Procesos de Itô y Esquema de Milnstein *

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Índice

1. Euler scheme revisted

The idea is to freeze the coefficients in small intervals in order to produce a sum that approximate the solution of the SDE.

- Determine n, and define $\delta = T/n$.
- Consider the mesh $\{\delta i\colon i = 0,\dots,n\}$, to produce a discretization $X(\delta i)\colon i = 0,\dots,n$.
- Set $X(0) = x_0$, where x_0 is the result of a simulation of X_0.
- While $i < n$ set

$$X_{i+1} = X_i + b(X_i)\delta + \sigma(X_i)[W(\delta(i+1)) - W(\delta i)].$$

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1.1. Geometric Brownian motion

We consider the SDE

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0.$$

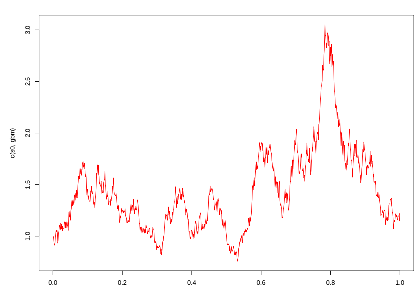
The Euler discretization is, with $S_i = S(t_i^n)$, $\Delta = T/n$:

$$\begin{aligned} S_{i+1} &= S_i + rS_i\delta + \sigma S_i[W(t_{i+1}^n + \Delta) - W(t_i^n + \Delta)] \\ &= S_i[1 + r\delta + \sigma \mathbf{N}(0, \delta)]. \end{aligned}$$

So, the code to simulate and plot one trajectory follows:

```
# Euler scheme for the GBM
n<-100 # time discretization
r<-1
sigma<-0.1
t<-1
delta<-t/n
steps<-seq(0,t,length=n)
s0<-1
gbm<-rep(0,n)
gbm[1]<-s0
for(j in 2:n){
gbm[j]<-gbm[j-1]* (1+r*delta+sigma*rnorm(1,0,sqrt(delta)))
}
plot(steps,gbm,col=‘‘red",type=‘‘l")
```

The code give us the following plot



1.2. Ornstein-Uhlenbeck process (OU)

The OU process is the solution of the following SDE:

$$dX(t) = a(b - X(t))dt + \sigma dW(t), \quad X(0) = X_0$$

where X_0 can be:

- a number, in this case we get Vasicek model for interest rates,
- $X_0 \sim \mathbf{N}(b, \sigma^2/2a)$ independent from W: in this case we have the *stationary* OU process.

1.3. Asymptotic behavior of the OU processes

The OU process has an asymptotic distribution[1], i.e.

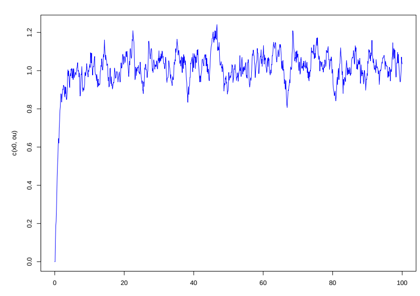
$$\lim_{t\to\infty} X(t) \stackrel{d}{=} \mathbf{N}(b, \sigma^2/2a)$$

independent of the initial condition. We now check this property by simulation. For the Euler scheme we choose n, T, define $\delta = T/n$, and

$$\begin{aligned} X_{i+1} &= X_i + a(b - X_i)\delta + \sigma[W(t_i^n + \delta) - W(t_i^n)] \\ &= ab\delta + X_i(1 - a\delta) + \sigma\mathbf{N}(0, \delta). \end{aligned}$$

We first simulate and print one trajectory for the OU process, for parameters

$$a = b = 1, \quad \sigma = 0{,}1.$$



Here we see the asymptotic behavior of the process around $b = 1$.

1.4. Code for OU asymptotic distribution

```
n<-5000 # time discretization
m<-1000 # nr of trajectories
a<-1;b<-1;sigma<-0.1;t<-100;delta<-t/n
x0<-0
ou<-rep(0,n) #one trajectory
final<-rep(0,m) # final values of each trajectory
ou[1]<-x0
for(i in 1:m){ for(j in 2:n){
ou[j]<-a*b*delta+ou[j-1]*(1-a*delta)
      +sigma*rnorm(1,0,sqrt(delta)) }
```

[1] this is not the case for the Wiener process, nor the GBM

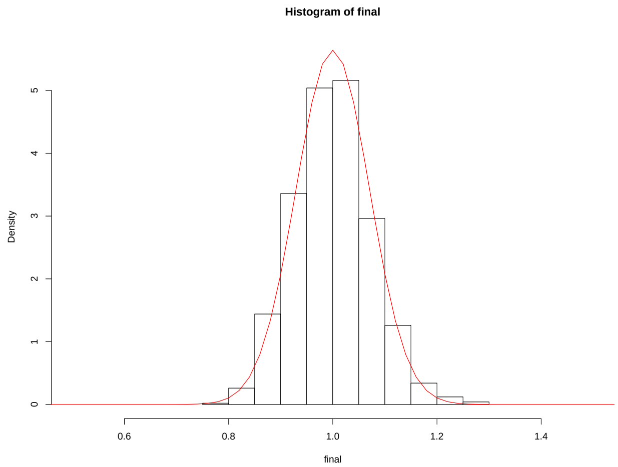


Figura 1: Plot of the OU asymptotic distribution

```
final[i]<-ou[n]
}
sigma2<-sigma/sqrt(2*a)
hist(final,freq=F,ylim = c(0,0.4/sigma2),xlim = c(0.5,1.5))
curve(dnorm(x,b,sigma2),add=T,col=‘‘red")
```

2. Milnstein scheme

It consists in a sharper approximation of the stochastic integral part:

- In Euler scheme we approximate

$$\int_t^{t+h} \sigma(X(r))dW(r) \sim \sigma(X(t))[W(t+h) - W(t)]$$

- Milstein proposes

$$\int_t^{t+h} \sigma(X(r))dW(r) \sim \sigma(X(t))[W(t+h) - W(t)] + \frac{1}{2}\sigma'(X(t))\sigma(X(t))\left([W(t+h) - W(t)]^2 - h\right).$$

Note: If $\sigma(x) = \sigma$ is a constant function, both schemes coincide.

3. Itô processes

We consider the class of processes of the form

$$X(t) = X_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s),$$

where X_0 is a r.v. independent of the Brownian motion W, and $\{b(t)\colon 0 \le t \le T\}$ and $\{\sigma(t)\colon 0 \le t \le T\}$ are processes in the class $\mathcal{H}$. If

$$b(t) = b(t, X(t)), \quad \sigma(t) = \sigma(t, X(t)),$$

then X is the solution of a SDE, but the class of processes is more general. In differential form, we write

$$dX(t) = b(t)dt + \sigma(t)dW(t).$$

4. Itô Formula II

Now, given an Itô diffusion X, we this second formula gives an expansion of $f(t, X(t))$, when f is a smooth function, by the form

$$\begin{aligned} f(t, X(t)) - f(0, X(0)) &= \int_0^t \frac{\partial f}{\partial x}(s, X(s))dX(s) \\ &\quad + \int_0^t \left(\frac{\partial f}{\partial t}(s, X(s)) + \frac{1}{2}\sigma(s)^2 \frac{\partial^2 f}{\partial x^2}(s, W(s)) \right) ds \\ &= \int_0^t \frac{\partial f}{\partial x}(s, X(s))\sigma(s)dW(s) \\ &\quad + \int_0^t \left(\frac{\partial f}{\partial t}(s, X(s)) + \frac{\partial f}{\partial x}(s, X(s))b(s) \right. \\ &\qquad \left. + \frac{1}{2}\sigma(s)^2 \frac{\partial^2 f}{\partial x^2}(s, W(s)) \right) ds. \end{aligned}$$

Note: If $b = 0$ and $\sigma = 1$, then the Itô diffusion is $dX = dW$ and we recover the first formula.

5. Milstein method

The idea behind the method is to apply Itô Lemma to obtain a second-order expansion and increase accuracy. In Euler method we assume that in a small interval the coefficients are constant, i.e. for $t \le r \le t + h$, we have

$$b(X(r)) = b(X(t)), \quad \sigma(X(r)) = \sigma(X(t)).$$

producing an increment

$$X(t + h) = X(t) + b(X(t))h + \sigma(X(t))[W(t + h) - W(t)].$$

We know that

$$W(t+h) - W(t) \sim \sqrt{h} \gg h$$

because h is small. So Milstein[2] proposed to better approximate $\sigma(X(r))$. So, applying Itô Formula, for $t < r < t+h$,

$$\begin{aligned}\sigma(X(r)) &\sim \sigma(X(t)) + \sigma'(X(t))(X(r) - X(t)) + \frac{1}{2}\sigma''(X(t))(r-t) \\ &\sim \sigma(X(t)) + \sigma'(X(t))\sigma(X(t))[W(r) - W(t)],\end{aligned}$$

where we keep the largest increment dW. Now

$$\begin{aligned}\int_t^{t+h} \sigma(X(r))dW(r) &\sim \sigma(X(t))[W(t+h) - W(t)] \\ &\quad + \sigma'(X(t))\sigma(X(t)) \int_t^{t+h} [W(r) - W(t)]dW(r).\end{aligned}$$

Now, using the integral $\int_t^{t+h} W(r)dW(t)$,

$$\begin{aligned}\int_t^{t+h} [W(r) - W(t)]dW(r) &= \int_t^{t+h} W(r)dW(r) \\ &\quad - W(t)(W(t+h) - W(t)) = \frac{1}{2}\left([W(t+h) - W(t)]^2 - h\right).\end{aligned}$$

[2]A comprehensive reference is Kloeden-Platen (1999). Numerical Solution of Stochastic Differential Equations. Springer.