Ecuaciones diferenciales estocásticas
Clases 13-14. *

18 de octubre de 2017

Índice

1. Stochastic integration

Consider the class of processes

$$\mathcal{H} = \{h = (h(s))_{0\leq t\leq T}\}$$

that satisfy (A) and (B):

*Notas preparadas por E. Mordecki para el curso de Simulación en procesos estocásticos 2017.

(A) $h(t), W(t+h) - W(t)$ are independent, $\forall 0 \le t \le t+h \le T$,

(B) $\int_0^T \mathbf{E}(h(t)^2)dt < \infty$.

Example: If $\mathbf{E}[f(W(t))^2] \le K$, then $h(t) = f(W(t)) \in \mathcal{H}$: In fact,

$$f(W(t)), W(t+h) - W(t) \text{ are independent},$$

and

$$\int_0^T \mathbf{E}(f(W(t))^2)dt < KT.$$

1.1. Properties

The stochastic integral has the following properties

(P1) $\mathbf{E}\int_0^T h(t)dW(t) = 0$

(P2) Itô isometry:

$$\mathbf{E}\left(\int_0^T h(t)dW(t)\right)^2 = \int_0^T \mathbf{E}(h(t))^2 dt.$$

1.2. Comments

- $I(h) = \int_0^T h(t)dW(t)$, is a random variable.
- For $0 \le t \le T$ we define

  $$I(h,t) = \int_0^T \mathbf{1}_{[0,t)} h(t)dW(t) \stackrel{nt.}{=} \int_0^t h(s)dW(s),$$

  to obtain a *stochastic process*.
- The proofs of properties (P1) and (P2) are based on the independence of h_k and $W(t_k) - W(t_{k-1})$.
- The approximation is in the *complete* space $L^2(\Omega, \mathcal{F}, \mathbf{P})$ of random variables with second finite moment, using Cauchy sequences arguments.

2. Simulation of stochastic integrals: Euler scheme

The definition of a stochastic integral suggest a way to simulate an approximation of the solution. Given a process h, we choose n and we define a mesh $t_k^n = kT/n,\ k = 0, \dots, n$.

$$\sum_{k=0}^{n-1} h(t_i^n)[W(t_{i+1}^n) - W(t_i^n)].$$

It is very important to estimate the integrand in t_i^n and to take the increment of W in $[t_i^n, t_{i+1}^n]$ to have the independence. Otherwise a wrong result is obtained.

3. Itô Formula

Theorem 1. *Let $\{W(t)\colon t \geq 0\}$ be a Brownian motion, and consider a smooth*[1] *function $f = f(t,x)\colon [0,\infty) \times \mathbf{R} \to \mathbf{R}$. Then*

$$f(t,W(t)) - f(0,W(0)) = \int_0^t \frac{\partial f}{\partial x}(s,W(s))dW(s) + \int_0^t \left(\frac{\partial f}{\partial t}(s,W(s)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,W(s))\right) ds.$$

If $f(t,x) = f(x)$ (i.e. the function does not depend on time) the formula takes the simpler form

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(s))dW(s) + \frac{1}{2}\int_0^t f''(W(s))ds.$$

3.1. Example: $f(x) = x^2$

If $f(t,x) = x^2$, then

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2,$$

and Itô formula gives

$$W(t)^2 = 2\int_0^t W(s)W(s) + \int_0^t ds = 2\int_0^t W(s)W(s) + t,$$

If $t = 1$:

$$\int_0^1 W(s)W(s) = \frac{1}{2}(W(1)^2 - 1),$$

our previous result.

3.2. Example: Geometric Brownian motion

If $f(t,x) = S_0 e^{\sigma x + \mu t}$, then

$$S(t) = f(t,W(t)) = S_0 e^{\sigma W(t) + \mu t},$$

is the Geometric Brownian motion. To apply Itô Formula, we compute:

$$\frac{\partial f}{\partial t} = \mu f, \quad \frac{\partial f}{\partial x} = \sigma f, \quad \frac{\partial^2 f}{\partial x^2} = \sigma^2 f.$$

and

$$f(t,W(t)) - f(0,W(0) = \int_0^t \left(\mu + \frac{1}{2}\sigma^2\right) f(s,W(s))ds + \int_0^t \sigma f(s,W(s))dW(s).$$

[1] By smooth we mean $f(t)$ differentiable in t and twice differentiable in x.

As $f(t, W(t)) = S(t)$, we obtain

$$S(t) - S(0) = \int_0^t \left(\mu + \frac{1}{2}\sigma^2\right) S(s)ds + \int_0^t \sigma S(s)dW(s).$$

This same expression, in differential form, is

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0. \tag{1}$$

where we used that $\mu = r - \sigma^2/2$. So, as the the two assets in Black-Scholes model satisfy the equations

$$\begin{cases} dB(t) = B(t)(rdt), & B(0) = B_0, \\ dS(t) = S(t)(rdt + \sigma dW(t)), & S(0) = S_0. \end{cases}$$

The second is a modification of the first + noise.

4. Introduction to SDE

With the purpose of constructing a large class of stochastic processes, we consider *stochastic differential equations (SDE)* driven by a Brownian motion. Given then two functions:

$$b\colon \mathbf{R} \to \mathbf{R}, \quad \sigma\colon \mathbf{R} \to \mathbf{R},$$

a driving Brownian motion $\{W(t)\colon 0 \leq t \leq T\}$ and an independent random variable X_0 (the initial condition) our purpose is to construct a process $X = \{X(t)\colon 0 \leq t \leq T\}$ such that, for $t \in [0, T]$, the following equation is satisfied:

$$X(t) = X_0 + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s). \tag{SDE}$$

4.1. Comments

- As usual ordinary differential equations, the SDE just introducen can be written in differential form

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t),\ X(0) = X_0, \quad t \in [0, T].$$

- We can think that we add "noise" $\sigma(X(t))dW(t)$ to an ordinary differential equation

$$dX(t) = b(X(t))dt,\ X(0) = x_0, \quad t \in [0, T].$$

- As for fixed ω the trajectories of W are not smooth, we use the stochastic integral

$$\int_0^t \sigma(X(s))dW(s)$$

- We should precise conditions on X_0, b, σ for this equation to have one unique solution.

5. Existence and uniqueness of solutions of SDE

We know that a GBM satisfies the SDE in (1). The following theorem answers the inverse question: if a stochastic process satisfies the equation (1), then, it is a GBM.

Theorem 2. *Let $W = \{W(t)\colon 0 \leq t \leq T\}$ be a BM and X_0 an independent rv with finite second moment. Consider two functions*

$$b(t,x), \sigma(t,x)\colon [0,T] \times \mathbf{R} \to \mathbf{R}.$$

Assume that there exists K such that, for all $t \in [0,T]$ and x, y in $\mathbf{R}$

(L) $|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K|x-y|$,

(G) $|b(x)| + |\sigma(x)| \leq K(1+|x|)$.

Then, there exists a unique stochastic process

$$X = \{X(t)\colon 0 \leq t \leq T\}$$

that satisfies

$$dX(t) = b(t, X(t))dt + \sigma(X(t))dW(t), \quad X(0) = X_0.$$

i.e., the integral equation (SDE). Furthermore,

$$\mathbf{E}\left(\max_{0\leq t\leq T} X(t)^2\right) < \infty.$$

By uniqueness we mean that if another process Y satisfies the SDE, then

$$\mathbf{P}(\{\omega\colon \exists t \text{ such that } X(t) \neq Y(t)\}) = 0.$$

6. Euler scheme

The idea is to freeze the coefficients in small intervals in order to produce a sum that approximate the solution of the SDE.

- Determine n, and define $\delta = T/n$.
- Consider the mesh $\{\delta i\colon i = 0, \dots, n\}$, to produce a discretization $X(\delta i)\colon i = 0, \dots, n$.
- Set $X(0) = x_0$, where x_0 is the result of a simulation of X_0.
- While $i < n$ set

$$X_{i+1} = X_i + b(X_i)\delta + \sigma(X_i)[W(\delta(i+1)) - W(\delta i)].$$

6.1. Comments on the Euler scheme

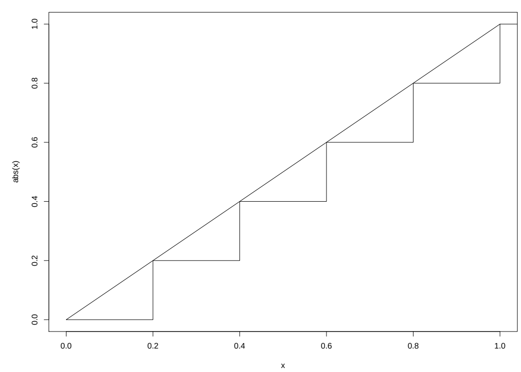
Consider the step function (time change)

$$\tau_n(t) = \begin{cases} 0, & \text{if } 0 \leq t < T/n, \\ T/n, & \text{if } T/n \leq t < 2T/n, \\ \quad\vdots & \qquad\vdots \\ kT/n, & \text{if } kT/n \leq t < (k+1)T/n, \\ \quad\vdots & \qquad\vdots \\ (n-1)T/n, & \text{if } (n-1)T/n \leq t \leq (k+1)T/n. \end{cases}$$

It can be seen that

$$\tau_n(t) = \left\lfloor \frac{nt}{T} \right\rfloor \frac{T}{n}.$$

Below, we see a plot of the time change $\tau_n(t)$ for $n = 5$ and $T = 1$



It should be noticed that

$$\text{máx}\, |\tau_n(t) - t| = \frac{T}{n}.$$

Consider now the equation, for $t \in [0, T]$:

$$X(\tau_n(t)) = X_0 + \int_0^t b(X(\tau_n(s)))ds + \int_0^t \sigma(X(\tau_n(s)))dW(s), \qquad (2)$$

As the coefficients $b(X(\tau_n(t)))$ and $\sigma(X(\tau_n(t)))$ are constant over the time intervals $(kT/n, (k+1)T/n)$, we can solve this equation recursively: $X(\tau_n(0)) = X_0$.

For $t < T/n$, we have $\tau_n(t) = 0$, so

$$
\begin{aligned}
X(\tau_n(T/n)) & \\
&= X(0) + \int_0^{T/n} b(X(0))ds + \int_0^{T/n} \sigma(X(0))dW(s) \\
&= X(0) + b(X(0))\delta + \sigma(X(0)) \int_0^{T/n} dW(s) \\
&= X(0) + b(X(0))\delta + \sigma(X(0))W(\delta),
\end{aligned}
$$

and this gives $X_1 = X(\tau_n(T/n))$, is the first step of Euler scheme. Let us analyze the second setp:

$$
\begin{aligned}
X_2 = X(\tau_n(2T/n)) & \\
&= X_1 + \int_{T/n}^{2T/n} b(X(T/n))ds + \int_{T/n}^{2T/n} \sigma(X(T/n))dW(s) \\
&= X_1 + b(X_1)\delta + \sigma(X_1)[W(2\delta) - W(\delta)].
\end{aligned}
$$

We conclude that the sequence provided by the Euler scheme is the solution to the stochastic difference equation (2) .

As $\tau_n(t) \to t$ uniformly, we hope that the approximation provided by the Euler scheme is close to the true solution.

6.2. Simulation of the solution of an SDE

We consider the SDE

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0.$$

The Euler discretization is, with $S_i = S(t_i^n)$, $\Delta = T/n$:

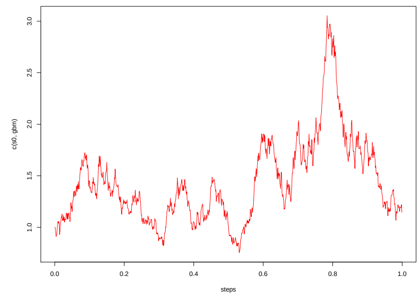
$$
\begin{aligned}
S_{i+1} &= S_i + rS_i\delta + \sigma S_i[W(t_{i+1}^n + \Delta) - W(t_i^n + \Delta)] \\
&= S_i[1 + r\delta + \sigma \mathbf{N}(0, \delta)].
\end{aligned}
$$

So, the code to simulate and plot one trajectory follows:

```
# Euler scheme for the GBM
n<-100 # time discretization
r<-1
sigma<-0.1
t<-1
delta<-t/n
steps<-seq(0,t,length=n)
s0<-1
gbm<-rep(0,n)
```

```
gbm[1]<-s0
for(j in 2:n){
gbm[j]<-gbm[j-1]* (1+r*delta+sigma*rnorm(1,0,sqrt(delta)))
}
plot(steps,gbm,col=‘‘red",type=‘‘l")
```

The code give us the following plot



6.3. Lookback options

A lookback option pays the maximum of the stock price in a prescribed process. Its price is given by

$$L(K) = e^{-rT}\mathbf{E}\left(\max_{0\leq t\leq T} S(t) - K\right)^{+},$$

where $S(t)$ is the Geometric Brownian motion. We price the option by simulation. In each run, we save the payoff, defined as

$$\left(\max_{0\leq t\leq T} S(t) - K\right)^{+} = \max\left[(\max_{0\leq t\leq T} S(t) - K), 0\right].$$

6.4. Code for Lookback options

```
# Loockback options Euler scheme for the GBM
n<-100 # time discretization
m<-10 # nr. of paths
r<-0.01; sigma<-0.02; t<-1; k<-250; delta<-t/n
steps<-seq(0,t,length=n)
s0<-100
payoff<-rep(0,m)
for(i in 1:m){
      gbm<-rep(0,n)
      gbm[1]<-s0
      for(j in 2:n){
       gbm[j]<-gbm[j-1]*
```

```
         (1+r*delta+sigma*rnorm(1,0,sqrt(delta)))
        }
        payoff[i]<-max(max(gbm)-k,0)
}
cat("L-k price:", exp(-r*t)*mean(payoff))
```

6.5. Random Genetic Drift model

The continuous version of the discrete-time Wright-Fisher model is the solution of the following SDE:

$$dX(t) = \sqrt{X(t)(1-X(t))}dW(t), \quad X(0) = x, \quad t \geq 0.$$

Here $X(t)$ represents the proportion of A alleles in a gene population at time t (is the continuous limit of a proportion that has binomial distribution). Observe that once $X(t) = 0$ or $X(t) = 1$ the variation $dX(t) = 0$, meaning that the process remains at this fixed values. This corresponds to the fact that the SDE has *absobing* end points.

It always reach one of the two limits. Defining

$$\tau_0 = \text{ínf}\{t \geq 0\colon X(t) = 0\}, \quad \tau_1 = \text{ínf}\{t \geq 0\colon X(t) = 1\},$$

interesting questions are:

- what is $\mathbf{P}_x(\tau_0 < \tau_1) = a$
- How are the distribution of $X(t)$ for given $t \geq 0$
- How fast does this distribution converge to

$$\ell(x) = a\delta_0(dx) + (1-a)\delta_1(dx)$$