilled. If*

(i) $X^n \xrightarrow{d} X \quad$ *($\mathcal{F}$-stably),*

(ii) $(Var(B^n)_t | \bar{\mathbf{P}})$ *is tight for all $t \geq 0$,*

then

$$([X^n, X^n], X^n) \xrightarrow{d} ([X, X], X) \quad (\mathcal{F}\text{-}stably)$$

and, in particular,

$$[X^n, X^n] \xrightarrow{d} [X, X] \quad (\mathcal{F}\text{-}stably)$$

Corollary 1. *Under the assumptions of Theorem 1*

$$[X^n, X^n]_t \xrightarrow{\mathbf{P}} C_t \quad \forall t \geq 0.$$

Theorem 2. *Assume conditions* **(S)** *and* **(L)** *are fulfilled. If*

(i) $X^n \xrightarrow{d} X \quad (\mathcal{F}\text{-}stably)$,

(ii) $(Var(B^n)_t|\bar{P})$ *is tight;*

then

(1) $[\delta - \mathbf{R}^+]: \quad \mathbb{I}_{\{|x|>\varepsilon\}} * \nu_t^n \xrightarrow{\mathbf{P}P} 0 \quad \forall t \in \mathbf{R}^+ \quad \forall \varepsilon > 0,$

(2) $[\gamma - \mathbf{R}^+]: \quad \tilde{C}_t^n \xrightarrow{P} C_t \quad \forall t \in \mathbf{R}^+.$

Corollary 2. *Assume conditions* **(S)**, **(L)** *and* **(N)** *are fulfilled. Suppose that X^n's are local martingales satisfying $|\Delta X_t^n| \le c$ for some constant c, and X is a continuous $\mathcal{F}$-conditionally Gaussian martingale (for each fixed ω, X is a Gaussian martingale).*

Then the following statements are equivalent:

(1) $X^n \xrightarrow{d} X \quad (\mathcal{F}\text{-}stably)$,

(2) $[X^n, X^n]_t \xrightarrow{\mathbf{P}} [X,X]_t \quad \forall t \in \mathbf{R}^+,$

(3) $[\gamma - \mathbf{R}^+] + [\delta - \mathbf{R}^+]$.

2. Proofs.

Let us first characterize stable convergence in a convenient way. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Denote by $\mathcal{P}(\mathcal{G})$ the set of probability measures $\tilde{\mathbf{P}}$ on $(\Omega, \mathcal{F})$ such that $d\tilde{\mathbf{P}} = Y d\mathbf{P}$ for any $\mathcal{G}$-measurable, positive and bounded random variable Y. Given X a random variable; let $\sigma(X)$ denote the σ-field generated by X.

Lemma 1. *Suppose on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is given a sequence of random elements X^n and X with values in a metric space (E, δ). Let $\mathcal{G} \subset \mathcal{F}$. Then (1) and (2) are equivalent:*

(1) $X^n \xrightarrow{d} X \quad (\mathcal{G}\text{-}stably)$,

(2) $X^n \xrightarrow{\mathcal{L}(\tilde{P})} X \quad \forall \tilde{P} \in \mathcal{P}(\mathcal{G})$

If $E = \mathbf{R}^d$, δ is the Euclidean metric in $\mathbf{R}^d$, and $\sigma(X) \subset \mathcal{G}$, then (1) or (2) are equivalent to (3)

(3) $X^n \xrightarrow{\mathbf{P}} X$

Proof. Equivalence (1) $\Leftrightarrow$ (2) follows from linearity and normalization.

(1) $\Rightarrow$ (3). We know that $\mathbf{E}(\exp\{i(\lambda X + \mu X^n)\}) \to \mathbf{E}(\exp\{i(\lambda X + \mu X)\})$ for the real and continuous functions $\cos(\lambda x)$, $\cos(\mu x)$, $\sin(\lambda x)$, $\sin(\mu x)$ and the bounded and $\mathcal{G}$-measurable random variables $\cos(\lambda X)$, $\cos(\mu X)$, $\sin(\lambda X)$, $\sin(\mu X)$.

From this fact it follows that

$$(X, X^n) \xrightarrow{\mathcal{L}(\mathbf{P})} (X, X) \quad \text{and} \quad X^n - X \xrightarrow{\mathbf{P}} 0.$$

(3) $\Rightarrow$ (2). If $\tilde{\mathbf{P}} \in \mathcal{P}$ then $\mathbf{P} \sim \tilde{\mathbf{P}}$. So $X^n \xrightarrow{\tilde{\mathbf{P}}} X$ and $X^n \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} X$.

Proof of Theorem 1. According to Lemma 1 it is suffices to verify

$$([X^n, X^n], X^n) \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} ([X,X], X),$$

where $\tilde{\mathbf{P}} \in \mathcal{P}$.

We know that the quadratic variation is invariant under an absolutely continuous change of measure, so it is natural to apply Theorem VI.6.1 of (Jacod and Shiryaev, 1987) under $\tilde{\mathbf{P}}$. Let us verify its hypothesis. We know

$$X^n \xrightarrow{\mathcal{L}(\tilde{\mathbf{P}})} X$$

((1) $\Rightarrow$ (2) in Lemma 1). In order to prove $(Var(B^n)_t|\tilde{\mathbf{P}})$ is tight, let us fix an $\varepsilon > 0$. As $\tilde{\mathbf{P}} \sim \bar{\mathbf{P}}$, there exists a $\delta > 0$ such that

$$\bar{\mathbf{P}}(A) < \delta \Rightarrow \tilde{\mathbf{P}}(A) < \varepsilon, \quad A \in \mathcal{F}.$$

Let K_ε be such that

$$\bar{\mathbf{P}}(Var(B^n)_t > K_\varepsilon) < \delta \quad \forall n \in \mathbb{N}$$

then

$$\tilde{\mathbf{P}}(Var(B^n)_t > K_\varepsilon) < \varepsilon \quad \forall n \in \mathbb{N}$$

Now the application of Theorem VI.6.1 of (Jacod and Shiryaev, 1987) completes the proof

$$([X^n, X^n], X^n) \xrightarrow{\mathcal{L}(\tilde{P})} ([X, X], X)$$

Proof of Corollary 1. In view of Theorem 1 and implication (1) $\Rightarrow$ (3) in Lemma 1 the result follows from the fact that C_t is $\mathcal{F}$-measurable.

Proof of Theorem 2. (1) As X is continuous,

$$\sup_{s \leq t} |\Delta X_s^n| \xrightarrow{\mathbf{P}} 0 \tag{2.1}$$

which is equivalent to $[\delta - \mathbf{R}^+]$ in virtue of the Lenglart-Rebolledo inequalities (see (Jacod and Shiryaev, 1987) section VI.4.22).

(2) Let

$$X^n = M^n + B^n + \check{X}^n(h)$$

be the semimartingale decomposition of X^n:

$\check{X}^n(h) = \sum_{s \leq t} \Delta X_s^n - h(\Delta X_s^n)$

M^n is a local martingale with bounded jumps and $M_0^n = 0$,

B^n is a predictable process with finite variation in intervals.

By (2.1) we know $\mathbf{P}(\sup_{s \leq t} |\check{X}^n(h)_s| > 0) \to 0$

Furthermore on the set $\{\sup_{s \leq t} |\Delta \check{X}_s^n| = 0\}$ we have

$$[X^n, X^n] = [M^n, M^n] + [B^n, B^n] + 2[M^n, B^n]$$

and

$$\sup_{s \leq t} |[X^n, X^n] - [M^n, M^n]| \leq Var(B^n)_t \sup_{s \leq t} (|\Delta B_s^n| + 2|\Delta M_s^n|).$$

We know that $Var(B^n)_t$ is tight. Let us first check that

$$\sup_{s\le t}|\Delta B^n_s| \xrightarrow{\mathbf{P}} 0. \tag{2.2}$$

Observe that

$$|\Delta B^n_s| = h(x) * \nu^n(\{s\}, dx) \le \varepsilon + \Delta \mathbb{I}_{\{|x|>\varepsilon\}} * \nu^n_s \le \varepsilon + \mathbb{I}_{\{|x|>\varepsilon\}} * \nu^n_t$$

(assuming $h(x) = x$ on the set $\{|x| < \varepsilon\}$ and $|h(x)| \le 1$). So

$$\sup_{s\le t}|\Delta B^n_s| \le \varepsilon + \mathbb{I}_{\{|x|>\varepsilon\}} * \nu^n_t$$

wich yields (2.2) in virtue of $[\delta - \mathbf{R}^+]$.

Let us then prove

$$\sup_{s\le t}|\Delta M^n_s| \xrightarrow{\mathbf{P}} 0 \tag{2.3}$$

We have

$$\Delta M^n_s = \Delta X^n_s - \Delta B^n_s - \Delta \check{X}(h)_s = h(\Delta X^n_s) - \Delta B^n_s$$

so, as $|h(x)| \le |x|$,

$$|\Delta M^n_s| \le |\Delta X^n_s| + |\Delta B^n_s|$$

and (2.3) follows.

Thus, we have then proved that:

$$\sup_{s\le t}|[X^n, X^n] - [M^n, M^n]| \xrightarrow{\mathbf{P}} 0$$

and as Corollary 1 gives $[X^n, X^n]_t \xrightarrow{\mathbf{P}} C_t$ we obtain $[M^n, M^n]_t \xrightarrow{\mathbf{P}} C_t$. Now $[\gamma - \mathbf{R}^+]$ follows from a standard argument given in (Rebolledo, 1980) that establishes, that under (2.1)

$$\sup_{s\le t}| < M^n, M^n > -[M^n, M^n]| \xrightarrow{\mathbf{P}} 0. \tag{2.4}$$

Proof of Corollary 2. Let us first take a truncation function h that satisfies $h(x) = x$ for $|x| \le c$: we then have $B^n(h) = 0$.

(1) $\Rightarrow$ (2) is a particular case of Corollary 1.

(3) $\Rightarrow$ (1) is a particular case of Feigin's Theorem (1985), see also section VIII.5.42 in (Jacod and Shiryaev, 1987).

(2) $\Rightarrow$ (3). Since (2.1) $\forall t \in \mathbf{R}^+$ is equivalent to $[\delta - \mathbf{R}^+]$ and under (2.1) convergence (2.4) takes place, it is enough to prove that (2) $\Rightarrow$ (2.1).

Now it follows from (2) that

$$\sup_{s\le t}|\Delta[X^n, X^n]_s| \xrightarrow{\mathbf{P}} 0$$

Since $|\Delta X^n|^2 = \sum_{i=1}^d \Delta[X^{n,i}, X^{n,i}]$, (2.1) holds.

Aknowledgements:. The author is indebted to A.N. Shiryaev, D.O. Kramkov and A.A. Gushchin for helpful discussions.

References

Feigin P.D. (1985), *Stable convergence of semimartingales*, Stochastic Process. Appl. **19**, 125–134.

Jacod J. and Shiryaev A.N. (1987), *Limit theorems for stochastic processes*, Springer-Verlag, Berlin, Heidelberg, New York.

Liptser R.S. and Shiryaev A.N. (1983), *On weak convergence of semimartingales to stochastically continuous processes with independent and conditionally independent increments*, Math. USSR Sbornik **44**, 299–323.

Mordecki E. (1993), *About necessary conditions for stable convergence of semimartingales*, Uspekhi Mat. Nauk **48**, 193–194.

Rebolledo R. (1980), *Central limit theorems for local martingales*, Z. Wahrscheinlichkeitstheor. verw. Geb. **51**, 269-286.