Clases 11-12: Integración estocástica. Fórmula de Itô [*]

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Índice

1. Introduction to Stochastic integrals

With the purpose of constructing a large class of stochastic processes, we consider *stochastic differential equations (SDE)* driven by a Brownian motion. Given then two functions:

$$b\colon \mathbf{R} \to \mathbf{R}, \quad \sigma\colon \mathbf{R} \to \mathbf{R},$$

a driving Brownian motion $\{W(t)\colon 0 \leq t \leq T\}$ and an independent random variable X_0 (the initial condition) our purpose is to construct a process $X = \{X(t)\colon 0 \leq t \leq T\}$ such that, for $t \in [0, T]$, the following equation is satisfied:

$$X(t) = X_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s).$$

Comments

- The differential notation for an SDE is

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = X_0.$$

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- The integral
$$\int_0^t b(X(s))ds$$
is a usual Riemann integral, as we expect $X(s)$ to be a continuous function
- As for fixed ω the trajectories of W are not smooth, we must define precisely the integral
$$\int_0^t \sigma(X(s))dW(s)$$

The three main properties to define the *stochastic integral* are:

(A) If X is independent from W
$$\int_a^b X dW(s) = X \int_a^b dW(s),$$

(B) The integral of the function $\mathbf{1}_{[a,b)}$ is the increment of the process:
$$\int_0^T \mathbf{1}_{[a,b)} dW(s) = \int_a^b dW(s) = W(b) - W(a).$$

(C) Linearity:
$$\int_0^T (f+g)dW(s) = \int_0^T f dW(s) + \int_0^T g dW(s).$$

2. Stochastic integration

Consider the class of processes
$$\mathcal{H} = \{h = (h(s))_{0 \le t \le T}\}$$
that satisfy (A) and (B):

(A) $h(t), W(t+h) - W(t)$ are independent, $\forall 0 \le t \le t+h \le T$,

(B) $\int_0^T \mathbf{E}(h(t)^2)dt < \infty$.

Example: If $\mathbf{E}[f(W(t))^2] \le K$, then $h(t) = f(W(t)) \in \mathcal{H}$: In fact,
$$f(W(t)), W(t+h) - W(t) \text{ are independent},$$
and
$$\int_0^T \mathbf{E}(f(W(t))^2)dt < KT.$$

Step processes

The stochastic integral is first defined for a subclass of *step processes* in $\mathcal{H}$: A *step processes* h is of the form

$$h(t) = \sum_{k=0}^{n-1} h_k \mathbf{1}_{[t_k, t_{k+1})}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0, T]$, and

(A) $h_k, W(t_k + h) - W(t_k)$ are independent for $1 \le k \le n$, $h > 0$.

(B) $\mathbf{E}(h_k^2) < \infty$.

For a step process $h \in \mathcal{H}$ we define the stochastic integral applying properties (A), (B) and (C):

$$\begin{aligned}
\int_0^T h(t)dW(t) &= \int_0^T \sum_{k=0}^{n-1} h_k \mathbf{1}_{[t_k, t_{k+1})}(t) dW(t) \\
&\overset{(C)}{=} \sum_{k=0}^{n-1} \int_0^T h_k \mathbf{1}_{[t_k, t_{k+1})}(t) dW(t) \\
&\overset{(A)}{=} \sum_{k=0}^{n-1} h_k \int_0^T \mathbf{1}_{[t_k, t_{k+1})}(t) dW(t) \\
&\overset{(B)}{=} \sum_{k=0}^{n-1} h_k [W(t_{k+1}) - W(t_k)].
\end{aligned}$$

Notation:

$$I(h) \overset{nt.}{=} \int_0^T h(t)dW(t).$$

Properties

The stochastic integral defined for step processes is a random variable that has the following properties

(P1) $\mathbf{E} \int_0^T h(t)dW(t) = 0$

(P2) Itô isometry:

$$\mathbf{E}\left(\int_0^T h(t)dW(t)\right)^2 = \int_0^T \mathbf{E}(h(t)^2)dt.$$

Proof of (P1):

We compute the expectation:

$$\mathbf{E}\left(\int_0^T h(t)dW(t)\right) = \mathbf{E}\left(\sum_{k=0}^{n-1} h_k[W(t_{k+1}) - W(t_k)]\right)$$
$$= \sum_{k=0}^{n-1} \mathbf{E}\left(h_k[W(t_{k+1}) - W(t_k)]\right)$$
$$\stackrel{(A)}{=} \sum_{k=0}^{n-1} \mathbf{E}(h_k)\mathbf{E}[W(t_{k+1}) - W(t_k)] = 0,$$

because $W(t_{k+1}) - W(t_k) \sim \mathbf{N}(0, t_{k+1} - t_k)$.

Proof of (P2):

We first compute the square:

$$\left(\int_0^T h(t)dW(t)\right)^2 = \left(\sum_{k=0}^{n-1} h_k[W(t_{k+1}) - W(t_k)]\right)^2$$
$$= \sum_{k=0}^{n-1} h_k^2[W(t_{k+1}) - W(t_k)]^2$$
$$+ 2\sum_{0 \le j < k \le n-1} h_j h_k[W(t_{j+1}) - W(t_j)][W(t_{k+1}) - W(t_k)]$$

We have, by independence, as $t_j < t_{j+1} \le t_k$:

$$\mathbf{E}\left(h_j h_k[W(t_{j+1}) - W(t_j)][W(t_{k+1}) - W(t_k)]\right)$$
$$= \mathbf{E}\left(h_j h_k[W(t_{j+1}) - W(t_j)]\right)\mathbf{E}[W(t_{k+1}) - W(t_k)] = 0.$$

Furthermore, also by independence:

$$\mathbf{E}\left(h_k^2[W(t_{k+1}) - W(t_k)]^2\right) = \mathbf{E}(h_k^2)\mathbf{E}[W(t_{k+1}) - W(t_k)]^2$$
$$= \mathbf{E}(h_k^2)(t_{k+1} - t_k) = \int_{t_k}^{t_{k+1}} \mathbf{E}(h(s)^2)ds.$$

Summarizing,

$$\mathbf{E}\left(\int_0^T h(t)dW(t)\right)^2 = \sum_{k=0}^{n-1} \mathbf{E}\left(h_k[W(t_{k+1}) - W(t_k)]\right)^2$$
$$= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbf{E}(h(s)^2)ds = \int_0^T \mathbf{E}(h(s)^2)ds,$$

concluding the proof of (P2). This property is an *isometry*, because it can be stated as:

$$\|I(h)\|^2_{L^2(\Omega)} = \|h\|^2_{L^2(\Omega\times[0,T])},$$

where

$$\|I(h)\|^2_{L^2(\Omega)} = \mathbf{E}(I(h)^2), \quad \|h\|^2_{L^2(\Omega\times[0,T])} = \int_0^T \mathbf{E}(h(s)^2)ds,$$

This makes possible to extend the integral to the whole set $\mathcal{H}$ by approximation:

- Given $h \in \mathcal{H}$ we find a sequence of steps processes (h_n) such that

$$\|h_n - h\|^2_{L^2(\Omega\times[0,T])} \to 0, \ (\ell \to \infty).$$

- We define

$$I(h) = \text{lím}\, I(h_n) \quad \text{as } n \to \infty.$$

It is necessary to prove that $I(h_n)$ is a Cauchy sequence in $L^2(\Omega)$ to prove that the limit exists.

Comments

- $I(h) = \int_0^T h(t)dW(t)$, is a random variable.
- For $0 \le t \le T$ we define

$$I(h,t) = \int_0^T \mathbf{1}_{[0,t)} h(t)dW(t) \overset{nt.}{=} \int_0^t h(s)dW(s),$$

  to obtain a *stochastic process*.

Covariance of two stochastic integrals

Proposition 1. *Consider two process h and g in the class $\mathcal{H}$. Then*

$$\mathbf{E}\left(\int_0^T g(t)dW(t)\int_0^T h(t)dW(t)\right) = \int_0^T \mathbf{E}(g(t)h(t))dt.$$

The proof is based on the *polarization* identity:

$$ab = \frac{1}{4}[(a+b)^2 - (a-b)^2].$$

Demostración. Observe that

- $I(h+g) = I(h) + I(g)$,
- $\mathbf{E}(I(f)^2) = \int_0^T \mathbf{E}(f(t)^2)dt$.

Then

$$\begin{aligned}4\mathbf{E}(I(g)I(h)) &= \mathbf{E}(I(h+g)^2 - I(h-g)^2)\\ &= \int_0^T \mathbf{E}((h+g)^2 - (h-g)^2)dt\\ &= 4\int_0^T \mathbf{E}(hg)dt.\end{aligned}$$

□

3. Simulation of stochastic integrals: Euler scheme

The definition of a stochastic integral suggest a way to simulate an approximation of the solution. Given a process h, we choose n and we define a mesh $t_k^n = kT/n,\ k = 0, \ldots, n.$

$$\sum_{k=0}^{n-1} h(t_i^n)[W(t_{i+1}^n) - W(t_i^n)].$$

It is very important to estimate the integrand in t_i^n and to take the increment of W in $[t_i^n, t_{i+1}^n]$ to have the independence. Otherwise a wrong result is obtained.

Example: Wiener integrals

Suppose that $h(t)$ is a deterministic continuous function. Then, the approximating processes are also deterministic:

$$h_n = \sum_{k=0}^{n-1} h(t_i^n)\mathbf{1}_{[t_i^n, t_{i+1}^n)}.$$

So, we must find the limit of

$$I_n = \int_0^T h_n dW(t) = \sum_{k=0}^{n-1} h(t_i^n)[W(t_{i+1}^n) - W(t_i^n)].$$

As I_n is a sum of independent normal variables, it is normal. Furthermore (applying the properties or computing directly), we obtain that

$$\mathbf{E}(I_n) = 0, \quad \mathrm{var}(I_n) = \sum_{k=0}^{n-1} h(t_i^n)^2(t_{i+1}^n - t_i^n) \to \int_0^T h(t)^2 dt.$$

We conclude that

$$\int_0^T h(t)dW(t) \sim \mathbf{N}\left(0, \int_0^T h(t)^2 dt\right).$$

Simulation of a Wiener integral

Consider $f(t) = \sqrt{t}$ over $[0,1]$. According to our results:

$$\int_0^1 \sqrt{t} dW(t) \sim \mathbf{N}\left(0, \int_0^1 (\sqrt{t})^2 dt\right)$$

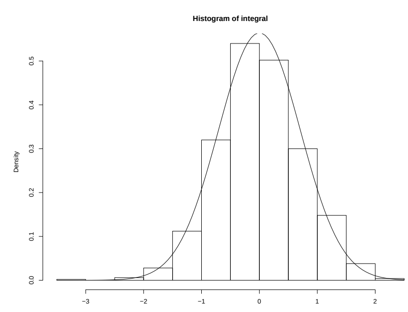
$$= \mathbf{N}\left(0, \int_0^1 t dt\right) = \mathbf{N}\left(0, 1/2\right).$$

We check this result by simulation, writing the following code:

```
# Wiener integral of sqrt(t)
n<-1e3 # discretization to simulate W
m<-1e3 # nr of variates
steps<-seq(0,1-1/n,1/n)
integral<-rep(0,m)
for(i in 1:m){
integral[i]<-sum(sqrt(steps)*rnorm(n,0,sqrt(1/n)))
}
>mean(integral) # Theoretical mean is 0
[1] 0.02194685
>var(integral) # Theoretical variance is 1/2
[1] 0.527872
```

We now plot our results and compare with the theoretical density $\mathbf{N}(0, 1/2)$:

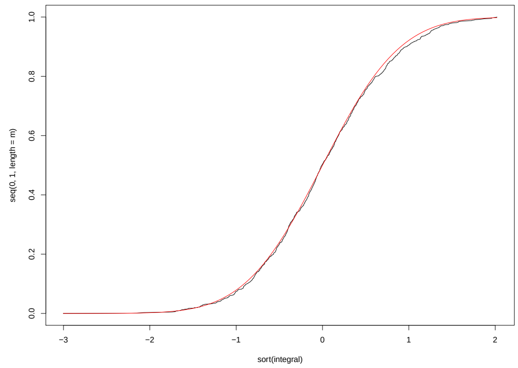
```
>hist(integral,freq=FALSE)
>curve(dnorm(x,mean=0,sd=sqrt(1/2)),add=TRUE)
```



Comparison of distributions

To perform a statistical comparison of our results, we compare the distributions:

```
plot(ecdf(integral),type="l")
curve(pnorm(x,0,sqrt(1/2)), add=TRUE, col=‘‘red")
```



To assess whether the empirical distribution (black) is not far from the theoretical (red) a Kolmogorov-Smirnov test should be performed.

```
>ks.test(integral,"pnorm",0,sqrt(1/2))
```

The result is

```
        One-sample Kolmogorov-Smirnov test
data: integral
D = 0.025065, p-value = 0.5562
alternative hypothesis: two-sided
```

Here we reject when our statistic D is such that

$$\sqrt{m}D > k_0 = 1{,}36,$$

and this is true if and only if

$$\texttt{p} - \texttt{value} < 0{,}05.$$

Simulation of a stochastic integral

Suppose $h(t) = f(W(t)) = W(t)^2$. The R code is

```
# Stochastic integral of f(W(t))
n<-1e3 # for the grid of BM
t<-1
f<-function(x) x^2
m<-1e4 # generate m samples of the integral
integral<-rep(0,m)
for(i in 1:m){
   increments<-rnorm(n,0,sqrt(t/n))
   bm<-c(0,cumsum(increments[1:(n-1)]))
   integral[i]<-sum(f(bm)*increments)
}
```

Remember

$$\sum_{k=0}^{n-1} f(W(t_i^n))[W(t_{i+1}^n) - W(t_i^n)].$$

We obtain the results:

```
>mean(integral)
[1] 0.00691243
>var(integral)
[1] 1.007898
```

We compute the theoretical variance with the isometry property. First, if $Z \sim \mathbf{N}(0,1)$:

$$\mathbf{E}(W(t)^4) = \mathbf{E}(\sqrt{t}Z)^4 = t^2\mathbf{E}(Z^4) = 3t^2.$$

Then

$$\begin{aligned}\mathbf{E}\left(\int_0^1 W(t)^2 dW(t)\right)^2 &= \int_0^1 \mathbf{E}(W(t)^4)dt \\ &= \int_0^1 3t^2 = 1.\end{aligned}$$

Theoretical example

Consider $h(t) = W(t)$. It is not difficult to verify that $h \in \mathcal{H}$. We want to integrate

$$\int_0^T W(t)dW(t).$$

To define the approximating process, denote $t_i^n = Ti/n$. The approximating processes are defined by

$$h_n = \sum_{k=0}^{n-1} W(t_i^n)\mathbf{1}_{[t_i^n, t_{i+1}^n)}.$$

So, we must find the limit of

$$\int_0^T h_n dW(t) = \sum_{k=0}^{n-1} W(t_i^n)[W(t_{i+1}^n) - W(t_i^n)].$$

We have

$$\begin{aligned}W(t_i^n)[W(t_{i+1}^n) - W(t_i^n)] &= W(t_{i+1}^n)W(t_i^n) - W(t_i^n)^2. \\ &= -\frac{1}{2}[W(t_{i+1}^n) - W(t_i^n)]^2 + \frac{1}{2}[W(t_{i+1}^n)^2 - W(t_i^n)^2]\end{aligned}$$

When we sum, the second term is telescopic:

$$\sum_{k=0}^{n-1} W(t_i^n)[W(t_{i+1}^n) - W(t_i^n)] = W(t_{i+1}^n)W(t_i^n) - W(t_i^n)^2.$$

$$= -\frac{1}{2}\sum_{k=0}^{n-1}[W(t_{i+1}^n) - W(t_i^n)]^2 + \frac{1}{2}W(T)^2.$$

We have seen that

$$\sum_{k=0}^{n-1}[W(t_{i+1}^n) - W(t_i^n)]^2 \to T,$$

We then conclude that

$$\int_0^T W(t)dW(t) = \frac{1}{2}\left(W^2(T) - T\right).$$

Observe, that if $x(t)$ is a differentiable function,

$$\int_0^T x(t)dx(t) = \frac{1}{2}x^2(T).$$

The extra term T is a characteristic of the stochastic integral, discovered by *Kiyoshi Itô* in the decade of 1940-1950.

4. Itô Formula

Theorem 1. *Let $\{W(t)\colon t \geq 0\}$ be a Brownian motion, and consider a smooth*[1] *function $f = f(t,x)\colon [0,\infty) \times \mathbf{R} \to \mathbf{R}$. Then*

$$f(t, W(t)) - f(0, W(0)) = \int_0^t \frac{\partial f}{\partial x}(s, W(s))dW(s) + \int_0^t \left(\frac{\partial f}{\partial t}(s, W(s)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W(s))\right) ds.$$

If $f(t,x) = f(x)$ (i.e. the function does not depend on time) the formula takes the simpler form

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(s))dW(s) + \frac{1}{2}\int_0^t f''(W(s))ds.$$

[1] By smooth we mean $f(t)$ differentiable in t and twice differentiable in x.

On the proof of Itô formula.

We explain more in detail the proof for $f(t,x) = f(x)$ (i.e. the function does not depend on time). The key moment is the Taylor expansion:

$$f(W(t+h)) - f(W(h)) = f'(W(t))[W(t+h) - W(t)] + \frac{1}{2}f''(W(t))[W(t+h) - W(t)]^2 + o([W(t+h) - W(t)]^2).$$

Then, due to the property of the quadratic variation, when h is small

$$[W(t+h) - W(t)]^2 \sim h. \tag{1}$$

giving the second integral in the formula.

Example

If $f(t,x) = x^2$, then

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2,$$

and Itô formula gives

$$W(t)^2 = 2\int_0^t W(s)W(s) + \int_0^t ds = 2\int_0^t W(s)W(s) + t,$$

If $t = 1$:

$$\int_0^1 W(s)W(s) = \frac{1}{2}(W(1)^2 - 1),$$

our previous result.

Example: Geometric Brownian motion

If $f(t,x) = S_0 e^{\sigma x + \mu t}$, then

$$S(t) = f(t, W(t)) = S_0 e^{\sigma W(t) + \mu t},$$

is the Geometric Brownian motion. To apply Itô Formula, we compute:

$$\frac{\partial f}{\partial t} = \mu f, \quad \frac{\partial f}{\partial x} = \sigma f, \quad \frac{\partial^2 f}{\partial x^2} = \sigma^2 f.$$

and

$$f(t, W(t)) - f(0, W(0) = \int_0^t \left(\mu + \frac{1}{2}\sigma^2\right) f(s, W(s))ds + \int_0^t \sigma f(s, W(s))dW(s).$$

As $f(t, W(t)) = S(t)$, we obtain

$$S(t) - S(0) = \int_0^t \left(\mu + \frac{1}{2}\sigma^2\right) S(s)ds + \int_0^t \sigma S(s)dW(s).$$

This same expression, in differential form, is

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0. \tag{2}$$

where we used that $\mu = r - \sigma^2/2$. So, as the the two assets in Black-Scholes model satisfy the equations

$$\begin{cases} dB(t) = B(t)(rdt), & B(0) = B_0, \\ dS(t) = S(t)(rdt + \sigma dW(t)), & S(0) = S_0. \end{cases}$$

The second is a modification of the first + noise.