Fractional Convexity

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Outline

- Fractional convexity
- Fractional Convexity vs Convexity
- The first fractional eigenvalue
- The fractional convex envelope
- Open problem

Fractional convexity

is there a notion of convexity in the fractional setting?

Fractional convexity

Convexity

A function $u\colon \Omega \to \mathbb{R}$ is said to be convex in Ω if, for any two points $x, y \in \Omega$ such that the segment $[x, y] := \{tx + (1-t)y : t \in (0,1)\}$ is contained in Ω, it holds that

$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(y), \qquad \forall t \in (0,1).$$

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$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(y), \qquad \forall t \in (0,1).$$

Notice that $v(tx + (1-t)y) := tu(x) + (1-t)u(y)$ is just the solution to the equation

$$v'' = 0 \text{ in the segment } [x, y]$$

that verifies $v(x) = u(x)$ and $v(y) = u(y)$.

Fractional convexity

Definition

Given $s \in (0,1)$,

$$L_s(\mathbb{R}) := \left\{ f \in L^1_{loc}(\mathbb{R}) \colon \int_{\mathbb{R}^N} \frac{|f(\tau)|}{(1+|\tau|)^{1+2s}} d\tau < \infty \right\}.$$

A function $u \in L^\infty_{loc}(\mathbb{R}^N)$ is said to be s−convex in Ω if

- $t \to u(x+tz) \in L_s(\mathbb{R})$ for any $x \in \Omega$ and $z \in \mathbb{R}^N$ with $|z| = 1$;
- For any two points $x, y \in \Omega$, such that the segment $[x,y]$ is contained in Ω, it holds that

$$u(tx + (1-t)y) \leq v(tx + (1-t)y), \qquad \forall t \in (0,1) \tag{1}$$

where v is a viscosity solution of

$$\begin{cases} \Delta^s_1 v(tx+(1-t)y) := C_s \int_{\mathbb{R}} \frac{v(rx+(1-r)y) - v(tx+(1-t)y)}{|r-t|^{1+2s}} dr = 0, & \forall t \in (0,1), \\ v(tx+(1-t)y) = u(tx+(1-t)y) & \forall t \notin (0,1). \end{cases}$$

As usual, the integral is to be understood in the principal value sense.

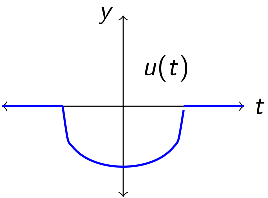
Fractional convexity
Definition

Is there a fractional convex function?

Fractional convexity
Example

Let $u\colon \mathbb{R} \to \mathbb{R}$

$$u(t) := \begin{cases} -(1-t^2)^s & \text{if } t \in [-1,1], \\ 0 & \text{if } |t| > 1. \end{cases}$$



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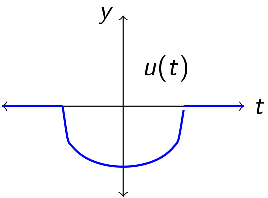

2012, B. Dyda proved that

$$\Delta_1^s u(t) = \Gamma(2s+1) \geq 0 \text{ in } (-1,1)$$

Fractional convexity

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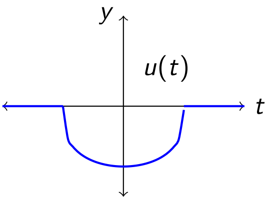

2012, B. Dyda proved that

$$\Delta_1^s u(t) = \Gamma(2s+1) \geq 0 \text{ in } (-1,1)$$

Then, given two points in $x, y \in (-1,1)$,

$$\Delta_1^s u(tx + (1-t)y) = |x-y|^{2s}\Gamma(2s+1) \quad \text{for } t \in (0,1).$$

Fractional convexity

Example

Thus, by the maximum principle, if v is the viscosity solution of

$$\begin{cases} \Delta_1^s v(tx + (1-t)y) = 0 & t \in (0,1), \\ v(tx + (1-t)y) = u(tx + (1-t)y) & t \in \mathbb{R} \setminus (0,1), \end{cases}$$

then

$$u(tx + (1-t)y) \le v(tx + (1-t)y) \quad \forall t \in (0,1).$$

Fractional convexity

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Thus, by the maximum principle, if v is the viscosity solution of

$$\begin{cases} \Delta_1^s v(tx+(1-t)y) = 0 & t \in (0,1), \\ v(tx+(1-t)y) = u(tx+(1-t)y) & t \in \mathbb{R} \setminus (0,1), \end{cases}$$

then

$$u(tx+(1-t)y) \leq v(tx+(1-t)y) \quad \forall t \in (0,1).$$

We conclude that u is $s-$convex in $(-1,1)$.

Fractional convexity

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Thus, by the maximum principle, if v is the viscosity solution of

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then

$$u(tx+(1-t)y) \leq v(tx+(1-t)y) \quad \forall t \in (0,1).$$

We conclude that u is $s-$convex in $(-1,1)$.
The same argument shows that if u is a viscosity solution to

$$\Delta_1^s u(t) \geq 0 \quad \text{in } (a,b)$$

then u is $s-$convex in (a,b).

Fractional convexity
Definition

Is there a fractional convex function? Yes! ☺

Fractional convexity
Definition

Is there a fractional convex function? Yes! ☺

is there a notion of convexity in the fractional setting? Yes! ☺

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Fractional Convexity vs Convexity

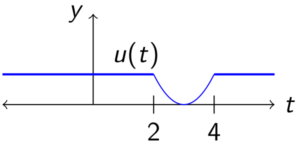
u is a convex function $\overset{?}{\Longrightarrow}$ u is a $s-$convex function.

Fractional Convexity vs Convexity

Example

Let $u\colon \mathbb{R} \to \mathbb{R}$ be given by

$$u(t) := \begin{cases} (t-3)^2 & \text{if } t \in [2,4], \\ 1 & \text{if } |t-3| > 1. \end{cases}$$



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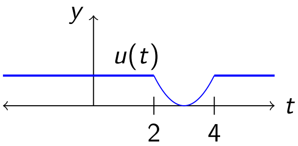

Observe that u is a convex function in $[-1,1]$.

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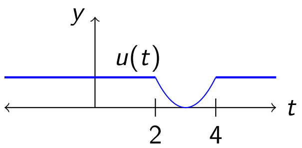

Observe that u is a convex function in $[-1,1]$.
On the other hand, for any $x, y \in [-1,1]$, if v is the viscosity solution of

$$\begin{cases} \Delta_1^s v(tx+(1-t)y) = 0 & \forall t \in (0,1), \\ v(tx+(1-t)y) = u(tx+(1-t)y) & \forall t \in \mathbb{R} \setminus (0,1), \end{cases}$$

By the strong maximum principle, we have that

$$1 > v(tx+(1-t)y) \quad \forall t \in (0,1).$$

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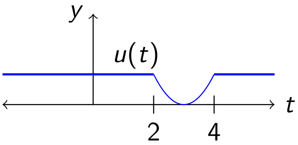

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By the strong maximum principle, we have that

$$1 > v(tx+(1-t)y) \quad \forall t \in (0,1).$$

Therefore u is not s−convex.

Fractional Convexity vs Convexity

u is a convex function $\not\Rightarrow$ u is a $s-$convex function.

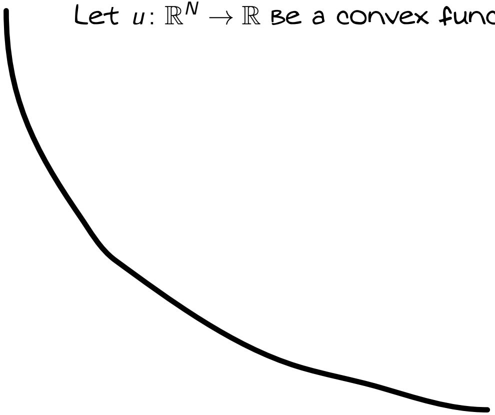
Fractional Convexity vs Convexity

Proposition (LMDP, A. Quaas and J. Rossi).

Let $s > \frac{1}{2}$, and u be a convex function in $\mathbb{R}^N$ such that $t \mapsto u(x+tz) \in L_s(\mathbb{R})$ for any $x \in \mathbb{R}^N$ and any $z \in \mathbb{R}^N$ with $|z| = 1$. Then u is $s-$convex in $\mathbb{R}^N$.

Fractional Convexity vs Convexity

Idea of the proof



Let $u\colon \mathbb{R}^N \to \mathbb{R}$ be a convex function.

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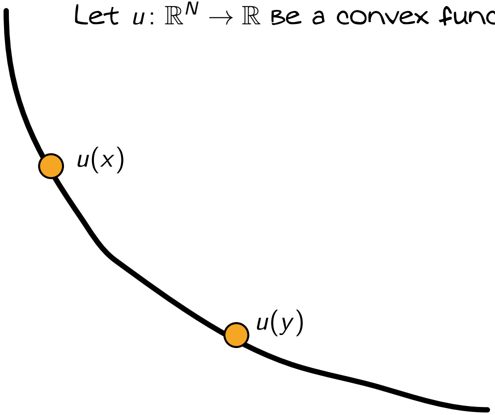

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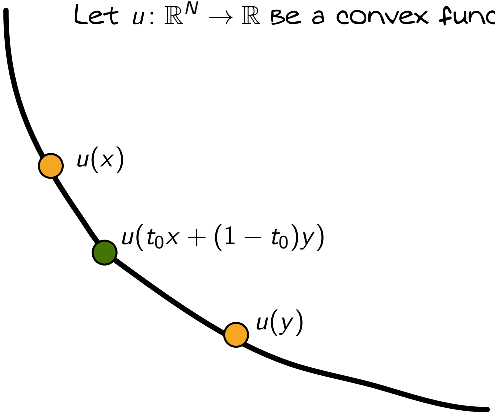

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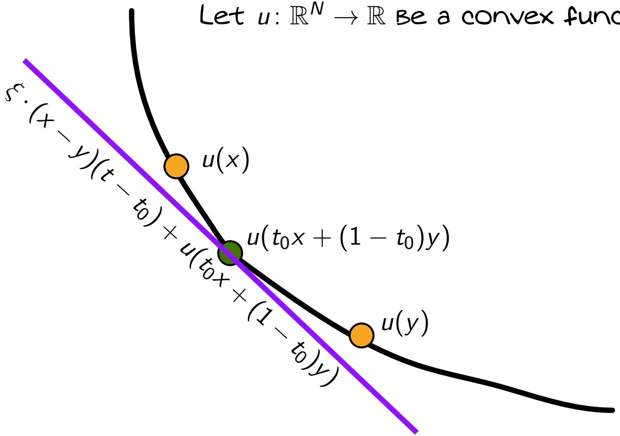
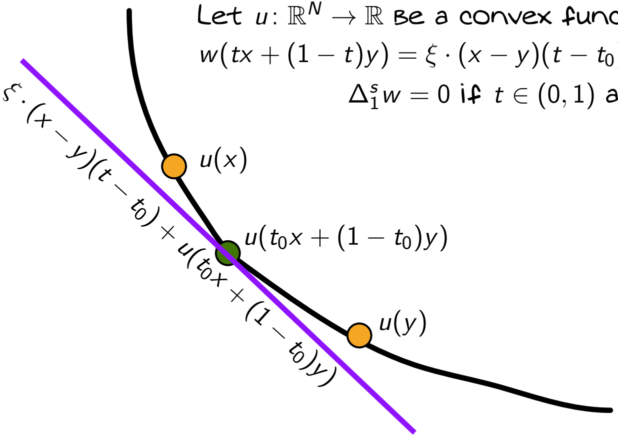


Fractional Convexity vs Convexity

Idea of the proof

Let $u\colon \mathbb{R}^N \to \mathbb{R}$ be a convex function. Then, since $s > \frac{1}{2}$,
$w(tx + (1-t)y) = \xi \cdot (x-y)(t-t_0) + u(t_0x + (1-t_0)y)$ is a solution of

$$\Delta_1^s w = 0 \text{ if } t \in (0,1) \text{ and } w \leq u(tx + (1-t)y) \text{ if } t \notin (0,1)$$



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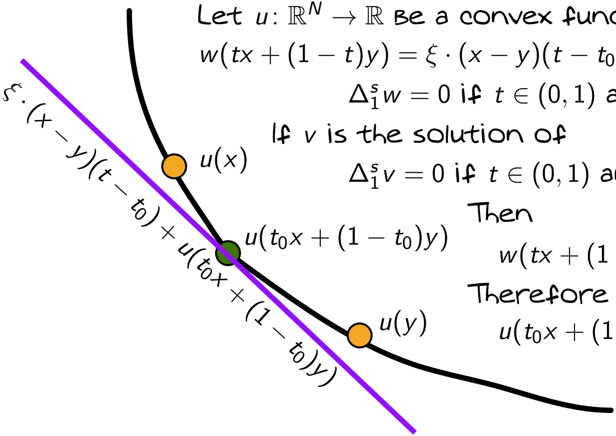

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If v is the solution of

$$\Delta_1^s v = 0 \text{ if } t \in (0,1) \text{ and } v = u(tx + (1-t)y) \text{ if } t \notin (0,1).$$

Then

$$w(tx + (1-t)y) \le v(tx + (1-t)y) \quad \forall t \in \mathbb{R}.$$

Therefore

$$\begin{aligned} u(t_0x + (1-t_0)y) &= w(t_0x + (1-t_0)y) \\ &\le v(t_0x + (1-t_0)y). \end{aligned}$$

Fractional Convexity vs Convexity

u is a convex function $\overset{?}{\Longleftarrow}$ u is a $s-$convex function.

Fractional Convexity vs Convexity

Example

Let $u\colon \mathbb{R} \to \mathbb{R}$ be the solution to

$$\begin{cases} \Delta_1^s u(x) = 0 & \forall x \in (0,1), \\ u(x) = f(x) & \forall x \in \mathbb{R} \setminus (0,1) \end{cases}$$

where f is a bounded smooth function such that $f(0) = f(1) = 1$, $f \geq 1$ with at least one x such that $f(x) > 1$.

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Observe that u is a $s-$convex function in $(0,1)$.

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where f is a bounded smooth function such that $f(0) = f(1) = 1$, $f \geq 1$ with at least one x such that $f(x) > 1$.
Observe that u is a $s-$convex function in $(0,1)$.
On the other hand, u is smooth, continuous up to the boundary and, by the strong maximum principle, it holds that

$$1 < u(x) \quad \forall x \in (0,1)$$

together with $u(0) = u(1) = 1$.

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On the other hand, u is smooth, continuous up to the boundary and, by the strong maximum principle, it holds that

$$1 < u(x) \quad \forall x \in (0,1)$$

together with $u(0) = u(1) = 1$.
Therefore, u is not convex in $(0,1)$.

Fractional Convexity vs Convexity

u is a convex function $\not\Leftarrow$ u is a $s-$convex function.

Outline

- Fractional convexity
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- The first fractional eigenvalue
- The fractional convex envelope
- Open problem

The first fractional eigenvalue

Local case

Let $u \in C^2(\Omega)$. Then u is convex in Ω if only if $D^2u(x)$ is positive semidefinite in Ω, that is

$$\langle D^2u(x)z, z\rangle \geq 0 \quad \forall z \in \mathbb{R}^N, x \in \Omega. \tag{$\star$}$$

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$$\langle D^2u(x)z, z\rangle \geq 0 \quad \forall z \in \mathbb{R}^N, x \in \Omega. \qquad (\star)$$

In terms of the eigenvalue of D^2u, $(\star)$ can be written as

$$\begin{aligned}\Lambda_1(D^2u(x)) &:= \min\left\{\lambda\colon \lambda \text{ is an eigenvalue of } D^2u(x)\right\}\\ &= \inf_{\theta\in\mathbb{S}^{N-1}}\langle D^2u(x)\theta, \theta\rangle \geq 0\end{aligned}$$

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Theorem (A. Oberman).

A continuous function u is convex if only if u is a viscosity solution of

$$\Lambda_1(D^2u(x)) \geq 0.$$

The first fractional eigenvalue

We define the first fractional egienvalue of u as

$$\Lambda_1^s u(x) = \inf \left\{ C_s \int_{\mathbb{R}} \frac{u(x+tz) - u(x)}{|t|^{1+2s}} \, dt : z \in \mathbb{S}^{N-1} \right\}$$

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A function $u \colon \mathbb{R}^N \to \mathbb{R}$ is a viscosity solution of

$$\Lambda_1^s u(x) \geq 0 \quad \text{in } \Omega.$$

if for any $x \in \Omega$, any $z \in \mathbb{S}^{N-1}$, any $\phi \in C^2(\mathbb{R}^{\mathbb{N}})$ such that $\phi(x) = w(x)$ and $\phi(y) \geq w(y)$ in $B_\delta(x)$ we have that $t \to w(x+tz) \in L_s(\mathbb{R})$ and

$$C_s \left(\int_{-\delta}^{\delta} \frac{\phi(x+tz)-\phi(x)}{|t|^{1+2s}}\, dt + \int_{\mathbb{R}\setminus(-\delta,\delta)} \frac{w(x+tz)-w(x)}{|t|^{1+2s}}\, dt \right) \geq 0,$$

where w is the upper semicontinuous envelope of u in $\mathbb{R}^N$.

The first fractional eigenvalue

Theorem (LMDP, A. Quaas, and J. Rossi).

Let Ω be a bounded strictly convex C^2 domain (that is, a bounded domain with C^2-boundary such that all the principal curvatures of the surface $\partial\Omega$ are positive everywhere). Then u is $s-$convex in Ω if only if u is a viscosity solution of

$$\Lambda_1^s u(x) \geq 0 \quad \text{in } \Omega.$$

The first fractional eigenvalue

Fractional truncated laplacian

2022, Birindelli, Galise, and Topp studied the following operator

$$\mathcal{I}_n^- u(x) := \inf\left\{ \sum_{i=1}^{n} C_s \int_{\mathbb{R}} \frac{u(x+tz_i)-u(x)}{|t|^{1+2s}}\, dt \colon \{z_i\}_{i+1}^{n} \in \mathcal{V}_n \right\}$$

where $\mathcal{V}_n$ is the family of $n-$dimensional orthonormal set in $\mathbb{R}^N$.

The first fractional eigenvalue

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where $\mathcal{V}_n$ is the family of $n-$dimensional orthonormal set in $\mathbb{R}^N$.
Observe that

$$\mathcal{I}_1^- u(x) = \Lambda_1^s u(x).$$

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Observe that

$$\mathcal{I}_1^- u(x) = \Lambda_1^s u(x).$$

They show

$$\mathcal{I}_n^- u(x) \to \mathcal{P}_n^- u(x) = \sum_{j=1}^{n} \Lambda_j(D^2 u(x)) \text{ as } s \to 1^-.$$

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The fractional convex envelope
Local case

Given a function $g\colon \partial\Omega \to \mathbb{R}$, the convex envelope of the boundary datum g in Ω is

$$u^\star(x) := \sup\{u(x)\colon u \text{ is convex in } \overline{\Omega} \text{ and } u \le g \text{ on } \partial\Omega\}$$

That is, $u^\star$ is the largest convex function in Ω that is below g on $\partial\Omega$. Moreover $u^\star$ is the largest viscosity solution of the

$$\begin{cases} \Lambda_1(D^2u(x)) = 0 & \text{in } \Omega, \\ u(x) \le g(x) & \text{on } \partial\Omega. \end{cases}$$

The convex envelope
Local case

Theorem (A. Oberman and L. Silvestre).

If Ω is strictly convex and g is continuous then u^* is the unique viscosity solution of

$$\begin{cases} \Lambda_1(D^2u(x)) = 0 & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega. \end{cases}$$

The fractional convex envelope

Let $s \in (0,1)$ and $g\colon \mathbb{R}^N \setminus \Omega \to \mathbb{R}$. Let us call $H(g)$ the set of s−convex functions that are below g outside Ω,

$$H(g) := \left\{ u\colon u \text{ is } s\text{−convex in } \overline{\Omega} \text{ and verifies } u|_{\mathbb{R}^N\setminus\Omega} \le g \right\}.$$

Lemma (LMDP, A. Quaas, and J. Rossi).

Let $u \in L^\infty_{loc}(\mathbb{R}^N)$ be such that $t \to u(x+tz) \in L_s(\mathbb{R})$ for any $x \in \Omega$ and any $z \in \mathbb{R}^N$ with $|z| = 1$. Then, $u \in H(g)$ if only if u is a viscosity solution to

$$\begin{aligned} \Lambda_1^s u(x) &\ge 0 && \text{in } \Omega, \\ u(x) &\le g(x) && \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

The fractional convex envelope

The $s-$convex envelope of an exterior datum $g\colon \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ is given by

$$u_s^*(x) = \sup \Big\{ u(x) \colon u \in H(g) \Big\}.$$

Observe that u_s^* is $s-$convex.

The fractional convex envelope

The s–convex envelope of an exterior datum $g\colon \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ is given by

$$u_s^*(x) = \sup\Big\{u(x)\colon\, u \in H(g)\Big\}.$$

Observe that u_s^* is s–convex.
Is u^* a viscosity solution of

$$\Lambda_1^s u(x) = 0 \text{ in } \Omega, \quad u(x) = g(x) \text{ in } \mathbb{R}^N \setminus \Omega?$$

The fractional convex envelope

Viscosity solution

A bounded upper semicontinuous function $u\colon \mathbb{R}^N \to \mathbb{R}$ is a viscosity subsolution to the Dirichlet problem

$$\Lambda_1^s u(x) = f(x) \text{ in } \Omega, \quad u(x) = g(x) \text{ in } \mathbb{R}^N \setminus \Omega,$$

if $u \leq g$ in $\mathbb{R}^N \setminus \overline{\Omega}$ and for each $\delta > 0$ and $\phi \in C^2(\mathbb{R}^N)$ such that x_0 is a maximum point of $u - \phi$ in $B_\delta(x_0)$, then

$$-E_\delta(u^g, \phi, x_0) \leq 0 \text{ in } \Omega, \quad \min\{-E_\delta(u^g, \phi, x_0), u(x_0) - g(x_0)\} \leq 0 \quad \text{on } \partial\Omega.$$

where

$$E_\delta(u^g, \phi, x_0) := C_s \inf_{z \in \mathbb{S}^{N-1}} \left\{ \int_{-\delta}^{\delta} \frac{\phi(x_0 + tz) - \phi(x_0)}{|t|^{1+2s}} dt + \int_{\mathbb{R}\setminus(-\delta,\delta)} \frac{u^g(x_0 + tz) - u(x_0)}{|t|^{1+2s}} dt - f(x_0) \right\}$$

$$u^g(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in \mathbb{R}^N \setminus \overline{\Omega}, \\ \max\{u(x), g(x)\} & \text{if } x \in \partial\Omega. \end{cases}$$

The fractional convex envelope

Attainability of the exterior datum

Theorem (LMDP, A. Quaas, and J. Rossi).

Let Ω be a bounded strictly convex C^2–domain, $f \in C(\overline{\Omega})$, $g \in C(\mathbb{R}^N \setminus \Omega)$ be bounded, and $u, v\colon \mathbb{R}^N \to \mathbb{R}$ be viscosity sub and supersolution of

$$\begin{cases} \Lambda_1^s w(x) = f(x) & \text{in } \Omega, \\ w(x) = g(x) & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then

$$u \le g \text{ on } \partial\Omega \text{ and } v \ge g \text{ on } \partial\Omega.$$

The fractional convex envelope

Comparison principle

Theorem (LMDP, A. Quaas, and J. Rossi).

Let Ω be a bounded strictly convex C^2–domain, $f \in C(\overline{\Omega})$, $g \in C(\mathbb{R}^N \setminus \Omega)$ be bounded and $u, v \colon \mathbb{R}^N \to \mathbb{R}$ be viscosity sub and supersolution of

$$\begin{cases} \Lambda_1^s w(x) = f(x) & \text{in } \Omega, \\ w(x) = g(x) & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then

$$u \leq v \text{ in } \mathbb{R}^N.$$

The fractional convex envelope

Existence and uniqueness of solution

Theorem (LMDP, A. Quaas, and J. Rossi).

Let Ω be a bounded strictly convex C^2–domain, $f \in C(\overline{\Omega})$ and $g \in C(\mathbb{R}^N \setminus \Omega)$ be bounded. Then, there is a unique viscosity solution u to

$$\begin{cases} \Lambda_1^s u(x) = f(x) & \text{in } \Omega, \\ u(x) = g(x) & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

This solution is continuous in $\overline{\Omega}$ and the datum g is taken with continuity, that is, $u|_{\partial\Omega} = g|_{\partial\Omega}$.

The fractional convex envelope

Existence and uniqueness of solution

Corollary (LMDP, A. Quaas, and J. Rossi).

Let Ω be a bounded strictly convex C^2–domain and $g \in C(\mathbb{R}^N \setminus \Omega)$ be bounded. Then, u_s^* is the unique viscosity solution to

$$\begin{cases} \Lambda_1^s u(x) = 0 & \text{in } \Omega, \\ u(x) = g(x) & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover $u_s^* \in C(\overline{\Omega})$ and the datum g is taken with continuity, that is, $u_s^*|_{\partial\Omega} = g|_{\partial\Omega}$.

The fractional convex envelope

A regularity result

Theorem (B. Barrios, LMDP, A. Quaas, J. Rossi).

Let Ω be a bounded strictly convex C^2-domain and u be a viscosity solution of

$$\Lambda_1^s u(x) = f(x) \quad \text{in} \quad \Omega, \qquad u(x) = g(x) \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.$$

Assume that $s > 1/2$, f, g are bounded functions and g satisfies a Hölder bound, so that there exist M_g and $\beta \in (s, 2s)$ such that

$$|g(x) - g(y)| \leq M_g |x - y|^\beta, \ x, \ y \in \mathbb{R}^N \setminus \Omega.$$

Then $u \in C^\gamma(\overline{\Omega})$ where

$$\gamma \in \begin{cases} (0, 2s-1) & \text{if} \ g \equiv 0, \\ (0, \beta - s) & \text{if} \ g \not\equiv 0. \end{cases}$$

The fractional convex envelope

A regularity result

To prove the lasts result we have to take into account the geometry properties of our domain. In particular we use that,

$$\Omega = \bigcap_{z \in \partial\Omega} B_R(z - R\nu(z)),$$

for some $R > 0$ whose value it is related with the principal curvatures of $\partial\Omega$ and $\nu(z)$ denotes the outward normal unit vector of Ω in $z \in \partial\Omega$.

The fractional convex envelope

A regularity result

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2021 I. Birindelli, G. Galise and H. Ishii.

2022 I. Birindelli, G. Galise and D. Schiera.

The fractional convex envelope

The limit as $s \nearrow 1$

Theorem (B. Barrios, LMDP, A. Quaas, J. Rossi).

Given a continuous and bounded exterior datum $g\colon \mathbb{R}^N \setminus \Omega \mapsto \mathbb{R}$, let u_s^* be the sequence of $s-$convex enevelopes of g and u^* be the convex envelope of g. Then, u_s^* converges uniformly in $\overline{\Omega}$ to u^* as $s \nearrow 1$.

The fractional convex envelope

The limit as $s \nearrow 1$

Idea of the proof. Our strategy to show that u_s^* converge to the usual convex envelope as $s \nearrow 1$, is to use the well known half relaxed limits. These are given by

$$u^{\square}(x) := \sup \left\{ \limsup_{k\to\infty, s\nearrow 1} u_s^*(x_k) : x_k \to x \right\} \text{ and } u_{\square}(x) := \inf \left\{ \liminf_{k\to\infty, s\nearrow 1} u_s^*(x_k) : x_k \to x \right\}.$$

We show that $u^{\square}$ is a subsolution of

$$\Lambda_1^s u(x) = 0 \quad \text{in} \quad \Omega, \qquad u(x) = g(x) \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.$$

and $u_{\square}$ is a supersolution. From the comparison principle, we obtain $u^{\square} \le u_{\square}$ (notice that the reverse inequality trivially holds) and hence we conclude that $u^{\square} = u_{\square} = u^*$ proving the desired convergence result.

Outline

- Fractional convexity
- Fractional Convexity vs Convexity
- The first fractional eigenvalue
- The fractional convex envelope
- Open problem

Open problem

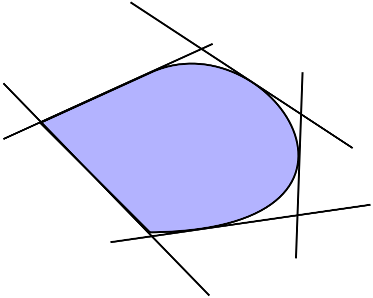
What is a fractional convex set?

Open problem

Our idea

Any closed convex set C is the intersection of all halfspaces that contain it:

$$C = \cap\{H : H \text{ halfspace s.t. } C \subseteq H\}.$$



Open problem

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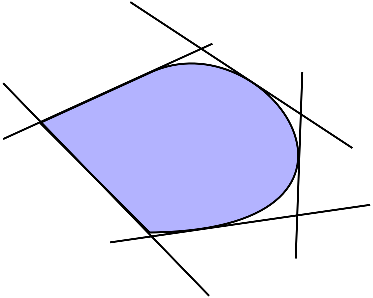

What is a nonlocal halfspace?

Thank you!