Clase 6: El método de Monte Carlo *

4 de septiembre de 2017

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1. The Monte Carlo method

The basis of the application of the Monte Carlo method (MC) are the limit theorems in probability. Given a sequence of random variables $\{X_n\} = \{X_1, X_2, \dots\}$ and another random variable X, we say that:

*Notas preparadas por E. Mordecki para el curso de Simulación en procesos estocásticos 2017.

- The sequence $\{X_n\}$ *converges almost sure* to X, when

$$\mathbf{P}\left(X_n \to_n X\right) = 1,$$

denoted $X_n \to X,\ a.s.$

- The sequence $\{X_n\}$ *converges in distribution* to X when

$$F_{X_n}(x) \to_n F_X(x) \text{ for all } x \text{ continuity points of } x,$$

denoted $X_n \xrightarrow{d} X$, and also $F_{X_n} \xrightarrow{d} F_X$.

1.1. Limit Theorems

Consider a sequence of independent random variables $\{X_n\}$ with common distribution F,

Theorem 1 (Law of large numbers) *If $\mu = \mathbf{E}X_1 < \infty$, then*

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n} \to \mu,\ a.s.$$

Theorem 2 (Central Limit Theorem (CLT)) *If $\mu = \mathbf{E}X_1$, and $\sigma^2 = \mathrm{var}X_1 < \infty$, we have*

$$\frac{\sqrt{n}}{\sigma}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}(0,1). \tag{1}$$

In other terms, we have

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq x\right) \to \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt.$$

1.2. Monte Carlo method

Suppose that we want to compute a quantity μ that can be written as the expectation of a certain random variable X, i.e.

$$\mu = \mathbf{E}X.$$

If we are able to simulate a large sequence $X_1, \ldots X_n$ of independent random variables, distributed as X, we will have

$$\bar{X}_n \approx \mu, \tag{2}$$

obtaining an approximation of the unknown quantity μ.

The CLT gives a way to assess the quality of this approximation.

2. Confidence interval for Monte Carlo method

The CLT allow us to evaluate the quality of the approximation in (2). Let $1-\alpha$ be a desired confidence level (usually 0,95, i.e. $\alpha = 0{,}05$), and $z_{1-\alpha/2}$ the quantile such that

$$\mathbf{P}(-z_{1-\alpha/2} \leq Z \leq z_{1-\alpha/2}) = 1-\alpha.$$

Then, applying the CLT (assuming $\mathrm{var} X < \infty$), we have

$$\mathbf{P}\left(-\frac{\sigma z_{1-\alpha/2}}{\sqrt{n}} \leq \bar{X}_n - \mu \leq \frac{\sigma z_{1-\alpha/2}}{\sqrt{n}}\right) \to 1-\alpha.$$

We say that the true value μ lies in the *confidence interval (CI)*:

$$\left(\bar{X}_n - \frac{\sigma z_{1-\alpha/2}}{\sqrt{n}}, \bar{X}_n + \frac{\sigma z_{1-\alpha/2}}{\sqrt{n}}\right).$$

We also say that the error of estimation, at the desired confidence level, is

$$\varepsilon = \frac{\sigma z_{1-\alpha/2}}{\sqrt{n}}.$$

As usually σ is also unknown, to compute the error we estimate σ through

$$\widehat{\sigma} = s = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X}_n)^2}. \tag{3}$$

Usual values are:

α	0.1	**0.05**	0.01
$z_{1-\alpha/2}$	1.64	**1.96**	2.58.

2.1. Example 1: Computation of a probability

Given an event A we want to compute its probability

$$p = \mathbf{P}(A) = \mathbf{E}\mathbf{1}_A.$$

Suppose that we are able to simulate independently, for $k = 1, \dots, n$:

$$X_k = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Then, as $\mathbf{E}X_1 = p$,

$$\widehat{p}_n = \frac{X_1 + \cdots + X_n}{n} \approx p.$$

To construct the CI, we have $\sigma^2 = \text{var}X_1 = p(1-p)$. So we can estimate

$$\widehat{\sigma} = \sqrt{\widehat{p}_n(1-\widehat{p}_n)}.$$

and the error of estimation for p is

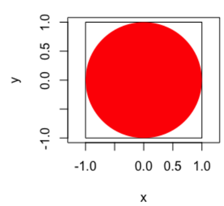
$$\varepsilon = \frac{z_{1-\alpha/2}}{\sqrt{n}}\sqrt{\widehat{p}_n(1-\widehat{p}_n)}.$$

An important observation is that, as $p(1-p) \leq 1/4$, we have the bound

$$\varepsilon \leq \frac{z_{1-\alpha/2}}{2\sqrt{n}}.$$

2.2. Example: Computation of π

We compute the area of a circle trough the acceptance - rejection method.



As the area of the circle is π (the radius being one) the probability of a uniform vector in $[-1,1]\times[-1,1]$ to hit the circle is $\pi/4$. We use the following code

```
> n<-1e7
# we produces a vector of T F read as 1 and 0:
> x<-(runif(n)^2+runif(n)^2<=1)
> 4*mean(x)
[1] 3.141376
> 1.96*4*sd(x)/sqrt(n)
[1] 0.001017929
```

So, our 95 % confidence interval for π with $n = 10^7$ is

$$[3{,}1403, 3{,}1424]$$

If $n = 10^8$ we get

$$[3{,}141352, 3{,}141995]$$

2.3. Application: computing integrals by the rejection method

Let $g\colon D(\subset \mathbf{R}^d) \to [0,\infty)$. We want to compute

$$\mu = \int_D g(x)dx.$$

If D is bounded, and g is also bounded, we can find a hyper-rectangle in $\mathbf{R}^{d+1}$, denote it

$$R = \prod_{k=1}^{d}[a_k, b_k] \times [0, b], \quad \lambda(R) = \prod_{k=1}^{d}(b_k, a_k)b.$$

such that $D \subset \prod_{k=1}^{d}[a_k, b_k]$, $0 \leq g(x) \leq b$. Define the set

$$S = \{x \in \mathbf{R}^{d+1} \colon (x_1, \ldots, x_n) \in D, 0 \leq x_{n+1} \leq g(x_1, \ldots, x_n)\}$$

we have $\lambda(S) = \mu$. If we simulate a uniform random vector $U = (U_1, \ldots, U_{n+1})$ in R, we have

$$\mathbf{P}(U \in S) = \frac{\lambda(S)}{\lambda(R)} = \frac{\mu}{\lambda(R)}.$$

We obtain a MC estimator $\widehat{p}_n$ of the probability $\mathbf{P}(U \in S)$, with a sample of size n. The confidence interval for μ of the form

$$\epsilon = \frac{z_{1-\alpha/2}\lambda(R)}{\sqrt{n}}\sqrt{\widehat{p}_n(1-\widehat{p}_n)}.$$

It is important to notice that the speed at which the error vanishes does not depend on the dimension d. MC is specially suited for large dimensions, as the speed of competing methods usually depends on d.

2.4. Example: Computation of a double integral

We want to compute

$$\mu = \iint_{[0,1]^2} xy\left(\sin\frac{1}{xy}\right)^2 dxdy.$$

As

$$0 \leq f(x,y) = xy\left(\sin\frac{1}{xy}\right)^2 \leq 1,$$

our box is $[0,1]^3$ (that has volume 1).

If $U_3 \leq f(U_1, U_2)$ we accept, otherwise, reject.

Our code follows:

```
# computation of a double integral
n<-1e7
u1<-runif(n)
u2<-runif(n)
u3<-runif(n)
x<-(u1<u2*u3*sin(1/(u2*u3))^2)
mu<-mean(x)
e<-1.96*sd(x)/sqrt(n)
cat(mu-e,mu+e)
```

Produces a 95 % confidence interval:

$$[0{,}15933,\ 0{,}15979]$$

For $n = 10^7$, the interval is

$$[0{,}159463,\ 0{,}159606]$$

3. Example 2: Computation of an integral by the sample mean method

Let $g\colon D(\subset \mathbf{R}^d) \to \mathbf{R}$. We want to compute

$$\mu = \int_D g(x)dx.$$

To perform the MC method, we need a density $f_X(x) > 0$ on D, and we should be able to simulate $X \sim f_X$. In this case we extend g (under the same notation) to $\mathbf{R}^d$ such that $g(x) = 0,\ x \notin D$. Finally

$$\mu = \int_D g(x)dx = \int_{\mathbf{R}^d} g(x)dx = \int_{\mathbf{R}^d} \left(\frac{g(x)}{f_X(x)}\right) f_X(x)dx = \mathbf{E}\left(\frac{g(X)}{f_X(X)}\right).$$

Denoting $Y = \frac{g(X)}{f_X(X)}$ we estimate $\bar{Y}_n \approx \mu$. To construct a confidence interval, we require

$$\mathbf{E}Y^2 = \int \frac{g(x)^2}{f_X(x)}dx < \infty.$$

In this case, we estimate s throgh

$$s^2 = \frac{1}{n-1}\sum(Y_k - \widehat{Y}_n)^2,$$

and compute the error with confidence $1-\alpha$ as

$$\epsilon = \frac{z_{1-\alpha/2}}{\sqrt{n}} s.$$

3.1. Example: Integrals in $[0,1]$

If our integral is defined in $[0,1]$ we simply take $f_X(x) = 1$ and sample uniform variables. For instance, we compute

$$\mu = \int_0^1 4\sqrt{1-x^2}dx = \pi.$$

```
> n<-1e7
> x<-4*sqrt(1-runif(n)^2)
> mu <-mean(x)
> e  <-1.96*sd(x)/sqrt(n)
> c(mu-e,mu+e)
[1]  3.141063  3.142169

> n<-1e8
  .   .   .

[1]  3.141449  3.141799
```

3.2. Example: Integral over $[0,\infty)$

We want to compute

$$\mu = \int_0^\infty e^{-x} x \sin x dx.$$

As $f(x) = e^{-x}$ is the density of $T \sim \exp(1)$, we have

$$\mu = \mathbf{E}(T \sin T),$$

and we can proceed by the average sample method.

3.3. Example: Double integral as a mean

$$\iint_{[0,1]^2} xy \left(\sin \frac{1}{xy}\right)^2 dxdy = \frac{1}{4} \iint_{[0,1]^2} \left(\sin \frac{1}{xy}\right)^2 f_{XY}(x,y)dxdy$$
$$= \frac{1}{4}\mathbf{E}\left(\sin \frac{1}{\sqrt{UV}}\right)^2.$$

We have to prove that if (U,V) is uniform in $[0,1]^2$, then $(\sqrt{U},\sqrt{V})$ has density

$$f_{XY}(x,y) = 4xy.$$

```
> n<-10e6
> x<-(1/4)*(sin(1/sqrt(runif(n)*runif(n))))^2
> mu<-mean(x)
> e<-1.96*sd(x)/sqrt(n)
> cat(mu-e,mu+e)
0.1594209 0.1595292
```

4. Example 3: Option pricing

Suppose that certain asset (as the CAC40) has a value S_0. We model its future value at time T of an asset S by

$$S_T = S_0 e^{\sigma W(T)+(r-\sigma^2/2)T}.$$

where

- $W(T) \sim \mathbf{N}(0,T)$.
- r is the interest rate in the market (for instance 0.01),
- σ is the volatility of the asset.

A *call option* is a contract that gives the right (but no the obligation) to buy this asset at time T by a *strike price* K.

The price of a Call option written on an asset S is computed through the formula

$$C = C(r,\sigma,K,T) = e^{-rT}\mathbf{E}(S_T - K)^+$$

(an expectation!)

Black-Scholes closed formula gives this price, by

$$C(r,\sigma,T,K) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

$$d_{1,2} = \frac{\log(S_0/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}}.$$

To evaluate the accuracy of MC, and also to prepare us for cases when Z is not normal, (and no closed formula exists) we can approximate the price as

$$\widehat{C} = \frac{1}{n}\sum_{k=1}^{d}\left(S_0 e^{\sigma\sqrt{T}Z_k + (r-\sigma^2/2)T} - K\right)^+$$

where $Z_1, \dots, Z_n$ are simulated independent standard normal random variables.

5. Exercises

When not specified confidence level is $1-\alpha = 0{,}95$ and sample size $n = 10^6$.

Exercise 1. *Empirical law of large numbers.* In this exercise we empirically test the Law of Large numbers with the help of R, for different distributions. Then simulate a vector $(X_1, ..., X_n)$ with the distribution F, compute the partial sums

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n,$$

and prove empirically that $S_n/n \to \mathbf{E}(X_1)$ a.s. by plotting the sequence

$$(S_0, S_1/1, S_2/2, ..., S_n/n).$$

(a) Use the following distributions: (i) Uniform in $[0,1]$, (ii) Standard normal distribution (i.e. zero mean one variance) (iii) Normal with mean 1 and variance 4, (iv) exponential with parameter 4, (v) Cauchy distribution, (vi) binomial distribution with parameters $(10, 1/2)$. Take $n = 10000$. The six plots should appear in the same window with the help of the command lines. Use different colors for each plot, and adjust the plots with `ylim=c(-n,n)` in such a way that all of them are visible.

(b) Does the law of large numbers hold for all examples? Can you justify your answer?

Exercise 2. *Empirical Central Limit Theorem (CLT)*

(a) Determine which of the following seven distributions F satisfy the CLT, computing the theoretical mean μ and standard deviation σ corresponding to F: (i) Uniform in $[0, 1]$, (ii) Normal with mean 1 and variance 4, (iii) exponential with parameter 4, (iv) Cauchy distribution, (v) Pareto with parameter 1/2, (vi) Pareto with parameter 3/2, (vii) Pareto with parameter 5/2.

(b) For the distributions that satisfy the CLT, consider a random sample $(X_1, \dots, X_n)$ with distribution F, and consider the standardized mean, defined as

$$Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

Prove that $\mathbf{E}Z = 0$ and $\text{var}Z = 1$.

(c) Generate now a sample $Z_1, \dots, Z_{100}$ of standardized means, taking $n = 10000$ for each Z_k and plot a histogram the sample.

(d) Plot, on the same picture, the standard normal density.

Exercise 3. *Computing an integral.* Compute

$$\mu = \int_0^1 x \left(\sin \frac{1}{x} \right)^2 dx,$$

by the following three methods:

(a) Acceptance-rejection.

(b) Sample mean method, as $\mathbf{E}(f(U))$ for U uniform.

(c) Sample mean method, as $\mathbf{E}(g(X))$ for X with density $2x\mathbf{1}_{\{0 \le x \le 1\}}$.

Compare the errors of the three methods.

Exercise 4. *The area under the e^{-x^2}.*

(a) Prove that

$$\int_0^1 \sqrt{-\log y}\, dy = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(b) Compute the first integral using exponential random variables, and the second, using uniform random variables. Compare the errors for the same number of variates.

Exercise 5. *The Basel problem*[1] consist in the summation of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We are going to approximate this quantity using simulation.

(a) Prove that if U is uniform in $[0,1]$, the discrete random variable $X = \lfloor 1/U \rfloor$ satisfies

$$\mathbf{P}(X = n) = \frac{1}{n(n+1)}.$$

(b) Prove that

$$\frac{\pi^2}{6} = \mathbf{E}\left(\frac{X+1}{X}\right).$$

(c) Provide an approximation by simulation of $\pi^2/6$ with the corresponding confidence intervals.

Exercise 6. *Options prices: Black-Scholes model* The price of a Call option written on an asset S is computed through the formula

$$C = C(r, \sigma, K, T) = e^{-rT}\mathbf{E}(S_T - K)^+,$$

where $S_T = S_0 \exp(\sigma \mathbf{N}(0,T) + (r - \sigma^2/2)T)$. Black and Scholes (1973) gave a formula to compute this value:

$$C(r, \sigma, T, K) = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

$$d_{1,2} = \frac{\log(S_0/K) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}}.$$

where Φ is the cumulative normal standard distribution.

(a) Write a code to compute the price of a call option according to Black-Scholes formula.

(b) Consider the corresponding values:

$$S_0 = 4930, \quad r = 0{,}01, \quad T = 1/12, \quad \sigma = 0{,}20.$$

and plot the option values as a function of K in an interval $[4000, 6000]$

[1] Solved by L. Euler in 1734. This is Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$ evaluated at $s = 2$.

(c) Compute the option value with the formula and with simulation when $K = S_0$ (option at the money), and check if the true value lies in the corresponding confidence interval.

Exercise 7. *Probabilities.* In the quarter-finals of a football competition take part 8 teams, named A,B,C,D,E,F,G,H, according to the following schedule:

- First round: A vs. B, C vs. D, E vs. F, G vs. H. We have four winners.
- Semi-finals: Winner of A,B vs winner of C,D; winner of E,F vs winner of G,H. We have two winners
- Final: between the two winners of the semi-finals.

The probabilities of winning matches of each couple are given in the following matrix:

win\loose	A	B	C	D	E	F	G	H
A	-	1/3	1/2	3/5	1/2	1/3	2/3	1/2
B		-	2/3	1/2	3/5	1/2	2/3	1/2
C			-	1/2	1/3	1/2	1/2	2/5
D				-	1/3	1/2	2/3	2/5
E					-	1/2	3/5	1/3
F						-	3/5	2/5
G							-	2/5
H								-

Cuadro 1: Probabilities of winning: $\mathbf{P}(\text{A wins B}) = 1/3$.

Compute the probabilities of winning for each team with its respective confidence intervals through MC. Present your results in a list of teams ordered w.r.t. winning probability in descending order.

Note: To declare a matrix

```
prob<-matrix(0,nrow=8,ncol=8) # matrix of probabilities
```

To enter an element

```
prob[1,2]<-1/3
```