Ricci flows which terminate in cones

Brett Kotschwar (Arizona State)

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Overview

We will show that noncompact solutions to the Ricci flow

$$\text{(RF)} \qquad \frac{\partial}{\partial t} g = -2\,\mathrm{Rc}(g)$$

on $M \times [0, T)$ which converge locally smoothly to a cone on an end as $t \nearrow T$ must be self-similar for $t < T$.

Thus, there is (effectively) at most one way in which a solution may flow into a cone.

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Motivation:

- Classification of asymptotically conical shrinkers/singularity models.
- "Strong" backward uniqueness of solutions to (RF).

Gradient Shrinking Solitons

(M, g) is a **gradient shrinking soliton (shrinker)** if

$$\text{(SGRS)} \qquad \mathrm{Rc}(g) + \nabla\nabla f = \frac{g}{2}$$

for some $f \in C^\infty(M)$.

- A complete shrinker gives rise to a *self-similar* solution $g(t) = -t\phi_t^* g$ on $M \times (-\infty, 0)$, where
$$\frac{\partial \phi}{\partial t} = -\frac{1}{t}\nabla f \circ \phi, \quad \phi_{-1} = \mathrm{Id}\,.$$
- Arise as models of finite-time singularities.

Gradient Shrinking Solitons

- Classified in dimensions $n \leq 3$ (Hamilton, Ivey, Perelman, Ni-Wallach, Cao-Chen-Zhu), and in the Kähler case in $n = 4$ (Bamler-Cifarelli-Conlon-Deruelle).
- All known complete noncompact shrinkers are asymptotic either to cones or to products at infinity.
- Work of Munteanu-Wang hints at possible dichotomy in $n = 4$ for geometry of ends of shrinkers of bounded curvature.

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Question: *What are the 4D asymptotically conical shrinkers?*

Asymptotically conical manifolds

For closed (Σ, g_Σ), write $\mathcal{C}^\Sigma = (\mathcal{C}_0^\Sigma, g_\mathcal{C})$, where

$$\mathcal{C}_a^\Sigma = (a, \infty) \times \Sigma, \quad g_\mathcal{C} = dr^2 + r^2 g_\Sigma,$$

and let $\rho : \mathcal{C}_0 \to \mathcal{C}_0$ be the map $\rho_\lambda(r, \sigma) = (\lambda r, \sigma)$. Note

$$\rho_\lambda^* g_\mathcal{C} = \lambda^2 g_\mathcal{C}, \quad \lambda > 0.$$

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Definition
(M, g) is C^k**-asymptotic** to $\mathcal{C}^\Sigma$ along $E \subset M$ if there is an $a > 0$ and a diffeomorphism $F : \mathcal{C}_a^\Sigma \to E$ such that $\lambda^{-2} \rho_\lambda^* F^* g \longrightarrow g_\mathcal{C}$ in C^k_{loc} as $\lambda \to \infty$.

Asymptotically conical shrinking solitons

Fact: A shrinker (M, g, f) with quadratic curvature decay on an end $E \subset M$ is C^k-asymptotic to some cone for all $k \geq 0$.

- Munteanu-Wang ('16): In 4D, if $R(x) \to 0$ at infinity on an end, then $|\operatorname{Rm}|(x) \leq Cr^{-2}(x)$ on that end.
- For $n \geq 4$, same is true if $|\operatorname{Rc}|(x) \to 0$.

Examples of asymptotically conical shrinkers

1. Gaussian soliton $(\mathbb{R}^n, g_{\text{euc}}, |x|^2/4)$.
2. Feldman-Ilmanen-Knopf ('03)
   - Kähler, $\mathrm{U}(m)$-invariant, on tautological line bundle over $\mathbb{CP}^{m-1}$, $m \geq 2$.
3. Dancer-Wang ('11), Yang ('08)
   - Kähler, generalizations of FIK on line bundles over products of KE manifolds with positive scalar curvature.
4. Angenent-Knopf ('22)
   - Doubly-warped products with fibers S^{m_1}, S^{m_2} where m_1, $m_2 \geq 2$, $m_1 + m_2 \leq 8$.

- Various families of incomplete examples.

The solution on an asymptotically conical end

If (M, g, f) is asymptotic to $\mathcal{C}^{\Sigma}$ along V, then (modulo gauge),

$$g(t) = \begin{cases} -t\phi_t^* g & t \in [-1, 0), \\ g_{\mathcal{C}} & t = 0, \end{cases}$$

is a *smooth* solution to (RF) on $W \times [-1, 0]$ for some $W \subset V$.

- Near infinity, g and $g_{\mathcal{C}}$ are time slices of a common smooth Ricci flow!
- *The properties of the asymptotic cone which are inherited by the shrinker are those which propagate backward in time along the Ricci flow.*

Asymptotically conical shrinking solitons

Theorem (with L. Wang, ('13))
If two shrinkers are asymptotic to the same cone along some ends of each, they are isometric near infinity on those ends.

Moreover:

- $\mathrm{Isom}(\Sigma)$ embeds in the isometry group of the end (with Wang, ('18)).
- If (M, g) is complete and $(\mathcal{C}_0^\Sigma, g_\mathcal{C})$ is Kähler, (M, g) is Kähler.

Corresponding statement for expanding solitons is *false* (e.g., Angenent-Knopf ('21)).

Prototype for linear parabolic inequalities

Model (Escauriaza-Seregin-Šverák ('03))
If $u : (\mathbb{R}^n \setminus \overline{B}_r) \times [0, T] \to \mathbb{R}$ *satisfies*

$$\begin{cases} |(\partial_t - \Delta)u| \leq N\,(|u| + |\nabla u|) & \textit{on } (\mathbb{R}^n \setminus \overline{B}_r) \times [0, T] \\ |u(x,t)| \leq Ne^{N|x|^2} & \textit{on } (\mathbb{R}^n \setminus \overline{B}_r) \times [0, T] \\ u(x, T) = 0 & \textit{on } \mathbb{R}^n \setminus \overline{B}_r, \end{cases}$$

then $u \equiv 0$*.*

Note discrepancy between forward and backward-time versions of the statements!

Solutions which terminate in cones

Stepping back: The solution $g(t)$ coincides with a cone near infinity at $t = 0$. What does this by itself tell us about $g = g(-1)$?

Solutions which terminate in cones

On one hand, a cone is a warped product. Warped products *do not* propagate backward in time in general, but:

- **Theorem:** If (M, g, f) is C^2-asymptotic to $\mathcal{C}^\Sigma$ along E where $\Sigma = \Sigma_1 \times \cdots \times \Sigma_k$ is a product of Einstein manifolds (Σ_i, g_i), then, up to isometry,

$$g = dr^2 + h_1^2(r)g_1 + \cdots + h_k^2(r)g_k$$

on a neighborhood of infinity on E.

Solutions which terminate in cones

On the other hand, a cone is a very *particular* warped product, characterized by its scaling invariance

$$\rho_\lambda^* g_{\mathcal{C}} = \lambda^2 g_{\mathcal{C}}, \quad \lambda > 0,$$

relative to the dilation map $\rho(r, \sigma) = (\lambda r, \sigma)$.

One would not expect a well-behaved solution to (RF) to simply *acquire* this invariance in finite time.

Solutions which terminate in cones

Question
What restrictions are there on a solution $g(t)$ *to* (RF) *on* $M \times [0, T)$ *which converges to a cone locally smoothly on some end* $E \subset M$ *as* $t \nearrow T$*?*

- A question of backward uniqueness at the singular time.
- Unknown even whether every complete solution on $M \times [0, T]$ which coincides with $\mathbb{R}^n$ at $t = T$ is static. (True if $\sup |\operatorname{Rm}| < \infty$.)

Main Theorem

Theorem (Global version)
Suppose $g(t)$ is a solution to (RF) on $M \times [-1, 0)$, and there is a diffeomorphism $F : \mathcal{C}_a^\Sigma \to E$ onto some end $E \subset M$ such that

- $|\operatorname{Rm}|(x, t) \leq K r^{-2}(x)$ *on* $E \times [-1, 0)$,
- $F^* g(t)$ *converges locally smoothly to* $g_{\mathcal{C}}$ *on* $\mathcal{C}_a^\Sigma$ *as* $t \nearrow 0$.

Then there exists $f \in C^\infty(M)$ *such that* $g = g(-1)$ *satisfies*

$$\operatorname{Rc}(g) + \nabla\nabla f = \frac{g}{2}.$$

Moreover, for $t \in [-1, 0)$, $g(t) = -t\Phi_t^* g$, *where*

$$\frac{\partial \Phi_t}{\partial t} = -\frac{1}{t} \nabla f \circ \Phi_t, \quad \Phi_{-1} = \operatorname{Id}.$$

Remarks

- We do not assume a priori the existence of a shrinker asymptotic to $\mathcal{C}_a^\Sigma$.
- No assumption on $g(t)$ made off of the end E; solution may elsewhere develop singularity at $t = T$.
- One consequence: a shrinking Ricci soliton

$$\mathrm{Rc}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{g}{2}$$

  that has an asymptotically conical end is gradient (cf. Naber, Chan-Ma-Zhang).
- Corresponding forward-time statement is false: e.g., compact perturbation of $\mathbb{R}^n$.

A local version

Let us write $\tau = -t$ and work with the *backward Ricci flow*

(BRF) $$\frac{\partial}{\partial \tau} g = 2\,\mathrm{Rc}(g).$$

Thus we will be interested in solutions to (BRF) which *emanate* from a cone at $\tau = 0$.

A local version of the main theorem

Theorem (Local version)
Suppose $g(\tau)$ *is a solution to* (BRF) *on* $\mathcal{C}_a^\Sigma \times [0,1]$ *satisfying*

$$g(0) = g_{\mathcal{C}} \quad \textit{and} \quad |\operatorname{Rm}|(x,\tau) \leq Kr^{-2}(x).$$

Then there is $f \in C^\infty(\mathcal{C}_a^\Sigma)$ *such that* $g = g(1)$ *satisfies*

$$\operatorname{Rc}(g) + \nabla\nabla f = \frac{g}{2} \quad \textit{and} \quad \operatorname{grad}_g f = \frac{r}{2}\frac{\partial}{\partial r}.$$

Moreover, if $\Phi_\tau(r,\sigma) = (r/\sqrt{\tau}, \sigma)$*, then*

$$g(\tau) = \tau \Phi_\tau^* g \ \textit{ for } \ \tau \in (0,1] \quad \textit{and} \quad \tau f \circ \Phi_\tau \longrightarrow \frac{r^2}{4} \ \textit{ as } \ \tau \searrow 0.$$

Recasting as a problem of uniqueness

Key Observation: For each $\lambda \geq 1$, the metrics

$$g_\lambda(\tau) = \lambda^{-2} \rho_\lambda^* g(\lambda^2 \tau)$$

solve (BRF) on $\mathcal{C}_a^\Sigma \times [0, \lambda^{-2}]$ and satisfy

$$g_\lambda(0) = \lambda^{-2} \rho_\lambda^* g_{\mathcal{C}} = g_{\mathcal{C}}.$$

That is, the g_λ all emanate from $g_{\mathcal{C}}$.

Recasting as a problem of uniqueness

With an appropriate principle of backward uniqueness, we could conclude $g(\tau) = g_\lambda(\tau)$, i.e.,

$$g(\tau) = \lambda^{-2}\rho_\lambda^* g(\lambda^2\tau)$$

on $\mathcal{C}_a^\Sigma \times [0, \lambda^{-2}]$ for each $\lambda \geq 1$.

Then

$$g(\tau) = \tau\Phi_\tau^* g \quad \text{on} \quad \mathcal{C}_a^\Sigma \times (0, 1],$$

where $\Phi_\tau(r, \sigma) = (r/\sqrt{\tau}, \sigma)$ and $g = g(1)$, and

$$\mathrm{Rc}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{g}{2}, \quad X = \frac{r}{2}\frac{\partial}{\partial r}.$$

Would remain only to show that X is gradient with respect to g.

Step 1: Uniqueness for Ricci flows which terminate in the same cone

Main claim: Two solutions to (BRF) on $\mathcal{C}_a^\Sigma \times [0, T]$ which emanate from the same cone at $\tau = 0$ and have quadratic curvature decay are identical.

- No assumptions are imposed on inner spatial boundary.
- Generalizes earlier result with Wang to non-self-similar solutions; gives agreement on entire domain.

Step 2: A general backward uniqueness principle

Theorem
Suppose that $g(\tau)$ is a family of metrics on $\mathcal{C}_a^\Sigma \times [0, T]$ which emanates smoothly from $g_\mathcal{C}$. If $\mathbf{X}$ and $\mathbf{Y}$ are smooth families of bounded sections over $\mathcal{C}_a^\Sigma$ which vanish at $\tau = 0$ and satisfy

$$
\begin{aligned}
|D_\tau \mathbf{X} + \Delta \mathbf{X}| &\leq \varepsilon\left(|\mathbf{X}| + |\nabla \mathbf{X}| + |\mathbf{Y}|\right) \\
|D_\tau \mathbf{Y}| &\leq C\left(|\mathbf{X}| + |\nabla \mathbf{X}| + |\mathbf{Y}|\right),
\end{aligned}
\tag{1}
$$

where $\varepsilon(r) \longrightarrow 0$ as $r \longrightarrow \infty$, then $\mathbf{X}$ and $\mathbf{Y}$ vanish identically on $\mathcal{C}_b^\Sigma \times [0, cT]$ for some $b \geq a$ and $c = c(n)$.

- Proof uses method of Carleman estimates.

Step 3: The soliton structure is gradient

- We apply this theorem with

$$\mathbf{X} = \nabla \operatorname{Rm} - \widetilde{\nabla}\widetilde{\operatorname{Rm}}, \quad \mathbf{Y} = (g - \tilde{g}, \nabla \tilde{g}, \nabla^{(2)} \tilde{g}).$$

  From the vanishing of **X** and **Y**, we obtain $g_\lambda(\tau) = g(\tau)$ on $\mathcal{C}_a^\Sigma \times [0, \lambda^{-2}]$ for all $\lambda \geq 1$.

- Hence

$$g(\tau) = \tau \Phi_\tau^* g \quad \text{on } \mathcal{C}_a^\Sigma \times (0, 1]$$

  and

$$\operatorname{Rc}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{g}{2},$$

  where $g = g(1)$, $\Phi_\tau(r, \sigma) = (r/\sqrt{\tau}, \sigma)$, and $X = \frac{r}{2}\frac{\partial}{\partial r}$.

Step 3: The soliton structure is gradient

- The one-forms $X^\flat = g(\tau)(X, \cdot)$ satisfy

$$(D_\tau + \Delta)X^\flat = 0, \quad X^\flat(0) = d(r^2/4),$$

so $W = dX^\flat$ satisfies

$$(D_\tau + \Delta)W = \mathrm{Rm} * W, \quad W(0) = 0.$$

- Applying the general principle above with $\mathbf{X} = W$ and $\mathbf{Y} = 0$, we find that $X^\flat$ is closed.

Step 3: The soliton structure is gradient

The global exactness of $X^\flat$ follows with a little more work.

- In fact, $X = \nabla f$ where

$$f(r,\sigma) = \frac{r^2}{4}\left(1 + 8\int_r^\infty s^{-3} R(s,\sigma,1)\, ds\right)$$

This completes the proof of the local version.

Step 4: Extending the soliton structure

For the global version, we apply the local statement to obtain $\bar{f} \in C^\infty(E)$ such that $g = g(1)$ satisfies

$$\mathrm{Rc}(g) + \nabla\nabla\bar{f} = \frac{g}{2}$$

on E.

We claim that this extends to all of M.

Step 4: Extending the soliton structure

We can pull back this structure to a connected component of $\pi^{-1}(E)$ in the universal cover $(\tilde{M}, \tilde{g}(1))$. Our first task is to extend the structure to all of $\tilde{M}$.

Since $\tilde{g}(\tau) = \pi^* g(\tau)$ is real-analytic for $\tau \in (0, 1]$, we have a problem of analytic continuation.

Step 4: Extending the soliton structure

Theorem

Suppose (M, g) *is connected, simply-connected, and real-analytic. If* (U, g, X_U, λ) *is a soliton structure on a connected open set* $U \subset M$*, then* X_U *extends to a vector field* X *on* M *such that* (M, g, X, λ) *is a Ricci soliton. If* X_U *is gradient, so is* X*.*

- Here, (M, g, X, λ) is a soliton structure if

$$\mathrm{Rc}(g) + \frac{1}{2}\mathcal{L}_X = \frac{\lambda}{2} g.$$

Step 4: Extending the soliton structure

- When $X_U = \nabla f$ and $\mathrm{Rc} : TM \to TM$ is nonsingular, one can simply argue that since $W = (\frac{1}{2}\,\mathrm{Rc}^{-1}(\nabla R))^\flat$ coincides with df on U, $dW = 0$ on all of M by analyticity.
- In general, one can define a continuation of soliton structures along paths.
- As for Killing vectors, X and $A = \nabla X(\cdot)$ are determined by their values at a point: in fact, along any path γ,

$$\nabla_{\dot\gamma} X = A(\dot\gamma),$$
$$\nabla_{\dot\gamma} A = \mathrm{Rm}(\dot\gamma, X) + B(\dot\gamma),$$

where $B_{ij}^k = \nabla^k R_{ij} - \nabla_i R_j^k - \nabla_j R_k^i$.

Step 5: Descending from the universal cover

- Having found $\tilde{f}$ such that $(\tilde{M}, \tilde{g}(1), \tilde{f})$ is a shrinking soliton, the last step is to argue that $\tilde{f}$ descends to a smooth function f on M.
- Idea: If a deck transformation fails to preserve $\tilde{f}$, then $(\tilde{M}, \tilde{g})$ splits. This is incompatible with (M, g) having an asymptotically conical end unless it is flat.

Remarks

- It would be interesting to know whether the uniform quadratic bound on the end can be relaxed.
- Unknown in general whether forward or backward uniqueness holds for (RF) assuming only the completeness of the solutions.

Thank you!