Simulación Monte Carlo y procesos estocásticos: Segunda entrega*

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Contents

1 Variance reduction techniques

In the following exercises, when not specified confidence level is $1-\alpha = 0.95$ and sample size $n = 10^6$. Present your results in a table with values obtained and the corresponding error of estimation for each method used, computed as

$$\epsilon = \frac{1.96s}{\sqrt{n}}.$$

where $s = \hat{\sigma}$.

1. We want to estimate by simulation:

$$\mu = \int_0^1 \frac{1}{2\sqrt{x+x^2}} dx = \int_0^1 \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1+x}} dx.$$

In all cases provide the 95% error of estimation.

*Fecha límite para la entrega de ejercicios: Viernes 1 de diciembre (2017)

(a) Estimate μ using uniform random variables, by the sample mean method.

(b) Estimate μ using random variables with density $f(x) = 1/(2\sqrt{x})$ for $0 \le x \le 1$.

(c) Use antithetic variables.

(d) Use control variates, with $g(x) = 1 - x$. First find the optimal c^* with a small sample (10^3) and then run your algorithm with this estimation.

2. We want to compute the integral by simulation:

$$\mu = \int_0^1 (1-x)e^{-x^2} dx$$

In all cases provide the 95% error of estimation.

(a) Estimate μ using the acceptance-rejection method on the square $[0,1]^2$.

(b) Estimate μ using uniform random variables, by the sample mean method.

(c) Estimate μ using random variables with density $f(x) = 2(1-x)$ in $0 \le x \le 1$.

(d) Use antithetic variables.

(e) Use control variates, with $g(x) = 1 - x$. First find the optimal c^* with a small sample (10^3) and then run your algorithm with this estimation.

The general purpose of the following exercises is to estimate using different methods of variance reduction the following quantities

$$\begin{aligned}
\mu_1 &= 4\int_0^1 \sqrt{1-x^2}dx = \pi, \\
\mu_2 &= \int_0^1 \sqrt{-\log x}dx = \frac{1}{2}\sqrt{\pi}, \\
\mu_3 &= \mathbf{P}(Z > 4), \text{ where } Z \sim \mathcal{N}(0,1) \\
\mu_4 &= \mathbf{E}(e^Z - 5)^+, \text{ where } Z \sim \mathcal{N}(0,1)
\end{aligned}$$

3. Compute μ_1 with the following methods:

(a) Acceptance rejection on the square $[0,1]^2$.

(b) Sample mean method.

(c) Use antithetic variables.

(d) Use control variates, with $g(x) = 1 - x$. First find the optimal c^* with a small sample (10^3) and then run your algorithm with this estimation.

(e) Use the stratified method, i.e. using, for $\mu = \int_0^1 h(x)dx$ the formula

$$\hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} h\left(\frac{U_j + j - 1}{n}\right).$$

You can combine with antithetic variates (In this case it is no direct to obtain an error estimate).

(f) Do you know a deterministic method to compute this integral? For instance, compute a Riemann sum, or use the trapezoidal rule:

$$\hat{\mu} = \frac{1}{2n}(f(0) + f(1)) + \frac{1}{n} \sum_{j=1}^{n-1} f(j/n).$$

4. Compute μ_2 with the following methods:

(a) Can you use acceptance rejection method with uniform variables?

(b) Use the sample mean method.

(c) Use antithetic variables.

(d) Use control variates, with $g(x) = -\log x$, taking into account that $\int_0^1 -\log(x)dx = 1$. First find the optimal c^* with a small sample (10^3) and then run your algorithm with this estimation.

5. Compute μ_3 with the following methods:

(a) First use crude MC. As the probability is very small, a large n is necessary.

(b) Check if the antithetic variates method improves the situation.

(c) Use the importance sampling method, based in the following idenantity.

$$\begin{aligned}\mathbf{P}(Z > 4) &= \int_4^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \int_4^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{e^{-(x-4)^2/2}}{e^{-(x-4)^2/2}} dx \\ &= \int_4^\infty e^{-4x+8} \frac{e^{-(x-4)^2/2}}{\sqrt{2\pi}} dx = \mathbf{E} e^{-4X+8} \mathbf{1}_{\{X>4\}}.\end{aligned}$$

where $X \sim \mathcal{N}(4, 1)$.

6. Compute μ_4 with the following methods:

(a) First use crude MC.

(b) Use antithetic variates.

(c) Now we use control variates in the following way. Check the following identity:

$$\mathbf{E}(e^Z - K)^+ = \mathbf{E}(e^Z) - K + \mathbf{E}(K - e^Z)^+.$$

(here $x^+ = \max(0, x)$, and you can use that $x = x^+ - (-x)^+$). Then, computing $\mathbf{E}e^Z = e^{1/2}$, we find the price of the *put option*, given by

$$P(K) = \mathbf{E}(K - e^Z)^+.$$

You can use also antithetic variates in this situation.

(d) Importance sampling can be implemented in the following way. Check the following identity:

$$\mathbf{E}(e^Z - K)^+ = \int_{\mathbf{R}} (e^x - K)^+ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{1/2} \int_{\mathbf{R}} (1 - Ke^{-x})^+ \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx.$$

that is known as *put-call duality*.

$$P(K) = e^{1/2}\mathbf{E}(1 - Ke^{-X})^+.$$

where $X \sim \mathcal{N}(1, 1)$. You can use also antithetic variates in this situation.

2 Stochastic Integrals

7. *Wiener integral* (a) Use Itô's formula to prove that

$$\int_0^T e^{W(t)-t/2} dt = e^{W(T)-T/2} - 1.$$

(b) Use the Euler scheme to simulate the integral

$$I = \int_0^1 e^{W(t)-t/2} dW(t).$$

Plot a histogram and estimate the expectation and the variance of I.

(c) Consider the random variable

$$J = e^{W(1)-1/2} - 1.$$

Plot a histogram, compute the expectation and variance of J.

(d) Plot the two histograms in the same figure and comment the results.

8. We consider the stochastic integral

$$I = \int_0^1 e^t dW(t).$$

(a) Compute $\mu = \mathbf{E}(I)$ and $\sigma^2 = \mathbf{var}(I)$.

(b) Simulate values of I using the Euler scheme for stochastic integrals and estimate approximately μ and σ^2.

(c) Plot in the same figure a histogram of the sample for I with the corresponding normal density.

9. *Itô isometry.* We check the isometry property in an example using simulation. Consider the process $\{h(t) = e^{W(t)} \colon 0 \leq t \leq 1\}$. The property states

$$\mathbf{E}\left(\int_0^1 e^{W(t)} dW(t)\right)^2 = \int_0^1 \mathbf{E}[(e^{W(t)})^2]dt = \int_0^T \mathbf{E}(e^{2W(t)})dt.$$

(a) First, using that $\mathbf{E}(e^{\mathcal{N}(\mu,\sigma^2)}) = e^{\mu+\sigma^2/2}$, compute $\int_0^1 \mathbf{E}(e^{2W(s)})ds$.

(b) Compute by simulation

$$\mathbf{E}\left(\int_0^1 e^{W(s)} dW(s)\right)^2, \quad 0 \leq t \leq 1,$$

with the corresponding error, and check that the numbers coincide.

10. We want to check numerically that

$$\int_0^1 W(t)dW(t) = \frac{1}{2}(W(1)^2 - 1).$$

Write then a code to compute a Brownian trajectory, compute both the integral and the result. Repeat the previous experiment a reasonable number of times, and plot the results in an (x, y) plot.

11. *Hermite polynomial of degree three.* (a) Write a code to simulate the integral

$$I = \int_0^1 (W(t)^2 - t)dW(t).$$

Plot a histogram and compute the expectation and the variance of I.

(b) Consider the random variable

$$J = \frac{1}{3}W(1)^3 - W(1).$$

Plot a histogram, compute the expectation and variance of J.

(c) Use Itô formula with the function[1] $H_3(t,x) = \frac{1}{3}x^3 - tx$ to prove that in fact $I = J$.

12. *Hermite polynomial of degree four.* Use Itô's formula to prove that

$$H_4(t, W(t)) = 12 \int_0^t H_3(s, W(s))dW(s)$$

Conclude that $\mathbf{E}H_4(t, W(t)) = 0$.

13. *Brownian Bridge.* (a) Let $\{W(t)\colon 0 \le t \le 1\}$ be a Brownian motion. Prove that the process

$$R(t) = (1-t)\int_0^t \frac{1}{1-r}dW(r), \quad 0 \le t \le 1,$$

is a Brownian bridge.

(b) We are interested in the random variable

$$A = \int_0^1 R(t)dt.$$

Prove that $\mathbf{E}(A) = 0$, and device a simulation scheme using the representation of part (a) to estimate $\mathbf{var}(A)$ (True value $1/12$).

14. *The Ornstein-Uhlenbeck process and its maximum.* Let $X = \{X(t)\colon 0 \le t \le 1\}$ be an OU process with parameters $a = 1, b = 0, \sigma = 1$. departing from $X_0 \sim \mathcal{N}(0,1)$. Compute by simulation a confidence interval of

$$\mu = \mathbf{E}\left(\max_{0\le t\le 1} X(t)\right).$$

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[1] $3H_3(t,x)$ is the Hermite polynomial of degree 3.

3 Stochastic differential equations

15. The *CIR process* is the solution to the SDE given by

$$dX(t) = a(b - X(t))dt + \sigma\sqrt{X(t)}dW(t).$$

Define the parameters

$$\alpha = \frac{2ab}{\sigma^2}, \text{ (shape paramater)},$$
$$\beta = \frac{2a}{\sigma^2}, \text{ (rate parameter)}.$$

It is known that the asymptotic distribution of the CIR process is a Gamma distribution, with density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0.$$

(a) Consider $x_0 = \alpha/\beta$, and the parameters

$$a = b = 1, \ \sigma = 0.5.$$

Use the Euler scheme to simulate trajectories of the CIR process

(b) Simulate 100 trajectories and consider the final values. Plot a histogram of this values and compare it in the same plot with the Gamma density with the corresponding parameters.

16. *Geometric Brownian motion.* A *look-back option* pays the *maximum* of the observed price of the stock. Its price is given by

$$L(K) = e^{-rT}\mathbf{E}(\max_{0 \leq t \leq T} S(t) - K)^+,$$

where $S(t) = S_0 \exp(\sigma W(t) + (r - \sigma^2/2)t)$ is a geometric Brownian motion.

Provide a simulation code using the Euler scheme for the SDE

$$dS(t) = S(t)[rdt + \sigma dW(t)],$$

to compute prices of look-back options for $S_0 = 100$, $r = 0.01$, $\sigma = 0.2$, $T = 1$.

17. Consider the Stochastic differential equation

$$dX(t) = \frac{1}{|X(t)|}dt + dW(t), \quad x_0 = 1.$$

(a) Write a code to simulate a trajectory in an interval $[0, 1]$

(b) Define

$$M = \max_{0 \le t \le 1} X(t).$$

and use your code to estimate $\mathbf{E}(M)$ and $\mathbf{var}(M)$.

(c) Plot a histogram with a sample of variables with the same distribution as M.