Ruin probabilities for Lévy processes with mixed-exponential negative jumps *

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To José Luis Massera (1915 - 2002)

In Memoriam

Abstract

Closed form of the ruin probability for a Lévy processes, possible killed at a constant rate, with arbitrary positive, and mixed exponentially negative jumps is given.

Keywords: Ruin probability, closed form, Lévy process, mixed-exponential distributions.

1 Introduction

1.1 Let $X = \{X_t\}_{t\geq 0}$ be a real valued stochastic process defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ that satisfies the usual conditions. Assume that X is càdlàg, adapted, $X_0 = 0$, and for $0 \leq s < t$ the random variable $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$ with a distribution that only depends on the difference $t - s$. The stochastic process X is a process with stationary independent increments (PIIS), or a Lévy process. For $q \in \mathbf{R}$, $\psi(q)$ denotes the characteristic exponent of X given by Lévy-Khinchine formula

$$\psi(q) = \frac{1}{t}\log E(e^{iqX_t}) = ibq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbf{R}} (e^{iqy} - 1 - iqy\mathbf{1}_{\{|y|<1\}})\Pi(dy)$$

where b and $\sigma \geq 0$ are real constants, and Π is a positive measure on $\mathbf{R} - \{0\}$ such that $\int(1 \wedge y^2)\Pi(dy) < \infty$, called the Lévy measure. The function $\psi(q)$,

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$q \in \mathbf{R}$ completely determines the law of the process. For general reference on the subject see Jacod and Shiryaev (1987), Skorokhod (1991), or Bertoin (1996).

1.2 Consider now a Lévy process with measure Π given by

$$\Pi(dy) = \begin{cases} \pi(dy) & \text{if} \quad y > 0, \\ \\ \lambda \sum_{k=1}^{n} a_k \alpha_k e^{\alpha_k y} dy & \text{if} \quad y < 0, \end{cases} \tag{1}$$

where π is an arbitrary Lévy measure concentrated on $(0, \infty)$, $0 < \alpha_1 < \ldots < \alpha_n$, $a_k > 0$, for $k = 1, \ldots, n$ and $\sum_{i=1}^{n} a_k = 1$. The magnitude of the negative jumps of X is mixed exponentially distributed, with parameter α_k chosen with probability a_k, and they occur at poissonian times with rate λ. As the process considered has a finite number of negative jumps on $[0, t]$, we consider a truncation function

$$h(y) = y\mathbf{1}_{\{0<y<1\}}. \tag{2}$$

Simple computations give

$$\psi(q) = iaq - \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{iqy} - 1 - iqh(y))\pi(dy) - \lambda \sum_{k=1}^{n} a_k \frac{iq}{\alpha_k + iq}, \tag{3}$$

with

$$a = b + \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k} [1 - \frac{1 + \alpha_k}{e^{\alpha_k}}]$$

We introduce now the Laplace exponent of X. For $p \in \mathbf{R}$, $p \leq 0$, $p \neq -\alpha_k$, for $k = 1, \ldots, n$, denote

$$\kappa(p) = ap + \frac{1}{2}\sigma^2 p^2 + \int_0^\infty (e^{py} - 1 - ph(y))\pi(dy) - \lambda \sum_{k=1}^{n} a_k \frac{p}{\alpha_k + p}. \tag{4}$$

Considered as a function with complex domain, the characteristic exponent $iq \mapsto \psi(q)$ in (3) can be extended analytically to the complex strip $\{z = p + iq\colon p \in (-\alpha_1, 0]\}$ and, for $-\alpha_1 < p \leq 0$, we have $\kappa(p) = \psi(p)$. As the Laplace transform of X_t when $-\alpha_1 < p \leq 0$ is given by $E(e^{pX_t}) = e^{t\kappa(p)}$, following Bertoin (1996) we call

$$\kappa(p) = \frac{1}{t} \log E(e^{pX_t}) \tag{5}$$

the Laplace exponent of X.

Let us examine the roots of the equation

$$\kappa(p) = 0, \quad \text{for} \quad p \leq 0.$$

As the integrand in (4) is convex in p for fixed y, we obtain that $\kappa(p)$ is convex on $(-\alpha_1, 0]$. From this follows when $\sigma > 0$, that under the condition

$$\kappa'(0-) = \lim_{p \to 0-} \frac{1}{p}\kappa(p) = a + \int_1^\infty y\pi(dy) - \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k} > 0, \tag{6}$$

(where the integral can take the value ∞) the equation $\kappa(p) = 0$ has $n+1$ negative roots $-p_j$, $j = 1, \dots, n+1$, that satisfy

$$0 < p_1 < \alpha_1 < p_2 < \ldots < p_n < \alpha_n < p_{n+1}. \tag{7}$$

Furthermore, observe that when $\gamma > 0$, equation $\kappa(p) = \gamma$ has always $n+1$ negative roots $-p_j$ satisfying (7).

Denote by $\tau(\gamma)$ an exponential random variable with parameter $\gamma > 0$, independent of X, and for $\gamma = 0$, put $\tau(0) = \infty$. We are interested in the random variables

$$M = \sup_{0 \le t < \tau(\gamma)} X_t \qquad \text{and} \qquad I = \inf_{0 \le t < \tau(\gamma)} X_t \tag{8}$$

called the supremum and infimum of the process, killed at rate γ if $\gamma > 0$ (see Bertoin (1996)).

Next result gives a closed formula for the ruin probability,

$$R(x) = P(\exists t \in [0, \tau(\gamma)) \colon x + X_t \le 0) = P(I \le -x), \quad x \ge 0, \tag{9}$$

for the process with characteristic exponent given by (3).

Theorem 1.1 *Let X be a Lévy process with characteristic exponent given by (3) and $\sigma > 0$.*

(a) Assume that condition (6) holds. Then, $P(\lim_{t\to\infty} X_t = \infty) = 1$, i.e. the process drifts to infinity.

(b) Let $\gamma \ge 0$. Assume that condition (6) holds when $\gamma = 0$. Then, the ruin probability (9) is given by

$$R(x) = \sum_{j=1}^{n+1} A_j e^{-p_j x}, \quad x \ge 0, \tag{10}$$

where $-p_1, \dots, -p_{n+1}$ are the negative roots of the equation $\kappa(p) = \gamma$, and the constants $A_1, \dots, A_{n+1}$ are given by

$$A_j = \frac{\prod_{k=1}^{n} \left(1 - \frac{p_j}{\alpha_k}\right)}{\prod_{k=1, k\neq j}^{n+1} \left(1 - \frac{p_j}{p_k}\right)}, \quad j = 1, \dots, n+1. \tag{11}$$

Remark: As noticed by the referee (and can be confirmed reading the proof), the existence of a real negative root in the interval $(-\infty, -\alpha_n)$ ensures the result (10) also in the case $\sigma = 0$.

1.3 Example. In the following example, the distribution of the supremum and infimum are obtained simultaneously, when jumps are completely specified. Let the process $X = \{X_t\}_{t\ge 0}$ be given by

$$X_t = at + \sigma B_t + \sum_{k=1}^{N_t} Y_k \tag{12}$$

where $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion, $N = \{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ, $Y = \{Y_k\}_{k\in\mathbf{N}}$ is a sequence of independent and identically distributed random variables with density

$$d(y) = \begin{cases} \sum_{k=1}^{m} b_k\beta_k e^{-\beta_k y} & \text{if} \quad y > 0, \\ \\ \sum_{k=1}^{n} a_k\alpha_k e^{\alpha_k y} & \text{if} \quad y < 0, \end{cases} \tag{13}$$

with $0 < \alpha_1 < \ldots < \alpha_n$, $0 < \beta_1 < \ldots < \beta_m$, a_k and b_k strictly positive for all k, and $\sum_{k=1}^{m} b_k + \sum_{k=1}^{n} a_k = 1$. The processes B, N, and Y are independent. The Laplace exponent (5) is given by

$$\kappa(p) = ap + \frac{1}{2}\sigma^2 p^2 + \lambda\sum_{k=1}^{m} b_k \frac{p}{\beta_k - p} - \lambda\sum_{k=1}^{n} a_k \frac{p}{\alpha_k + p}.$$

A simple application of Theorem 1.1 to X and $-X$ gives the following

Corollary 1.1 *Let X be given in (12).*
(a) For $\gamma > 0$, the infimum I in (8) has density given by

$$f_I(x) = \sum_{k=1}^{n+1} A_k p_k e^{p_k x}, \qquad x \leq 0, \tag{14}$$

where $-p_1, \ldots, -p_{n+1}$ are the negative roots of the equation $\kappa(p) = \gamma$ and $A_1, \ldots, A_{n+1}$ are given by (11).
(b) For $\gamma > 0$, the supremum M in (8) has density given by

$$f_M(x) = \sum_{k=1}^{m+1} B_k r_k e^{-r_k x}, \qquad x \geq 0, \tag{15}$$

where $r_1, \ldots, r_{m+1}$ are the positive roots of the equation $\kappa(p) = \gamma$ and the coefficients are given by

$$B_j = \frac{\prod_{k=1}^{m}\left(1 - \frac{r_j}{\beta_k}\right)}{\prod_{k=1,k\neq j}^{n+1}\left(1 - \frac{r_j}{r_k}\right)}, \quad j = 1, \ldots, m+1.$$

(c) If $\kappa'(0) > 0$ (or $\kappa'(0) < 0$) the process drifts to ∞ (or to $-\infty$), and the density of I (of M) in (8) with $\gamma = 0$ is given by (14) (or (15)).

Remark: (a) and (b) in Corollary 1 can also be obtained from Rogozin (1966) by fractional expansion. In case of Theorem 1.1, in order to use this factorization methods, complex variable results must be considered in order to count and locate the roots of $\psi(z) = \gamma$.

1.4 Related results to Theorem 1.1 and applications can be found in Asmussen (1995). The distribution of the infimum of a process with no negative jumps was found by Zolotarev (1964), see also Prabhu (1980) or Bertoin (1996) and the references therein. Results giving double Laplace transform of finite time ruin can be found in Baxter and Donsker (1952), based on combinatorial methods and complex analysis. Rogozin (1966) gives a factorization concerning the ruin of a killed process, extending combinatorial identities for random walks by Spitzer (1956). Theorem 1.1 generalizes previous work in case of a Wiener process with negative exponential jumps (Mordecki (1997)) and of a Lévy process with negative exponential jumps (Mordecki (2000)). Within the possible applications, the results are used to give closed solutions to optimal stopping problems for Lévy processes (Mordecki (2002)).

The method of proof is direct, along the lines of the barrier problem for simple random walks, consisting in finding a martingale with value 1 in the stopping region that vanishes if this region is not attained.

2 Proofs

First, introduce some notation and necessary facts. The infinitesimal generator of the process X with exponent in (3) is

$$\begin{aligned}(L^X f)(x) = af'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_0^\infty \Big(f(x+y) - f(x) - f'(x)h(y)\Big)\pi(dy) \\ +\lambda \sum_{k=1}^n a_k \int_{-\infty}^0 \Big(f(x+y) - f(x)\Big)\alpha_k e^{\alpha_k y} dy \end{aligned} \tag{16}$$

with h in (2). The jump measure of X is

$$\mu^X = \mu^X(\omega, dt, dy) = \sum_s \mathbf{1}_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s)}(dt, dy).$$

Its compensator is a deterministic measure given by

$$\nu = \nu(dt, dy) = dt\Pi(dy)$$

with Π in (1). The process $M = \{M_t\}_{t\geq 0}$ given by

$$M_t = \sigma B_t + \int_{[0,t]\times\mathbf{R}} h(y)(\mu^X - \nu),$$

where $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion, is a martingale with no negative jumps, and X has the decompositions

$$X_t = M_t + at + \int_{[0,t]\times\mathbf{R}} (y - h(y))\mu^X \tag{17}$$

$$= M_t + at + \sum_{0<s\le t} \Delta X_s \mathbf{1}_{\{\Delta X_s \ge 1\}} - \sum_{k=1}^{N_t} Y_k, \tag{18}$$

where $N = \{N_t\}_{t\ge 0}$ is a Poisson process with parameter λ, $Y = \{Y_k\}_{k\in\mathbf{N}}$ is a sequence of positive identically distributed random variables with density

$$d(y) = \sum_{i=1}^{n} a_k \alpha_k e^{-\alpha_k y}.$$

and B, N and Y are independent.

The following result gives (20), the integro-differential equation that the ruin probability must satisfy.

Lemma 2.1 *(a) Let*

$$0 < p_1 < \alpha_1 < p_2 < \ldots < p_n < \alpha_n < p_{n+1},$$

and define the constants

$$A_j = \frac{\prod_{k=1}^{n} \left(1 - \frac{p_j}{\alpha_k}\right)}{\prod_{k=1,k\ne j}^{n+1} \left(1 - \frac{p_j}{p_k}\right)}, \quad j = 1, \ldots, n+1.$$

Then

$$\begin{cases} \sum_{j=1}^{n+1} A_j = 1, \\ \sum_{j=1}^{n+1} A_j \frac{p_j}{p_j - \alpha_k} = 0, \quad k = 1, \ldots, n. \end{cases} \tag{19}$$

(b) For the infinitesimal generator L^X in (16), the function $R(x)$ in (10), where now $-p_1, \ldots, -p_{n+1}$ are the negative roots of the equation $\kappa(p) = \gamma$ (such that (7) hold), and $x > 0$, we have

$$(L^X R)(x) = \gamma R(x) \tag{20}$$

with $R(x) = 1$ if $x < 0$.

Proof. (a) By the Theorem on simple fractional expansion, there exist unique constants $A_1, \ldots, A_{n+1}$ such that

$$\frac{\prod_{j=1}^{n} \left(1 + \frac{p}{\alpha_j}\right)}{\prod_{j=1}^{n+1} \left(1 + \frac{p}{p_j}\right)} = \sum_{j=1}^{n+1} A_j \frac{p_j}{p_j + p}. \tag{21}$$

Each equation of the linear system (19) is equation (21) evaluated at $p = 0$ and $p = -\alpha_k$ for $k = 1, \ldots, n$, respectively. On the other side, the formula for A_j $(j = 1, \ldots, n+1)$ is obtained by standard methods, i.e. multiply (21) by $(p_j + p)/p_j$ and substitute $p = p_j$.

(b) Denote $f_p(x) = e^{-px^+}$ where $x^+ = \max(x,0)$. As $\sum_{j=1}^{n+1} A_j = 1$, we can write $R(x) = \sum_{j=1}^{n+1} A_j f_{p_j}(x)$. For $x > 0$,

$$(L^X f_p)(x) = e^{-px}\Big(\frac{1}{2}\sigma^2 p^2 - ap + \int_0^\infty (e^{-py} - 1 + ph(y))\pi(dy)\Big) +$$

$$+\lambda \sum_{k=1}^n a_k \int_{-\infty}^0 (e^{-p(x+y)^+} - e^{-px})\alpha_k e^{\alpha_k y} dy$$

$$= e^{-px}\Big(\frac{1}{2}\sigma^2 p^2 - ap + \int_0^\infty (e^{-py} - 1 + ph(y))\pi(dy) + \lambda \sum_{k=1}^n a_k \frac{p}{\alpha_k - p}\Big)$$

$$+\lambda \sum_{k=1}^n a_k e^{-\alpha_k x} \frac{p}{p - \alpha_k} = e^{-px}\kappa(-p) + \lambda \sum_{k=1}^n a_k e^{-\alpha_k x} \frac{p}{p - \alpha_k}$$

In conclusion, for $x > 0$,

$$L^X R(x) = \sum_{k=1}^{n+1} A_j (L^X f_{p_j})(x)$$

$$= \sum_{j=1}^{n+1} A_j e^{-p_j x}\kappa(-p_j) + \lambda \sum_{k=1}^n a_k e^{-\alpha_k x} \sum_{j=1}^{n+1} A_j \frac{p_j}{p_j - \alpha_k} = \gamma R(x),$$

because $\kappa(-p_j) = \gamma$ for $j = 1, \ldots, n+1$, and (19). $\square$

Next Lemma summarizes the application of Meyer-Itô formula to the process X with ψ in (3) and the function R in (10). Denote $Y = \{Y_t = x + X_t\}_{t \geq 0}$.

Lemma 2.2 *Let $\tau_0 = \inf\{t \geq 0 \colon x + X_t \leq 0\}$. For R in (10)*

$$R(Y_{t\wedge\tau_0}) - R(x) = \int_0^{t\wedge\tau_0} (L^X R)(Y_{s-})ds + M(R)_{t\wedge\tau_0} \tag{22}$$

with L^X in (16) and $M(R) = \{M(R)_t\}_{t\geq 0}$ given by

$$M(R)_t = \int_0^t R'(Y_{s-})dM_s + \int_{[0,t]\times\mathbf{R}} W(s,y)(\mu^X - \nu), \tag{23}$$

is a local martingale, where $W(s,y) = R(Y_{s-} + y) - R(Y_{s-}) - R'(Y_{s-})h(y)$.

Proof. As the function R is not $\mathbf{C}^2(\mathbf{R})$ we will apply Meyer-Itô formula (IV.51 in Protter (1992)) denoting by $m(da)$ the signed measure that is the second derivative of R in the sense of distributions, when restricted to compacts:

$$R(Y_t) - R(x) = \int_0^t R'(Y_{s-})dY_s + \sum_{0<s\leq t} \Big(R(Y_s) - R(Y_{s-}) - R'(Y_{s-})\Delta Y_s\Big)$$

$$+\frac{1}{2}\int_{\mathbf{R}} l^Y(a,t)m(da) \tag{24}$$

where $l^Y(a,t)$ is the local time of the process Y at level a and time t.

First, observe that

$$m(da) = R''(a)da + R'(0+)\delta_0(da)$$

with $R''(a)$ the second derivative of R if $a \neq 0$, $R'(0+)$ the right derivative of R at the point $x = 0$ and $\delta_0(da)$ the point mass at $x = 0$. Applying Corollary 1 to Theorem IV.51 of Protter (1992),

$$\int_{\mathbf{R}} l^Y(a,t)m(da) = \int_0^t R''(Y_{s-})d\left\langle Y^c, Y^c\right\rangle_s + R'(0+)l^Y(0,t). \tag{25}$$

In view of (17) and (25), as $d\left\langle Y^c, Y^c\right\rangle_s = \sigma^2 ds$, (24) gives

$$\begin{aligned} R(Y_t) - R(x) &= \int_0^t R'(Y_{s-})dM_s + \int_0^t \Big(\frac{1}{2}\sigma^2 R''(Y_{s-}) + aR'(Y_{s-})\Big)ds \\ &+\frac{1}{2}R'(0+)l^Y(0,t) + \sum_{0<s\leq t} W(s,\Delta X_s) \end{aligned} \tag{26}$$

Now, we compensate the jumps in the last term. As R is bounded, the left side in (26) is a special semimartingale. On the right side, the first term is a local martingale and the second and third are continuous processes, and in consequence, locally integrable. We can apply II.1.28 in Jacod and Shiryaev (1978), obtaining

$$\begin{aligned} \sum_{0<s\leq t} W(s,\Delta X_s) &= \int_{[0,t]\times\mathbf{R}} W\mu^X = \int_{[0,t]\times\mathbf{R}} W(\mu^X-\nu) + \int_{[0,t]\times\mathbf{R}} W\nu \\ &= \int_{[0,t]\times\mathbf{R}} W(\mu^X-\nu) + \int_0^t \Big[\int_{\mathbf{R}} W(s,y)\Pi(dy)\Big]ds. \end{aligned} \tag{27}$$

In order to evaluate (26) at τ_0 observe that $l^Y(0,\tau_0) = 0$ because $x + X_t > 0$ on the set $[0,\tau_0)$ and the local time is continuous in t (p. 165 in Protter (1992)). In view of (26) and (27), (22) follows. □

Proof of Theorem 1.1. (a) In order to see that

$$P(\lim_{t\to\infty} X_t = \infty) = 1 \tag{28}$$

consider the decomposition (18). Denote $d = \lambda\sum_{k=1}^n \frac{a_k}{\alpha_k}$. By (6) exists $\varepsilon > 0$ such that

$$a + \int_1^\infty y\pi(dy) - (d+\varepsilon) > 0. \tag{29}$$

Consider $X^1 = \{X_t^1\}_{t\geq 0}$ and $X^2 = \{X_t^2\}_{t\geq 0}$ with

$$X_t^1 = at + M_t + \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s \geq 1\}} - (d+\varepsilon)t$$

$$X_t^2 = (d+\varepsilon)t - \sum_{k=1}^{N_t} Y_k$$

On one side, as X^1 has only positive jumps, by (29) and VII.1.2 in Bertoin (1996) we deduce $P(\lim_{t\to\infty} X_t^1 = \infty) = 1$. On the other side, as $E(X_t^2) = \varepsilon t > 0$, we have $P(\lim_{t\to\infty} X_t^2 = \infty) = 1$ and conclude (28).
Proof of (b). According to Lemma 2.2, $\{R(Y_{t\wedge\tau_0})\}$ is a semimartingale. So the quadratic covariation

$$[R(Y_{t\wedge\tau_0}), e^{-\gamma t}] = 0$$

because the deterministic process $\{e^{-\gamma t}\}$ is continuous and has bounded variation. This gives

$$e^{-\gamma(t\wedge\tau_0)} R(Y_{t\wedge\tau_0}) - R(x)$$

$$= \int_0^{\tau_0\wedge t} e^{-\gamma s} dR(Y_{s\wedge\tau_0}) - \int_0^{\tau_0\wedge t} R(Y_{s\wedge\tau_0 -})\gamma e^{-\gamma s}) ds$$

$$= \int_0^{\tau_0\wedge t} e^{-\gamma s}[L^X R(Y_{s-}) - \gamma R(Y_{s-})] ds + \int_0^{\tau_0\wedge t} e^{-\gamma s} dM(R)_s$$

$$= \int_0^{\tau_0\wedge t} e^{-\gamma s} dM(R)_s$$

with $M(R)$ in (23), by Lemma 2.1 (b), as $Y_{s-} = x + X_{s-} > 0$ on $[0, \tau_0]$ and (20). As the function R is bounded, taking expectations and limits as $t \to \infty$

$$R(x) = E(e^{-\gamma\tau_0} R(x + X_{\tau_0})) = E(e^{-\gamma\tau_0}) = P(\tau_0 < \tau(\gamma))$$

$$= P(\exists t \in [0, \tau(\gamma))\colon x + X_t \leq 0)$$

concluding the proof. $\square$

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