The Ricci flow – a minicourse

The Ricci flow is a deformation process for Riemannian metrics which, in a suitable “gauge”, formally resembles the heat equation, and indeed exhibits a number of phenomena which are shared by other diffusion processes. These diffusive properties are highly desirable from the point of view of geometric and topological applications — in principle, the Ricci flow smooths out rough metrics and diffuses their curvature, driving them towards ideal and canonical equilibrium states, thereby restricting the possible topologies which the initial metric can carry. Alas, life is never so straightforward: the Ricci flow equation (suitably interpreted) is degenerate and non-linear, and suffers singularities in finite time, all of which prevent the direct implementation of this programme. Nonetheless, it has proved itself to be one of the most fruitful tools available to the geometric analyst, leading (famously) to proofs of the Poincaré and Thurston conjectures, amongst manifold further important advances.

Geometric motivations aside, the Ricci flow is the canonical heat equation for Riemannian metrics, and gives rise to many remarkable and beautiful geometric structures (solitons, ancient solutions) and analytic features (differential Harnack inequalities, pseudolocality, a gradient-like structure) and as such is a fascinating area of study for topologists, geometers, and analysts alike.

These notes document material presented in a series of lectures at the summer school “geometric flows and relativity” hosted by the Centro de Matemática of the Universidad de la República in Montevideo, Uruguay, in March 2024. Our goal was to provide a fast-paced introduction to the Ricci flow, leading up to Perelman’s key breakthroughs. As such, we omit a number of important aspects of the theory and do not always provide a detailed proof of stated results. For a more comprehensive treatment, we refer the reader to the bibliography. Our main sources have been the books [**3**,**11**,**12**,**25**]; we have also found the lecture notes [**5**] to be a very useful overview of the field.

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Lecture 1. The fundamentals

A smooth one-parameter family $\{g_t\}_{t\in I}$ of smooth metrics g_t on a smooth[1] n-manifold M^n EVOLVES BY/SATISFIES/IS A RICCI FLOW[2] if

$$\frac{d}{dt}g_t = -2\,\mathrm{Rc}_{g_t}\,, \tag{1.1}$$

where Rc_t is the Ricci tensor associated to g_t and the time derivative is understood fibrewise, in the usual sense: for any $x \in M$,

$$\left(\frac{d}{dt}g_t\right)_x \doteqdot \lim_{h\to 0}\frac{(g_{t+h})_x-(g_t)_x}{h}\,.$$

If we introduce local coordinates $\{x^i : U \to \mathbb{R}\}_{i=1}^n$ in some neighbourhood $U \subset M$ of a point $x \in M$, then we may represent $(g_t)_x$ and $(\mathrm{Rc}_t)_x$ as

$$(g_t)_x = g_{ij}(x,t)dx^i\otimes dx^j \;\text{ and }\; (\mathrm{Rc}_{g_t})_x = \mathrm{Rc}_{ij}(x,t)dx^i\otimes dx^j\,,$$

and we see that

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= -2\,\mathrm{Rc}_{ij}\\
&= -2g^{k\ell}\,\mathrm{Rm}_{ikj\ell}\\
&= 2g^{k\ell}\left(\partial_i\Gamma_{kj\ell}-\partial_k\Gamma_{ij\ell}+\Gamma_{kj}^m\Gamma_{im\ell}-\Gamma_{ij}^m\Gamma_{km\ell}\right)\\
&= g^{k\ell}\left(\frac{\partial^2 g_{ij}}{\partial x^k\partial x^\ell}+\frac{\partial^2 g_{k\ell}}{\partial x^i\partial x^j}-\frac{\partial^2 g_{kj}}{\partial x^i\partial x^\ell}-\frac{\partial^2 g_{i\ell}}{\partial x^k\partial x^j}\right)\\
&\quad+\frac{1}{2}g^{k\ell}g^{mn}\Big[\left(\partial_k g_{jm}+\partial_j g_{km}-\partial_m g_{kj}\right)\left(\partial_i g_{n\ell}+\partial_n g_{i\ell}-\partial_\ell g_{in}\right)\\
&\qquad\qquad-\left(\partial_i g_{jm}+\partial_j g_{im}-\partial_m g_{ij}\right)\partial_k g_{n\ell}\Big]\,,
\end{aligned} \tag{1.2}$$

a system of nonlinear second order partial differential equations. Unappealing, certainly, but it does have the redeeming feature that it is weakly parabolic (which explains the choice of sign on the right hand side).

We can make this a little nicer (and gain some very important intuition) by being more selective in our choice of "gauge": about any point $x \in M$, the existence of a HARMONIC COORDINATE chart can be established using standard results on the existence and regularity of solutions to elliptic partial differential equations. These are coordinates satisfying

$$\Delta x^i = 0\,,$$

where Δ is the Laplace–Beltrami operator induced by g; as such, in harmonic coordinates, the Ricci flow system can be seen to take the form

$$\frac{dg_{ij}}{dt} = \sum_{k=1}^n\frac{\partial^2 g_{ij}}{\partial x^k\partial x^k}+\text{terms of lower order}\,. \tag{1.3}$$

[1] Henceforth, we shall stop saying "smooth" so annoyingly often, leaving it for the most part to the reader to decide how regular they wish a given object to be in order to make sense of a given statement.

[2] In fact, we shall soon replace this by a more abstract definition, which may appear more complicated at first but has many advantages. The two definitions are equivalent in the sense that there is a canonical bijection between their solutions.

This suggests that **we should view the Ricci flow as a kind of geometric heat equation for Riemannian metrics** (and also provides justification for the factor of 2 on the right hand side of the equation). We shall soon see that it is quite right to do so, but before pursuing this further, let us first establish some additional useful intuition, this time more geometric.

1.1 Invariance properties. The Ricci flow is invariant under certain canonical operations, in the (not at all precise) sense that these operations take one solution and produce another.

1.1.1 Pullback by diffeomorphisms. If $\{g_t\}_{t\in I}$ is a Ricci flow on M and ϕ a self-diffeomorphism of M, then (since the Ricci curvature is invariant under diffeomorphisms)

$$\left(\frac{d}{dt}\phi^* g_t\right)_x = \left(\frac{d}{dt} g_t\right)_{\phi(x)} = -2(\mathrm{Rc}_{g_t})_{\phi(x)} = -2(\mathrm{Rc}_{\phi^* g_t})_x\,.$$

That is, $\{\phi^* g_t\}_{t\in I}$ is a Ricci flow on M. This is not at all surprising.

On the other hand, if we allow the diffeomorphism to change with time[3]), then we pick up an extra term due to the chain rule:

$$\frac{d}{dt}\phi_t^* g_t = -2\,\mathrm{Rc}_{\phi_t^* g_t} + \mathcal{L}_V(\phi_t^* g_t)\,,$$

where V is the vector field defined by

$$V(\phi(x,t)) \doteqdot \frac{d}{dt}(t\mapsto \phi_t(x))\,.$$

The converse of this statement is that if g_t satisfies the equation

$$\frac{d}{dt} g_t = -2\,\mathrm{Rc}_{g_t} + \mathcal{L}_V g_t$$

for some vector field V, then the family of metrics $\phi_{-t}^* g_t$ satisfies Ricci flow, where ϕ_t is the flow of V.

1.1.2 Time translations. If $\{g_t\}_{t\in I}$ is a Ricci flow on M and $\tau\in\mathbb{R}$, then clearly $\{g_{t+\tau}\}_{t\in I-\tau}$ is a Ricci flow on M.

1.1.3 Parabolic rescaling. If $\{g_t\}_{t\in I}$ is a Ricci flow on M and $\lambda > 0$, then (since the Ricci tensor is scale invariant)

$$\left(\frac{d}{dt}\lambda^2 g_{\lambda^{-2}t}\right)_x = -2\big(\mathrm{Rc}_{g_{\lambda^{-2}t}}\big)_x = -2\big(\mathrm{Rc}_{\lambda^2 g_{\lambda^{-2}t}}\big)_x\,.$$

That is, $\{\lambda^2 g_{\lambda^{-2}t}\}_{t\in\lambda^2 I}$ is a Ricci flow on M.

1.1.4 Orthogonal sums with flat factors. If $\{g_t\}_{t\in I}$ is a Ricci flow on M and $k\in\mathbb{N}$, then $\{g_t + g_{\mathbb{R}^k}\}_{t\in I}$ is a Ricci flow on $M\times\mathbb{R}^k$.

1.1.5 Quotients and lifts. Let $\{g_t\}_{t\in I}$ be a Ricci flow on M and G a group whose elements are isometries of g_t for all t. If G acts freely and transitively, then the induced metrics on M/G evolve by Ricci flow. Conversely, if $\pi : N\to M$ is a covering map, then the lifts of g_t to N evolve by Ricci flow.

[3]We shall always assume the group property $\phi_{t_1}\circ\phi_{t_2} = \phi_{t_1+t_2}$ for one-parameter families of diffeomorphisms ϕ_t.

1.2 Self-similar solutions (a.k.a. solitons). The continuous symmetries of Ricci flow (diffeomorphism, time translation and scaling) give rise to special types of solutions: those that evolve purely by some combination of these symmetries. There are three primary types (but more generally one might consider combinations of these motions).

1.2.1 Steady self-similar solutions. A solution $\{g_t\}_{t\in\mathbb{R}}$ to Ricci flow on a manifold M is called a STEADY SELF-SIMILAR SOLUTION if there is a one-parameter family of diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ of M such that

$$\phi_\varepsilon^* g_{t-\varepsilon} = g_t$$

for all ε and t. Differentiating with respect to ε at $\varepsilon = 0$, we find that such a solution must satisfy the equation

$$0 = \mathrm{Rc}_{g_t} + \tfrac{1}{2}\mathcal{L}_V g_t$$

for all t.

Conversely, if a Riemannian manifold (M, g) satisfies

$$0 = \mathrm{Rc} + \tfrac{1}{2}\mathcal{L}_V g$$

for some vector field V, then the family of metrics $\{g_t \doteqdot \phi_t^* g\}_{t\in\mathbb{R}}$ satisfies

$$\frac{d}{dt} g_t = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g_{t+\varepsilon} = \mathcal{L}_V g_t = \mathcal{L}_V \phi_t^* g = \phi_t^* \mathcal{L}_V g = -2\phi_t^* \mathrm{Rc}_g = -2\,\mathrm{Rc}_{g_t} .$$

1.2.2 Shrinking/expanding self-similar solutions. A solution $\{g_t\}_{t\in\mathbb{R}}$ to Ricci flow on a manifold M is called a HOMOTHETIC SELF-SIMILAR SOLUTION if there is a one-parameter family of diffeomorphisms $\{\phi_t\}_{t\in I}$ of M such that

$$\mathrm{e}^{2\varepsilon} \phi_\varepsilon^* g_{\mathrm{e}^{-2\varepsilon} t} = g_t$$

for all $t \in I$ and ε such that $\mathrm{e}^{-2\varepsilon} t \in I$. Differentiating with respect to ε at $\varepsilon = 0$, we find that such a solution must satisfy the equation

$$0 = g_t + 2t\,\mathrm{Rc}_{g_t} + \tfrac{1}{2}\mathcal{L}_V g_t$$

for all t. There are two cases: if $I = (-\infty, 0)$, then $\{g_t\}_{t\in(-\infty,0)}$ is called a SHRINKING SELF-SIMILAR SOLUTION. If $I = (0, \infty)$, then $\{g_t\}_{t\in(0,\infty)}$ is called an EXPANDING SELF-SIMILAR SOLUTION

Conversely, if a Riemannian manifold (M, g) satisfies

$$0 = g - \mathrm{Rc} + \tfrac{1}{2}\mathcal{L}_V g$$

for some vector field V, then the family of metrics $\{g_t \doteqdot -2t\phi^*_{-\log\sqrt{-t}} g\}_{t\in\mathbb{R}}$ satisfies

$$\frac{d}{dt} g_t = \phi^*_{\log\sqrt{-t}}(-2g - \mathcal{L}_V g) = -2\,\mathrm{Rc}_{g_t} .$$

If a Riemannian manifold (M, g) satisfies

$$0 = g + \mathrm{Rc} + \tfrac{1}{2}\mathcal{L}_V g$$

for some vector field V, then the family of metrics $\{g_t \doteqdot 2t\phi^*_{\log\sqrt{t}} g\}_{t\in\mathbb{R}}$ satisfies

$$\frac{d}{dt} g_t = \phi^*_{\log\sqrt{t}}(2g + \mathcal{L}_V g) = -2\,\mathrm{Rc}_{g_t} .$$

1.2.3 Examples: Einstein metrics. Recall that a Riemannian manifold (M, g) is EINSTEIN if

$$\mathrm{Rc} = \lambda g$$

for some $\lambda \in \mathbb{R}$ ($\lambda \in \{-1, 0, 1\}$ modulo scaling). Einstein metrics provide examples of "trivial" soliton Ricci flows: if $\lambda = 0$, e.g. $(M^n, g) = (\mathbb{R}^n, g_{\mathbb{R}^n})$, then $\{g_t = g\}_{t\in(-\infty,\infty)}$ is a steady self-similar Ricci flow (in this case STATIC), if $\lambda = -1$, e.g. $(M^n, g) = (H^n, g_{H^n})$, then $\{g_t = 2tg\}_{t\in(0,\infty)}$ is an expanding self-similar Ricci flow, and if $\lambda = 1$, e.g. $(M^n, g) = (S^n, g_{S^n})$, then $\{g_t = -2tg\}_{t\in(-\infty,0)}$ is a shrinking self-similar Ricci flow.

Observe that the static Ricci flow $t \mapsto g_t \doteqdot g_{\mathbb{R}^n}$ on Euclidean space $\mathbb{R}^n$ (for example) may also be viewed, not quite trivially, as a steady Ricci flow by pulling back along the flow ϕ of any Killing vector field K (since $\mathcal{L}_K g_{\mathbb{R}^n} = 0$). Similarly, we may view Euclidean space as an expanding or shrinking Ricci flow by pulling back along the flow of the conformally Killing radial vector field $X \doteqdot x^i\partial_i$ or its negative (since $\mathcal{L}_X g_{\mathbb{R}^n} = 2g_{\mathbb{R}^n}$).

1.3 Explicit solutions. Certain "explicit" solutions can be constructed "by hand" by imposing suitable symmetry or algebraic ansätze. We present three examples here, but there are a great many more examples which have been discovered by analogous methods.

1.3.1 Maximal symmetry. By imposing a large enough symmetry group, the Ricci flow equation may be reduced to a (possibly highly complicated) system of ODE.

Example 1.1 (The shrinking sphere)**.** We seek a solution to Ricci flow on S^n starting from a round metric, $g_0 = r_0^2 g_{S^n}$. Since we *expect* roundness to be preserved, we suppose *a priori* that the timeslices are always round,

$$g_t = r^2(t) g_{S^n}\,.$$

The Ricci tensor of g_t is then

$$\mathrm{Rc}_{g_t} = \mathrm{Rc}_{r^2 g_{S^n}} = \mathrm{Rc}_{g_{S^n}} = g|_{S^n} = r^{-2} g_t,$$

while its time derivative is

$$\frac{d}{dt} g_t = 2rr' g_{S^n} = 2\frac{r'}{r} g_t\,.$$

Equating the two yields $rr' = 1$, and hence

$$r^2(t) = r_0^2 - 2t\,,\quad t \in (-\infty, \tfrac{r_0^2}{2})\,.$$

We can play a similar game with self-similar solutions. Though in this case, since the time evolution is already trivial, we may decrease the degree of symmetry by one degree of freedom.

Example 1.2 (Hamilton's cigar)**.** We seek a two dimensional steady soliton on the plane which is circle fibred. I.e. a metric on $\mathbb{R}^2$ which takes the form

$$g = dr^2 + \psi^2(r) d\theta^2$$

in polar coordinates and satisfies

$$-\mathrm{Rc} = \frac{1}{2}\mathcal{L}_V g$$

for some vector field $V = f(r)\partial_r$. In two dimensions, the Ricci curvature is just $\mathrm{Rc} = \mathrm{K}\, g$, where K is the GAUSS CURVATURE, which in our case is given by $\mathrm{K} = -\frac{\psi_{rr}}{\psi}$. The Lie derivative term is found to be

$$\frac{1}{2}\mathcal{L}_V g = f_r dr^2 + f\frac{\psi_r}{\psi}\psi^2\, d\theta^2 .$$

Equating the two, we find that

$$\frac{f_s}{f} = \frac{\psi_s}{\psi} .$$

Taking $f = \psi$, we then find that

$$\psi_{rr} = \psi\psi_r ,$$

which, under the polar coordinate compatibility conditions (ψ admits a smooth odd extension about $r = 0$), is solved by

$$\psi = \tanh r.$$

The resulting metric

$$g = dr^2 + \tanh^2 r\, d\theta^2$$

is called HAMILTON'S CIGAR.

Example 1.3 (Bryant's soliton)**.** We may play the same game in higher dimensions. We now seek an $O(n)$-invariant metric

$$g = dr^2 + \psi^2(r) g_{S^{n-1}}$$

on $\mathbb{R}^n$ ($n \geq 3$). This leads to the system

$$f' = (n-1)\frac{\psi_{rr}}{\psi} , \quad \psi\psi_r f + (n-2)(1-\psi_r^2) = \psi\psi_{rr} .$$

Upon making suitable substitutions, we again are able to obtain a solution satisfying the required compatibility conditions. When $n \geq 3$, it behaves as

$$\psi \sim \sqrt{r} \;\text{ as }\; r \to \infty .$$

These basic ideas have a vast generalization: recall that a HOMOGENEOUS SPACE may be regarded as a Riemannian manifold (M, g) whose isometry group acts transitively. In short, the manifold "looks the same from any vantage point". This degree of symmetry guarantees that the curvature tensor (at any given point) is determined *algebraically* by the metric (at that point), thereby reducing the Ricci flow to a (possibly very complicated) system of ordinary differential equations. For a much more comprehensive examination of the Ricci flow on homogeneous geometries, see Chow and Knopf [**11**].

1.4 Short time existence and uniqueness of solutions. We would like to exhibit the Ricci flow equation as an equation or system of equations for which standard methods of partial differential equations are known to apply. There is indeed a general short-time existence theory which applies to strictly parabolic second order partial differential equations in vector bundles over compact manifolds. Unfortunately, this cannot be applied to the Ricci flow due to the lack of *strict* parabolicity.

For non-linear equations, parabolicity is determined by the linearization.

Lemma 1.1 (Linearization of the Ricci flow)**.** *Suppose that the two parameter family of metrics g_t^ε, $t \in I$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, forms a one-parameter family of Ricci flows $\{g_t^\varepsilon\}_{t\in I}$. Set $g_t \doteqdot g_t^0$. The variation field $h_t \doteqdot \frac{d}{d\varepsilon}\big|_{\varepsilon=0} g_t^\varepsilon$ satisfies, in any local coordinate chart,*

$$\frac{d}{dt}h_{ij} = g^{k\ell}\left(\nabla_k\nabla_\ell h_{ij} + \nabla_i\nabla_j h_{k\ell} - \nabla_\ell\nabla_j h_{ik} - \nabla_k\nabla_i h_{j\ell}\right) . \tag{1.4}$$

Proof. We leave the proof as an exercise. □

The equation (1.4) is weakly but not strictly parabolic. It turns out that the lack of strict parabolicity is due precisely to the Bianchi identities. Treating the Bianchi identities as a constraint, Hamilton [**18**] is able to prove short-time existence using direct methods (including the Nash–Moser implicit function theorem). Soon after Hamilton's work, de Turck found a way to relate the Ricci flow to a strictly parabolic equation, to which the standard theory may be more readily applied.

Theorem 1.2 (Short time existence and uniqueness)**.** *Let M^n be a compact manifold. Given a metric g_0 on M there exists $\delta > 0$ and a Ricci flow $\{g_t\}_{t\in(0,\delta)}$ on M such that g_t converges uniformly to g_0 as $t \to 0$ (in the smooth sense if g_0 is smooth). Moreover, any other Ricci flow starting from g_0 agrees with g_t on their common interval of existence. Finally, the Ricci flow $\{g_t\}_{t\in(0,\delta)}$ depends continuously on g_0 (in the smooth sense if g_0 is smooth).*

Sketch of the de Turck argument. Fix some background metric $\overline{g}$ on M and consider, instead of the Ricci flow, the Ricci-harmonic map flow system

$$\begin{cases} \dfrac{d}{dt}\Phi_t = \Delta_{g_t,\overline{g}}\Phi_t \\ \dfrac{d}{dt}g_t = \; -2\,\mathrm{Rc}_{g_t} , \end{cases} \tag{1.5}$$

where $\Phi_t : M \to M$ and $\Delta_{g_t,\overline{g}}$ is the map Laplacian with the domain endowed with the metric g_t and the codomain endowed with $\overline{g}$. In fact, don't consider (1.5); consider instead the system

$$\begin{cases} \dfrac{d}{dt}\Phi_t = \Delta_{\Phi_t^*\tilde{g}_t,\overline{g}}\Phi_t \\ \dfrac{d}{dt}\tilde{g}_t = \; -2\,\mathrm{Rc}_{\tilde{g}_t} - \mathcal{L}_{(\Phi_t^{-1})^*\frac{d}{dt}\Phi_t}\tilde{g}_t , \end{cases} \tag{1.6}$$

which is related to (1.5) by $g_t \doteqdot \Phi_t^*\tilde{g}_t$. But the system (1.6) is *strictly* parabolic, and hence admits a (unique) solution $\{(\Phi_t, g_t)\}_{t\in[0,\delta)}$ for a short time (which depends continuously on g_0), thereby providing the desired Ricci flow $\{g_t \doteqdot \Phi_t^*\tilde{g}_t\}_{t\in[0,\delta)}$. □

For a more in-depth discussion of de Turck's argument, especially its relkation to the Bianchi identities, see [**3**].

1.5 The space-time formalism. A one parameter family $\{g_t\}_{t\in I}$ of metrics $t \mapsto g_t \in \Gamma(T^*M \otimes T^*M)$ may (perhaps more properly) be viewed as a map $(x,t) \mapsto g_{(x,t)} \doteqdot (g_t)_x \in T^*M \otimes T^*M$. This map may be exhibited as a section of a bundle over $M \times I$ whose fibres are those of $T^*M \otimes T^*M$. Indeed, if we introduce the SPATIAL TANGENT BUNDLE

$$\mathfrak{S} \doteqdot \{\xi \in T(M\times I) : dt(\xi) = 0\}$$

of $M \times I$, where here $t : M \times I \to \mathbb{R}$ denotes the projection onto the second factor, then any $(x,t) \mapsto g_{(x,t)}$ induces (canonically) a section g of $\mathfrak{S}^* \otimes \mathfrak{S}^*$ (which we shall refer to as a TIME-DEPENDENT METRIC). Similarly, the Ricci tensors Rc_t of the metrics g_t induce a section Rc of $\mathfrak{S}^* \otimes \mathfrak{S}^*$. From this point of view, the Ricci flow equation becomes

$$\mathcal{L}_{\partial_t} g = -2\,\mathrm{Rc}\,, \tag{1.7}$$

where ∂_t is the canonical vector field on I. Indeed, since the flow of ∂_t is $\phi_s(x,t) = (x, t+s)$,

$$\mathcal{L}_{\partial_t} g_{(x,t)} = \left.\frac{d}{ds}\right|_{s=0} (\phi_s^* g)_{(x,t)} = \left.\frac{d}{ds}\right|_{s=0} g_{(x,t+s)} = \frac{d}{dt}(g_t)_x\,.$$

This may seem like abstract nonsense (and it is), but it does have a more pragmatic purpose: it naturally gives rise to the canonical (and very useful) SPACETIME CONNECTION.

Proposition 1.3 (The spacetime connection)**.** *Given any metric g on the spatial tangent bundle $\mathfrak{S}$ of $M \times I$ there exists a unique connection $\nabla : T(M \times I) \times \Gamma(\mathfrak{S}) \to \mathfrak{S}$ on $\mathfrak{S}$ which is*

(1) METRIC*: for any $U, V \in \Gamma(\mathfrak{S})$ and $\xi \in T(M \times I)$,*

$$0 = \nabla_\xi g(U,V) \doteqdot \xi g(U,V) - g(\nabla_\xi U, V) - g(U, \nabla_\xi V)$$

(2) SPATIALLY SYMMETRIC*: for any $U, V \in \Gamma(\mathfrak{S})$,*

$$\nabla_U V - \nabla_V U = [U,V]\,,$$

and

(3) IRROTATIONAL*: the tensor $\mathcal{S} \in \Gamma(\mathfrak{S}^* \otimes \mathfrak{S})$ defined by*

$$\mathcal{S}(V) \doteqdot \nabla_t V - [\partial_t, V]$$

is g-self-adjoint.

Proof. Observe that the properties (1)–(3))yield

$$\begin{aligned} 0 &= \partial_t(g(U,V)) - g(\nabla_t U, V) - g(U, \nabla_t V) \\ &= \mathcal{L}_{\partial_t} g(U,V) - g(\mathcal{S}(U), V) - g(U, \mathcal{S}(V)) \\ &= \mathcal{L}_{\partial_t} g(U,V) - 2\mathcal{S}(U,V)\,, \end{aligned}$$

and hence

$$\mathcal{S} = \frac{1}{2}\mathcal{L}_{\partial_t} g\,.$$

In particular, along a Ricci flow,

$$\mathcal{S} = -\,\mathrm{Rc}\,,$$

and hence, for any time dependent vector field $V \in \Gamma(\mathfrak{S})$, we have the formula

$$\nabla_t V = [\partial_t, V] - \mathrm{Rc}(V)\,. \tag{1.8}$$

Since $T(M \times I) = \mathfrak{S} \oplus \mathbb{R}\partial_t$ and the properties (1) and (2) ensure that $\nabla_\xi V$ satisfies the Levi–Civita formula when $\xi \in \mathfrak{S}$, this completely determines ∇.

Conversely, the Levi–Civita formula combined with (1.8) defines a connection on $\mathfrak{S}$. □

In the sequel, when it is clear that we are working in the "time-dependent" setting, we shall conflate $\mathfrak{S}$ with TM and we will often use the data $(M \times I, g)$ (where g is a spacetime metric satisfying (1.7)) to denote a Ricci flow.

1.6 Exercises.

Exercise 1.1. Show that the system (1.2) is weakly parabolic.

Exercise 1.2. Prove that the system (1.2) does indeed take the form (1.3) in harmonic coordinates.

Exercise 1.3. Show that the sectional curvature of a metric of the form

$$g = dr^2 + \psi^2(r,\theta)d\theta^2$$

is given by

$$K = -\frac{\psi_{rr}}{\psi}\,.$$

Exercise 1.4. Consider metric on S^2 which takes the form

$$g = dr^2 + \psi^2(r)d\theta^2$$

in spherical polar coordinates. Suppose that g satisfies

$$\mathrm{Rc} = g + \frac{1}{2}\mathcal{L}_V g$$

for some radial vector field $V = f(r)\partial_r$. Find f, and hence determine g.

Exercise 1.5. Show that the map $V \mapsto \mathcal{S}(V)$ of Proposition 1.3 is indeed linear over the ring of smooth functions and takes values in $\Gamma(\mathfrak{S})$ (and hence induces a genuine tensor $\mathcal{S} \in \Gamma(\mathfrak{S}^* \otimes \mathfrak{S})$ as claimed).

Lecture 2. Long time behaviour

2.1 The maximum principle. The Ricci flow admits various useful maximum principles.

2.1.1 Maximum principle for scalars.

Proposition 2.1. *Let $(M^n \times [0,T), g)$ be a Ricci flow on a compact manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T)) \cap C^0(M^n \times [0,T))$ satisfies*

$$(\partial_t - \Delta - \nabla_b)u \le cu$$

for some time-dependent vector field b and some locally bounded function $c : M^n \times [0,T) \to \mathbb{R}$, where the Laplacian Δ is taken with respect to the spacetime metric induced by g. If $\max_{M^n\times\{0\}} u \le 0$, then

$$\max_{M^n\times\{t\}} u \le 0 \;\; \textit{for all} \;\; t \in [0,T]\,. \tag{2.1}$$

If $c \equiv 0$, then

$$\max_{M^n\times[0,T]} u = \max_{M^n\times\{0\}} u\,. \tag{2.2}$$

Proof. Given $\varepsilon > 0$ and $s \in (0,T)$, consider $u_{\varepsilon,s}(x,t) \doteqdot u(x,t) - \varepsilon \mathrm{e}^{(C+1)t}$, where $C \doteqdot \max_{M^n\times[0,s]} c$. We claim that $u_{\varepsilon,s} < 0$ in $M^n \times [0,s]$. Suppose, to the contrary, that $u_{\varepsilon,s}(x_0,t_0) \ge 0$ for some point $(x_0,t_0) \in M^n \times [0,s]$. Since $u_{\varepsilon,s}(\cdot,0) < 0$, there exists a positive earliest such time, which we take to be t_0, in which case $u(x_0,t_0)$. At the point (x_0,t_0),

$$\begin{aligned} 0 \le (\partial_t - \Delta - \nabla_b)u_{\varepsilon,s} &= cu - \varepsilon(C+1)\mathrm{e}^{(C+1)t} \\ &= \varepsilon\mathrm{e}^{(C+1)t}c - \varepsilon(C+1)\mathrm{e}^{(C+1)t} \\ &\le \; -\varepsilon\mathrm{e}^{(C+1)t} < 0\,, \end{aligned}$$

which is absurd. We conclude that $u_{\varepsilon,s} < 0$ in $M^n \times [0,s]$. But $\varepsilon > 0$ and $s \in (0,T)$ were arbitrary. Taking $\varepsilon \to 0$ and then $s \to T$ yields the claim. □

Of course, the same argument applies with the inequalities reversed, leading to a **minimum principle**.

The following ODE COMPARISON PRINCIPLE is an immediate consequence of the maximum principle.

Lemma 2.2 (ODE comparison principle)**.** *Let $(M^n \times [0,T), g)$ be a Ricci flow on a compact manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T)) \cap C^0(M^n \times [0,T))$ satisfies*

$$\partial_t u \le \Delta u + \nabla_b u + F(u)\,, \tag{2.3}$$

for some time-dependent vector field b and some locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$, where the Laplacian Δ and covariant derivative ∇ are taken with respect to the induced metric. If $u \le \phi_0$ at $t = 0$ for some $\phi_0 \in \mathbb{R}$, then $u(x,t) \le \phi(t)$ for all $x \in M^n$ and $0 \le t < T$, where ϕ is the solution to the ODE

$$\begin{cases} \dfrac{d\phi}{dt} = F(\phi) \quad \textit{in} \;\; (0,T), \\ \phi(0) = \phi_0\,. \end{cases} \tag{2.4}$$

Proof. Fix $s \in (0,T)$. Since F is locally Lipschitz, there exists some $L < \infty$ such that

$$
\begin{aligned}
(\partial_t - \Delta - \nabla_b)(u-\phi) &\le F(u) - F(\phi) \\
&\le L|u-\phi| = L\,\mathrm{sign}(u-\phi)(u-\phi)
\end{aligned}
$$

in $M^n \times (0,s]$, where $\mathrm{sign}(u-\phi)$ is the sign of the expression $u-\phi$. The claim now follows, within $M^n \times [0,s]$, from Theorem 2.1. Taking $s \to T$ completes the proof. ☐

Again, one can reverse the inequalities to obtain the corresponding ODE comparison from below.

The strong maximum principle also passes to the geometric setting.

Theorem 2.3. *Let $(M^n \times (0,T), g)$ be a Ricci flow on a connected manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T))$ is non-positive and satisfies*

$$
(\partial_t - \Delta - \nabla_b)u \le cu \tag{2.5}
$$

for some time-dependent vector field b and some function $c : M^n \times (0,T) \to \mathbb{R}$, where the Laplacian Δ and covariant derivative ∇ are taken with respect to the induced metric. If $u(x_0,t_0) = 0$ for some $(x_0,t_0) \in M^n \times (0,T)$, then $u(x,t) = 0$ for all $(x,t) \in M^n \times (0,t_0]$.

Proof. In local coordinates $\{x^i\}_{i=1}^n$ for a connected coordinate patch $U \subset M^n$ about x_0, u satisfies

$$
\partial_t u \le g^{ij}u_{ij} + (b^k + g^{ij}\Gamma_{ij}{}^k)u_k + cu\,.
$$

The classical strong maximum principle then implies that $u \equiv 0$ in $U \times (0,t_0]$. Since M^n is connected, the claim follows from a standard 'open-closed' argument. ☐

2.1.2 A maximum principle for symmetric bilinear forms. Hamilton [**18**] discovered the following beautiful maximum principle for symmetric bilinear forms.

Theorem 2.4 (Tensor maximum principle)**.** *Let g be a time-dependent metric on a compact manifold M^n. Suppose that $S \in \Gamma(T^*M^n \odot T^*M^n)$ satisfies*

$$
(\nabla_t - \Delta - \nabla_b)S_{(x,t)}(v,v) \ge F(x,t,S_{(x,t)})(v,v) \;\; \text{for all} \;\; (x,t,v) \in TM^n
$$

for some time-dependent vector field $b \in \Gamma(TM)$ and some time-dependent vertical section F of $\pi^*(T^*M^n \odot T^*M^n)$ which is Lipschitz in the fibre and satisfies the NULL EIGENVECTOR CONDITION*:*

$$
F(x,t,T_{(x,t)})(v,v) \ge 0 \;\; \text{whenever} \;\; T_{(x,t)}(v) = 0\,.
$$

If $S_{(x,0)} \ge 0$ for all $x \in M^n$, then $S_{(x,t)} \ge 0$ for all $(x,t) \in M^n \times [0,T)$.

Proof. Fix $\varepsilon > 0$ and $\sigma \in (0,T)$. We claim that the tensor

$$
S^{\varepsilon,\sigma} \doteqdot S + \varepsilon \mathrm{e}^{(C_\sigma+1)t} g
$$

is positive definite in $M^n \times [0,\sigma]$, where $C_\sigma \doteqdot \max_{M^n\times[0,\sigma]} \mathrm{Lip} F(x,y,\cdot)$. By hypothesis, $S^{\varepsilon,\sigma}_{(x,0)} > 0$ for all $x \in M^n$. So suppose, contrary to the claim, that there exist $x_0 \in M^n$, $t_0 \in (0,\sigma]$ and $V_0 \in T_{x_0}M^n \setminus \{0\}$ such that $S^{\varepsilon,\sigma}_{(x,t)} > 0$ for each $(x,t) \in M^n \times [0,t_0)$ but $S^{\varepsilon,\sigma}_{(x_0,t_0)}(V_0,V_0) = 0$. Extend V_0 locally in space by solving

$$
\nabla_{\gamma'} V \equiv 0
$$

along radial g_{t_0}-geodesics γ emanating from x_0 and then extend the resulting local vector field in the time direction by solving

$$\nabla_t V \equiv 0\,.$$

Then $\nabla V(x_0, t_0) = 0$ and $\nabla_t V(x_0, t_0) = 0$. We claim that we also have $\Delta V(x_0, t_0) = 0$. To see this, let $\{e_i\}_{i=1}^n$ be an orthonormal frame at x_0 and parallel translate it along geodesics emanating from x_0, all of this respect to g_{t_0}. We then may compute using $e_i = \gamma_i'$ along γ_i with $\gamma_i'(0) = e_i$ that

$$\Delta V(x_0, t_0) = \sum_{i=1}^{n} \Big(\nabla_{e_i}(\nabla_{e_i} V) - \nabla_{\nabla_{e_i} e_i} V\Big)(x_0, t_0) = 0\,.$$

Now set

$$s_{\varepsilon,\sigma}(x,t) \doteqdot S^{\varepsilon,\sigma}_{(x,t)}(V_{(x,t)}, V_{(x,t)})$$

for (x,t) near (x_0, t_0). Then $s_{\varepsilon,\sigma}(x,t) \geq 0$ for (x,t) in a small parabolic neighborhood $B_r(x_0,t_0) \times (t_0 - r^2, t_0]$ of (x_0,t_0) and $s_{\varepsilon,\sigma}(x_0,t_0) = 0$, and hence

$$\begin{aligned}
0 &\geq (\partial_t - \Delta - \nabla_b) s_{\varepsilon,\sigma}|_{(x_0,t_0)} \\
&= (\nabla_t - \Delta - \nabla_b) S^{\varepsilon,\sigma}|_{(x_0,t_0)}(V_0, V_0) \\
&\geq F(x_0, t_0, S_{(x_0,t_0)})(V_0,V_0) + \varepsilon(C_\sigma + 1)\mathrm{e}^{(C_\sigma+1)t} g_{(x_0,t_0)}(V_0,V_0) \\
&\geq F(x_0, t_0, S^{\varepsilon,\sigma}_{(x_0,t_0)})(V_0,V_0) - C_\sigma\big(S^{\varepsilon,\sigma}_{(x_0,t_0)} - S_{(x_0,t_0)}\big)(V_0,V_0) \\
&\quad + \varepsilon(C_\sigma+1)\mathrm{e}^{(C_\sigma+1)t} g_{(x_0,t_0)}(V_0,V_0) \\
&\geq \varepsilon \mathrm{e}^{(C_\sigma+1)t_0} g_{(x_0,t_0)}(V_0,V_0) \\
&> 0\,,
\end{aligned}$$

which is absurd. So $S^{\varepsilon,\sigma}$ indeed remains positive definite in $[0,\sigma]$. The claim follows since $\varepsilon > 0$ and $\sigma \in (0,T)$ are arbitrary. □

2.1.3 The vector bundle maximum principle. There is even a version of the maximum principle for sections of a vector bundle.

Theorem 2.5 (Vector bundle maximum principle)**.** *Let $\pi : E \to M^n \times [0,T)$ be a time-dependent vector bundle over a compact manifold M^n which is equipped with a metric h and a metric connection ∇, and let $\Omega \subset E$ be a closed subset which is* CONVEX IN THE FIBRE *and* INVARIANT UNDER PARALLEL TRANSPORT*. Given any time-dependent vector field V on M and any* TIME-DEPENDENT VERTICAL VECTOR FIELD $F \in \Gamma(\pi^*E \to E)$ *which* POINTS INTO Ω*, any solution $u \in \Gamma(E)$ to*

$$(\nabla_t - \Delta - \nabla_V)u = F(u)$$

satisfying $u(x,0) \in \Omega$ for all $x \in M$ satisfies $u(x,t) \in \Omega$ for all $x \in M$ and all $t \in [0,T)$.

The proof of Theorem 2.5 uses different tools, but is very similar in nature to that of Theorem 2.4 see e.g. [**3**]; we omit it for brevity.

2.2 Evolution of geometry under Ricci flow. The Ricci flow equation induces diffusion equations of various types for the geometric features of the evolving geometry.

2.2.1 Distance distortion estimates. Let $(M \times I, g)$ be a Ricci flow. Given any curve $\gamma : [0, L] \to M$ in M,

$$\begin{aligned}\frac{d}{dt}\operatorname{length}(\gamma) &= \frac{d}{dt}\int_0^L |\gamma'(s)|\,ds\\ &= \frac{d}{dt}\int_0^L \sqrt{g(\gamma'(s),\gamma'(s))}\,ds\\ &= -\int_0^L \mathrm{Rc}\left(\frac{\gamma'(s)}{|\gamma'(s)|},\frac{\gamma'(s)}{|\gamma'(s)|}\right)\,ds\,.\end{aligned}$$

Thus, the Ricci curvature determines the rate of change of lengths of curves. Applying this at minimizing geodesics yields the following elementary distance distortion estimates.

Proposition 2.6. *If $\underline{K}g \le \mathrm{Rc} \le \overline{K}g$ along a complete Ricci flow $(M \times [t_1, t_2], g)$, then*

$$\underline{K}\operatorname{dist}(x,y,t) \le -\frac{d}{dt}\operatorname{dist}(x,y,t) \le \overline{K}\operatorname{dist}(x,y,t)$$

in the (forward and backward, respectively) barrier sense, and in the classical sense almost everywhere. Furthermore,

$$\mathrm{e}^{-\overline{K}(t_2-t_1)} \le \frac{\operatorname{dist}(x,y,t_2)}{\operatorname{dist}(x,y,t_1)} \le \mathrm{e}^{-\underline{K}(t_2-t_1)}\,.$$

Proof. Given any two distinct points $x, y \in M$ and any time $t_0 \in [t_1, t_2]$, we can find a distance minimizing geodesic $\gamma : [0, L] \to M$ with respect to the metric at time t_0 which joins x and y. We may assume that γ is parametrized by arclength, so that the distance $d(x, y, t_0)$ between x and y with respect to the metric at time t_0 is equal to L. By the above computation and the hypotheses,

$$\underline{K}\operatorname{length}(\gamma) \le -\frac{d}{dt}\operatorname{length}(\gamma) \le \overline{K}\operatorname{length}(\gamma)\,.$$

Since $\operatorname{length}(\gamma) \ge d(x, y, \cdot)$ with equality at time $t = t_0$, we have found a (forward resp. backward) barrier satisfying the inequalities. The a.e. classical inequality then follows because $t \mapsto \operatorname{dist}(x, y, t)$ is Lipschitz (and hence admits a classical derivative at a.e. time, which must be equal to that of the barrier because of the first order contact). We may then integrate to obtain the distance distortion estimates. □

These estimates are quite crude. The following argument (inspired by the proof of the Bonnet–Meyers theorem) provides a much sharper estimate on long geodesics.

Proposition 2.7. *If $\mathrm{Rc} \le (n-1)Kg$ for some $K > 0$ along a complete Ricci flow $(M^n \times [t_1, t_2], g)$, then*

$$-\frac{d}{dt}\operatorname{dist}(x,y,t) \le 10K^{\frac{1}{2}}$$

in the (forward and backward, respectively) barrier sense, and in the classical sense almost everywhere. Thus,

$$\operatorname{dist}(x,y,t_2) \ge \operatorname{dist}(x,y,t_1) - 10K^{\frac{1}{2}}(t_2-t_1)$$

Proof. Given any two distinct points $x, y \in M$ and any time $t_0 \in [t_1, t_2]$, we can find a distance minimizing geodesic $\gamma : [0, L] \to M$ with respect to the metric at time t_0 which joins x and y. We may assume that γ is parametrized by arclength, so that the distance

$d(x, y, t_0)$ between x and y with respect to the metric at time t_0 is equal to L. By the above computation and the hypotheses,

$$-\frac{d}{dt}\,\mathrm{length}(\gamma, t) \le K\,\mathrm{length}(\gamma, t)\,.$$

In case $L \le 2K^{-\frac{1}{2}}$, we have

$$-\frac{d}{dt}\,\mathrm{length}(\gamma, t) \le 2K^{\frac{1}{2}} \le 10K^{\frac{1}{2}}$$

at $t = t_0$. The interesting case is $L \ge 2K^{\frac{1}{2}}$. Choose a parallel orthonormal frame $\{E_i\}_{i=1}^n$ along γ such that $E_1 = \gamma'$ and let $\varphi : [0, L] \to \mathbb{R}$ be a smooth function satisfying

$$0 \le \varphi \le 1\,, \quad \varphi|_{[K^{-\frac{1}{2}}, L-K^{-\frac{1}{2}}]}\,, \quad |\varphi'|^2 \le 4K^{\frac{1}{2}}\,.$$

For $i = 2, \ldots, n$, the second variation formula for length yields

$$\begin{aligned} 0 &\le \int_0^L \left[|\nabla_s(\varphi E_i)|^2 - \mathrm{Rm}(\gamma', \varphi E_i, \gamma', \varphi E_i)\right]\,ds \\ &= \int_0^L \left[|\varphi'|^2 - \varphi^2\,\mathrm{Rm}(\gamma', E_i, \gamma', E_i)\right]\,ds \end{aligned}$$

at $t = t_0$, since γ is minimizing at $t = t_0$. Tracing, we obtain

$$0 \le \int_0^L \left[(n-1)|\varphi'|^2 - \varphi^2\,\mathrm{Rc}(\gamma'\gamma')\right]\,ds\,.$$

Thus,

$$\begin{aligned} \int_0^L \mathrm{Rc}(\gamma', \gamma')\,ds &= \int_0^L \left(\varphi^2\,\mathrm{Rc}(\gamma', \gamma') + (1-\varphi^2)\,\mathrm{Rc}(\gamma', \gamma')\right)\,ds \\ &\le (n-1)\int_0^L \left(|\varphi'|^2 + (1-\varphi^2)K\right)\,ds \\ &= (n-1)\int_{[0,K^{-\frac{1}{2}}]\cup[d-K^{-\frac{1}{2}},d]} \left(|\varphi'|^2 + (1-\varphi^2)K\right)\,ds \\ &\le 10(n-1)K^{\frac{1}{2}}\,. \end{aligned}$$

The claims now follow as before. □

2.2.2 The first variation of volume. Recall that, on any Riemannian manifold (M^n, g), the Riemannian measure of any compact subset $K \subset M$ is defined by

$$\mu(K) = \int_K d\mu \doteqdot \sum_\alpha \int_{x_\alpha(U_\alpha)} (x_\alpha^{-1})^*\left(\rho_\alpha\sqrt{\det g_\alpha}\right)dx\,,$$

where $\{(U_\alpha, x_\alpha)\}_\alpha$ is any locally finite covering of K, $\{\rho_\alpha\}_\alpha$ is any subordinate partition of unity, dx is the Lebesgue measure on $\mathbb{R}^n$, and g_α is the component matrix of g induced by the α-th chart. If $\{g_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a one-parameter family of metrics on M^n with $g_0 = g$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\,g_\varepsilon = h$, then, with respect to any coordinate chart,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\sqrt{\det g_\varepsilon} = \frac{1}{2}\sqrt{\det g}\,\mathrm{tr}_g\,h\,.$$

We thus obtain the FIRST VARIATION FORMULA:

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\mu_\varepsilon(K) = -\frac{1}{2}\int_K \mathrm{tr}_g\, h\, d\mu\,.$$

In particular,

Proposition 2.8. *along a Ricci flow $(M \times I, g)$,*

$$\frac{d}{dt}\mu(K) = -\int_K \mathrm{R}\; d\mu$$

for any compact $K \subset M$.

2.2.3 Evolution of the Ricci and scalar curvatures. Given a Ricci flow $\{g_t\}_{t\in I}$, applying Lemma 1.1 to the one parameter family $\{g_t^\varepsilon \doteqdot g_{t+\varepsilon}\}_{t+\varepsilon\in I}$ of time translated Ricci flows yields

$$\begin{aligned}\frac{d}{dt}\mathrm{Rc}_{ij} &= g^{k\ell}\left(\nabla_k\nabla_\ell \mathrm{Rc}_{ij} + \nabla_i\nabla_j \mathrm{Rc}_{k\ell} - \nabla_\ell\nabla_j \mathrm{Rc}_{ik} - \nabla_k\nabla_i \mathrm{Rc}_{j\ell}\right)\\ &= \Delta \mathrm{Rc}_{ij} + Q_{ij}\,,\end{aligned}$$

where

$$\begin{aligned}Q_{ij} &\doteqdot g^{k\ell}\left(\nabla_i\nabla_j \mathrm{Rc}_{k\ell} - \nabla_k\nabla_j \mathrm{Rc}_{i\ell} - \nabla_k\nabla_i \mathrm{Rc}_{j\ell}\right)\\ &= g^{k\ell}g^{pq}\left(\nabla_i\nabla_j \mathrm{Rm}_{kp\ell q} - \nabla_k\nabla_j \mathrm{Rm}_{ip\ell q} - \nabla_k\nabla_i \mathrm{Rm}_{jp\ell q}\right)\,.\end{aligned}$$

We can write this in a more natural way by applying the Bianchi identities and making use of the space-time connection.

Proposition 2.9. *Along a Ricci flow $(M^n \times I, g)$,*

$$(\nabla_t - \Delta)\,\mathrm{Rc} = Q(\mathrm{Rc})\,, \tag{2.6}$$

where, with respect to any local basis,

$$Q(\mathrm{Rc})_{ij} \doteqdot 2\,\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{k\ell}\,.$$

Proof. The second Bianchi identity and the definition and symmetries of curvature yield

$$\begin{aligned}g^{k\ell}g^{pq}\nabla_i\nabla_j \mathrm{Rm}_{kp\ell q} &= -\,g^{k\ell}g^{pq}\left(\nabla_i\nabla_k \mathrm{Rm}_{pj\ell q} + \nabla_i\nabla_p \mathrm{Rm}_{jk\ell q}\right)\\ &= -\,g^{k\ell}g^{pq}\left(\nabla_i\nabla_k \mathrm{Rm}_{pj\ell q} + \nabla_i\nabla_k \mathrm{Rm}_{jpq\ell}\right)\\ &= -\,2g^{k\ell}g^{pq}\nabla_i\nabla_k \mathrm{Rm}_{pj\ell q}\\ &= -\,2g^{k\ell}g^{pq}\left(\nabla_k\nabla_i \mathrm{Rm}_{pj\ell q} + (\mathrm{Rm}_{ki}\,\mathrm{Rm})_{pj\ell q}\right)\\ &= 2g^{k\ell}g^{pq}\left(\nabla_k\nabla_i \mathrm{Rm}_{jp\ell q} + (\mathrm{Rm}_{ik}\,\mathrm{Rm})_{pj\ell q}\right)\end{aligned}$$

The second and first Bianchi identities then yield

$$\begin{aligned}Q_{ij} &= g^{k\ell}g^{pq}\left(2(\mathrm{Rm}_{ik}\,\mathrm{Rm})_{pj\ell q} + \nabla_k\nabla_i \mathrm{Rm}_{jp\ell q} - \nabla_k\nabla_j \mathrm{Rm}_{ip\ell q}\right)\\ &= g^{k\ell}g^{pq}\left(2(\mathrm{Rm}_{ik}\,\mathrm{Rm})_{pj\ell q} - \nabla_k\nabla_p \mathrm{Rm}_{ij\ell q}\right)\\ &= g^{k\ell}g^{pq}\left(2(\mathrm{Rm}_{ik}\,\mathrm{Rm})_{pj\ell q} + \nabla_k\nabla_p \mathrm{Rm}_{iqj\ell} - \nabla_k\nabla_p \mathrm{Rm}_{i\ell jq}\right)\\ &= -\,2g^{k\ell}(\mathrm{Rm}_{ik}\,\mathrm{Rc})_{j\ell} + g^{k\ell}g^{pq}\left(\nabla_k\nabla_p \mathrm{Rm}_{iqj\ell} - \nabla_p\nabla_k \mathrm{Rm}_{iqj\ell}\right)\\ &= 2\,\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{k\ell} - 2\,\mathrm{Rc}^2_{ij} + g^{k\ell}g^{pq}(\mathrm{Rm}_{pk}\,\mathrm{Rm})_{iqj\ell}\,.\end{aligned}$$

Observe that[4]

$$\begin{aligned} g^{k\ell}g^{pq}(\mathrm{Rm}_{pk}\,\mathrm{Rm})_{iqj\ell} =& -g^{k\ell}g^{pq}g^{mn}\,(\mathrm{Rm}_{pkim}\,\mathrm{Rm}_{nqj\ell}+\mathrm{Rm}_{pkqm}\,\mathrm{Rm}_{inj\ell}\\ &+\mathrm{Rm}_{pkjm}\,\mathrm{Rm}_{iqn\ell}+\mathrm{Rm}_{pk\ell m}\,\mathrm{Rm}_{iqjn})\\ =&\, g^{k\ell}g^{pq}g^{mn}\,(\mathrm{Rm}_{kpim}\,\mathrm{Rm}_{nqj\ell}-\mathrm{Rm}_{n\ell iq}\,\mathrm{Rm}_{pkjm}\\ &\mathrm{Rm}_{mqpk}\,\mathrm{Rm}_{inj\ell}-\mathrm{Rm}_{pk\ell m}\,\mathrm{Rm}_{iqjn})\\ =&\,0\,. \end{aligned}$$

So the claim follows upon applying the identity

$$\nabla_t\,\mathrm{Rc}_{ij}=\frac{d}{dt}\,\mathrm{Rc}_{ij}+2\,\mathrm{Rc}^2_{ij}\;. \qquad \square$$

Taking the trace of (2.6), we find that

Corollary 2.10. *along a Ricci flow $(M^n\times I, g)$,*

$$(\partial_t-\Delta)\,\mathrm{R}=2|\mathrm{Rc}|^2\,. \tag{2.7}$$

Applying the maximum principle to these equations yields useful information about the behaviour of curvature under Ricci flow.

Proposition 2.11 (Scalar curvature tends towards positive)**.** *Let $\{g_t\}_{t\in[\alpha,\omega)}$ be a Ricci flow on a compact manifold M.*

(1) If $\min_{M\times\{\alpha\}}\mathrm{R}=0$ then either $\mathrm{Rc}\equiv 0$ or $\mathrm{R}>0$ for $t\in(\alpha,\omega)$.
(2) If $\min_{M\times\{\alpha\}}\mathrm{R}=r^{-2}>0$, then $\omega\le\alpha+\frac{n}{2}r^2$ and

$$\min_{M\times\{t\}}\mathrm{R}\ge\frac{1}{r^2-\frac{2}{n}(t-\alpha)}$$

for $t\in(\alpha,\omega)$.
(3) If $\min_{M\times\{\alpha\}}\mathrm{R}=-r^{-2}<0$, then

$$\min_{M\times\{t\}}\mathrm{R}\ge-\frac{1}{r^2+\frac{2}{n}(t-\alpha)}$$

for $t\in(\alpha,\omega)$.

Proof. In the first case, the maximum principle ensures that R remains non-negative due to (2.7). The strong maximum principle then guarantees that either $\mathrm{R}>0$ at interior times, or $\mathrm{R}\equiv 0$. But in the latter case, (2.7) implies $|\mathrm{Rc}|\equiv 0$.

Since

$$|\mathrm{Rc}|^2\ge\tfrac{1}{n}\,\mathrm{R}^2\,,$$

we may estimate

$$\partial_t\,\mathrm{R}\ge\Delta\,\mathrm{R}+\tfrac{2}{n}\,\mathrm{R}^2\;.$$

The ODE comparison principle then yields the remaining claims. $\square$

[4] There is an easier way to see that this term vanishes: since the terms $Q_{ij}=(\frac{d}{dt}-\Delta)\,\mathrm{Rc}_{ij}$ and $2(\mathrm{Rm}_{ik}\,\mathrm{Rm})_{pj\ell q}=2\,\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{k\ell}-2\,\mathrm{Rc}^2_{ij}$ are symmetric, so must be the remainder, $g^{k\ell}g^{pq}(\nabla_k\nabla_i\,\mathrm{Rm}_{jp\ell q}-\nabla_k\nabla_j\,\mathrm{Rm}_{ip\ell q})$. But this term is clearly skew-symmetric.

Proposition 2.11 tells us two important facts. First, a Ricci flow with positive scalar curvature on a compact manifold must become singular in finite time. Second if a Ricci flow on a compact manifold happens to exist on a very large time interval, then the scalar curvature is almost non-negative at the end time. In particular, if the flow has an infinite past, then the scalar curvature is non-negative in the present.

Corollary 2.12. *For any* ANCIENT *Ricci flow* $(M^n \times (-\infty, \omega), g)$ *on a compact manifold* M^n*, either* $\mathrm{R} > 0$ *or* $\mathrm{Rc} \equiv 0$.

2.2.4 Evolution of the curvature operator. It is also possible to derive an evolution equation for the full curvature tensor Rm.

Proposition 2.13. *Along a Ricci flow* $(M^n \times I, g)$,

$$(\nabla_t - \Delta)\,\mathrm{Rm} = \mathrm{Rm}^2 + \mathrm{Rm}^{\#}\,, \tag{2.8}$$

where, as operators on vector fields,

$$\mathrm{Rm}^2(X,Y) \doteqdot \mathrm{tr}\,\mathrm{Rm}(\mathrm{Rm}(X,Y)\cdot,\cdot)$$

and

$$\mathrm{Rm}^{\#}(X,Y) \doteqdot 2\,\mathrm{tr}\left[\,\mathrm{Rm}(X,\cdot),\mathrm{Rm}(Y,\cdot)\right],$$

or, with respect to a local orthonormal frame,

$$\mathrm{Rm}^2_{ijk\ell} = \mathrm{Rm}_{ijpq}\,\mathrm{Rm}_{k\ell pq}$$

and

$$\mathrm{Rm}^{\#}_{ijk\ell} = 2(\mathrm{Rm}_{ipkq}\,\mathrm{Rm}_{jp\ell q} - \mathrm{Rm}_{ip\ell q}\,\mathrm{Rm}_{jpkq})\,.$$

Proof. The identity (2.8) may be derived, with some effort, as an application of the curvature identity

$$\mathrm{Rm}(\partial_t, X, Y, Z) = \nabla_Y \mathrm{Rc}(Z,X) - \nabla_Z \mathrm{Rc}(Y,X)$$

and the second Bianchi identity

$$\nabla_t(\mathrm{Rm}(X,Y)) = \nabla_X(\mathrm{Rm}(\partial_t, Y)) - \nabla_Y(\mathrm{Rm}(\partial_t, X))$$

for the spacetime connection, whose proofs we shall omit. (See e.g. [**3**, **11**, **12**].) $\square$

The terms on the right hand side of (2.8) have a natural algebraic interpretation. Indeed, the term Rm^2 is at each point the square of Rm as an endomorphism of $\Lambda^2(TM)$, while $\mathrm{Rm}^{\#}$ is the "Lie algebra square" of Rm where at each point $\Lambda^2(TM)$ is identified with $\mathfrak{so}(n)$. I.e.

$$\mathrm{Rm}^{\#} = \mathrm{ad} \circ \mathrm{Rm} \wedge \mathrm{Rm} \circ \mathrm{ad}^*\,,$$

where

$$\mathrm{ad} : \Lambda^2(\mathfrak{so}(n)) \to \mathfrak{so}(n)$$

is the adjoint representation.

2.3 Global-in-space Bernstein estimates and long time existence.

The evolution equation for Rm immediately yields an evolution equation for $|\mathrm{Rm}|^2$:

$$\begin{aligned}(\partial_t-\Delta)|\mathrm{Rm}|^2 &= 2g((\nabla_t-\Delta)\,\mathrm{Rm},\mathrm{Rm})-2|\nabla\,\mathrm{Rm}|^2\\ &= 2g(\mathrm{Rm}^2+\mathrm{Rm}^\#,\mathrm{Rm})-2|\nabla\,\mathrm{Rm}|^2\,.\end{aligned}$$

The first term is formed from the metric contraction of a linear combination of terms which are cubic tensor products of Rm. In particular, by Young's inequality, $2g(\mathrm{Rm}^2+\mathrm{Rm}^\#,\mathrm{Rm})\le C|\mathrm{Rm}|^3$, where the constant C depends only on n.

Let us denote by $S*T$ any tensor which is a linear combination of metric contractions of the tensor product of S and T (of the same type).

Lemma 2.14. *Along a Ricci flow* $(M^n\times I,g)$,

$$[\nabla_t-\Delta,\nabla]T=\mathrm{Rm}*\nabla T+\nabla\,\mathrm{Rm}*T\,.$$

From this, we find that

$$\begin{aligned}(\partial_t-\Delta)|\nabla\,\mathrm{Rm}|^2 &= 2g((\nabla_t-\Delta)\nabla\,\mathrm{Rm},\nabla\,\mathrm{Rm})-2|\nabla^2\,\mathrm{Rm}|^2\\ &= 2g(\nabla(\nabla_t-\Delta)\,\mathrm{Rm}+\mathrm{Rm}*\nabla\,\mathrm{Rm}+\nabla\,\mathrm{Rm}*\mathrm{Rm},\nabla\,\mathrm{Rm})-2|\nabla^2\,\mathrm{Rm}|^2\\ &=\ \mathrm{Rm}*\nabla\,\mathrm{Rm}*\nabla\,\mathrm{Rm}-2|\nabla^2\,\mathrm{Rm}|^2\,.\end{aligned}$$

If $|\mathrm{Rm}|$ remains bounded on the time interval $[0,T]$, then we can estimate

$$(\partial_t-\Delta)|\nabla\,\mathrm{Rm}|^2\le C|\nabla\,\mathrm{Rm}|^2\,,$$

where C depends only on n and the bound for $|\mathrm{Rm}|$. The ODE comparison principle then implies that $|\nabla\,\mathrm{Rm}|^2$ grows at most exponentially on $[0,T]$:

$$|\nabla\,\mathrm{Rm}|^2\le\max_{t=0}|\nabla\,\mathrm{Rm}|^2\mathrm{e}^{CT}\,.$$

This estimate takes a more natural form if we exploit its scale invariance: since $|\mathrm{Rm}|$ scales (under parabolic rescaling of our Ricci flow) like the inverse square of distance, whereas t scales as distance squared, the constant CT will be scale invariant. If we introduce the scale parameter $r=\sqrt{T}$ and assume that $|\mathrm{Rm}|\le Kr^{-2}$ for $t\in[0,r^2]$ (a scale-invarant assumption), then the estimate becomes

$$|\nabla\,\mathrm{Rm}|^2\le C_1\max_{t=0}|\nabla\,\mathrm{Rm}|^2\,,$$

where C_1 depends only on K and n.

We can also obtain a time-interior version of this estimate: consider, for some to-be-determined constant a, the combination

$$Q\doteqdot 2t|\nabla\,\mathrm{Rm}|^2+a|\mathrm{Rm}|^2\,.$$

Observe that

$$\begin{aligned}(\partial_t-\Delta)Q &= 2|\nabla\,\mathrm{Rm}|^2+2t(\partial_t-\Delta)|\nabla\,\mathrm{Rm}|^2+a(\partial_t-\Delta)|\mathrm{Rm}|^2\\ &\le 2|\nabla\,\mathrm{Rm}|^2+2tC_1|\mathrm{Rm}||\nabla\,\mathrm{Rm}|^2+a(C_0|\mathrm{Rm}|^3-2|\nabla\,\mathrm{Rm}|^2)\\ &= 2(1+C_1t|\mathrm{Rm}|-a)|\nabla\,\mathrm{Rm}|^2+aC_0|\mathrm{Rm}|^3\,.\end{aligned}$$

If we know that $|\mathrm{Rm}|$ is bounded by Kr^{-2} on $M\times[0,r^2]$, then

$$(\partial_t-\Delta)Q\le 2(1+C_1K-a)|\nabla\,\mathrm{Rm}|^2+aC_0K^3r^{-6}$$

Thus, if we choose $a = 1 + C_1K$, then the ODE comparison principle yields

$$t|\nabla \mathrm{Rm}|^2 \leq Q \leq \max_{t=0} Q + aC_0K^3r^{-6}t \leq aK^2(1 + C_0K)r^{-4}\,.$$

That is,

$$|\nabla \mathrm{Rm}| \leq \frac{Dr^{-2}}{\sqrt{t}}\,,$$

where $D^2 \doteqdot aK^2(1+C_0K)$. This is another manifestation of the diffusive nature of the Ricci flow: even if the curvature is arbitrarily rough at the initial time, it becomes much more regular only a short time later.

An inductive extension of this argument yields the following estimates.

Theorem 2.15 ((Global-in-space) Bernstein estimates). *For every $n \in \mathbb{N}$, $K < \infty$ and $m \in \mathbb{N}$, there exists $C_m < \infty$ with the following property. Let $(M^n \times [0,T), g)$ be a Ricci flow on a compact manifold M^n. If*

$$|\mathrm{Rm}_{(x,t)}| \leq Kr^{-2} \;\; \text{for all} \;\; (x,t) \in M^n \times [0, r^2]\,,$$

then

$$|\nabla^m \mathrm{Rm}_{(x,t)}| \leq C_m \max_{M\times\{0\}} |\nabla^m \mathrm{Rm}| \;\; \text{for all} \;\; (x,t) \in M^n \times [0, r^2]$$

and

$$|\nabla^m \mathrm{Rm}_{(x,t)}| \leq \frac{C_m r^{-2}}{t^{\frac{m}{2}}} \;\; \text{for all} \;\; (x,t) \in M^n \times [0, r^2]\,.$$

There are two important applications of the global-in-space-Bernstein estimates. The first is the curvature blow-up characterization of finite time singularities.

Theorem 2.16 (Long time existence). *Let $(M \times [0,T), g)$ be a maximal Ricci flow on a compact manifold M^n. If $T < \infty$, then*

$$\limsup_{t\to T} \max_{M\times\{t\}} |\mathrm{Rm}| = \infty\,.$$

Sketch. Let $(M \times [0,T), g)$ with $T < \infty$ be a maximal Ricci flow on a compact manifold M^n and suppose, contrary to the claim, that

$$|\mathrm{Rm}| \leq K \;\; \text{on} \;\; M^n \times [0,T)\,.$$

Then, by the Bernstein estimates, we also have uniform bounds on $M^n \times [0,T)$ for $\nabla^m \mathrm{Rm}$ for all m. These geometric estimates can be converted, by an inductive argument, to uniform estimates in C^k for the metric coefficients in any local coordinate chart. The only subtlety is are the $k = 0$ and $k = 1$ cases. To control these derivatives, we observe that, for any $x \in M^n$ and any $v \in T_xM^n$,

$$\left|\frac{d}{dt}\log\left(g_{(x,t)}(v,v)\right)\right| = \left|\frac{2\,\mathrm{Rc}_{(x,t)}(v,v)}{g_{(x,t)}(v,v)}\right| \leq C\,;$$

integrating, we find that $g_{(x,t)}$ remains uniformly equivalent to $g_{(x,0)}$ under the evolution.

Cover M^n by compact sets which each lie to the interior of some coordinate chart. The Arzelá–Ascoli theorem now implies that in of these compact sets, we can find a sequence of times $t_j \to T$ such that the metric coefficients in the corresponding chart converge uniformly in the smooth topology to some limit. Since these limits must agree on overlaps, we obtain a global smooth metric on M, which we now evolve by the Ricci flow using our short time existence theorem. The thus extended family of metrics is smooth at each time and it is also smooth in time across the time T since time derivatives of g are related to spatial derivatives by the Ricci flow equation. But this is impossible since our Ricci flow was maximal. $\square$

Proposition 2.17. *Let $(M^n \times [0,T), g)$ be the maximal Ricci flow of a compact Riemannian manifold (M^n, g_0). If $T < \infty$, then*

$$\max_{M^n \times \{t\}} |\mathrm{Rm}| \geq \frac{C}{T-t}$$

where C depends only on n.

Proof. Since $\limsup_{t \nearrow T} \max_{M^n \times \{t\}} |\mathrm{Rm}| = \infty$ and

$$(\partial_t - \Delta)|\mathrm{Rm}|^2 \leq c(n)|\mathrm{Rm}|^3\,,$$

the claim follows from the ODE comparison principle. $\square$

2.4 Local-in-space Bernstein estimates and the compactness theorem. By introducing spatial cutoff functions into the above argument, one may derive the following local-in-space estimates.

Theorem 2.18 (Fully local Bernstein estimates)**.** *For every $n \in \mathbb{N}$, $K < \infty$ and $m \in \mathbb{N}$, there exists $C_m < \infty$ with the following property. Let $(M^n \times I, g)$ be a Ricci flow on a manifold M^n. If $B_r(p,t)$ has compact closure in M^n, $[t-r^2, t] \subset I$ and $\sup_{B_r(x,t)\times[t-r^2,t]} |\mathrm{Rm}| \leq Kr^{-2}$, then*

$$|\nabla^m \mathrm{Rm}_{(x,t)}| \leq C_m r^{-m-2}\,.$$

Theorem 2.19 (Compactness of the space of Ricci flows with bounded geometry)**.** *Let $\{(M_k \times I_k, o_k, g_k)\}_{k\in\mathbb{N}}$ be a sequence of pointed Ricci fows. Suppose the following conditions hold*

(1) $B_r(o_k, \alpha) \Subset M_k$ and $I \doteqdot [\alpha, \omega] \subset I_k$ for all k.

(2) $\max_{\overline{B_r(o_k,\alpha)}\times I} |\mathrm{Rm}| \leq C < \infty$ for all k.

(3) $\mathrm{inj}(o_k, \alpha) \geq \delta > 0$ for all k.

There exists a pointed Ricci flow $(M \times I, o, g)$ such that, after passing to a subsequence, the Ricci flows $(\overline{B_{\frac{r}{2}}(o_k, \alpha)} \times I_k, o_k, g_k)$ converge uniformly in the smooth sense to the Ricci flow $(\overline{B_{\frac{r}{2}}(o, 0)} \times I, o, g)$. That is, there exists a sequence of diffeomorphisms $\phi_k : \overline{B_{\frac{r}{2}}(o,0)} \to M_k$ with $\phi_k(o) = o_k$ such that $\phi_k^* g_k \to g$ uniformly in the smooth topology, where $(\phi_k^* g_k)_{(x,t)} \doteqdot (g_k)_{(\phi_k(x),t)}$.

By taking limits along diagonal subsequences, one can obtain a complete limit under global bounds on the curvature. Note though that the limit can lose or gain topology, and different subsequences can take different limits.

2.5 Perelman's curvature estimate. By Klingenberg's lemma, lower injectivity radius bounds are equivalent to lower volume bounds under the assumption of bounded curvature.

Proposition 2.20. *Given $\omega > 0$ and $K < \infty$, there exists $\delta > 0$ with the following property. Let (M^n, g) be a Riemannian manifold. If*

(1) $\sup_{B_r(x_0)} |\mathrm{Rm}| \leq Kr^{-2}$ and

(2) $\mathrm{volume}(B_r(x_0)) \geq \omega r^n$,

then

$$\mathrm{inj}(x_0) \geq \delta r\,.$$

Proof. See e.g. [**11**, **12**]. □

So the lower injectivity radius bound in the compactness theorem may be replaced by lower volume bounds for geodesic balls.

On the other hand, if the volume of a geodesic ball is bounded from below for some time under Ricci flow, then the curvature at the centre is bounded from above.

Theorem 2.21. *For any $n \geq 2$ and any $\kappa > 0$, there exists $C < \infty$ with the following property. Let $(M^n \times I, g)$ be a Ricci flow. Suppose that $B_r(x,t) \times (t - r^2, t] \Subset M^n \times I$. If*

$$\mathrm{Rm} \geq -r^{-2} g \text{ in } B_r(x,t) \times (t-r^2, t] \text{ and } \mathrm{volume}(B_r(x,t),t) \geq \kappa r^n\,,$$

then

$$|\mathrm{Rm}_{(x,t)}| \leq C r^{-2}\,.$$

Sketch. Suppose, to the contrary, that we can find $\kappa > 0$, a sequence $\{(M_j^n \times I_j, g_j)\}_{j \in \mathbb{N}}$ of Ricci flows $(M_j^n \times I_j, g_j)$ containing points $(x_j, t_j) \in M_j^n \times I_j$, and a sequence of scales r_j such that

$$\mathrm{Rm}_j \geq -r_j^{-2} g_j \text{ in } B_{r_j}(x_j,t_j) \times (t_j - r_j^2, t_j] \text{ and } \mathrm{volume}(B_{r_j}(x_j,t_j),t_j) \geq \kappa r_j^n\,,$$

but

$$|(\mathrm{Rm}_j)_{(x_j,t_j)}| > j^2 r_j^{-2}\,.$$

Set $Q_j(x,t) \doteqdot |(\mathrm{Rm}_j)_{(x,t)}|$. We claim that points $(\overline{x}_j, \overline{t}_j) \in M_j^n \times I_j$ can be found with the following properties:

(1) $(\overline{x}_j, \overline{t}_j) \in B_{\frac{2j}{Q_j(x_j,t_j)}}(x_j,t_j) \times (t_j - \frac{4j^2}{Q_j^2(x_j,t_j)} t_j]$.

(2) $Q_j(\overline{x}_j, \overline{t}_j) \geq Q_j(x_j,t_j)$.

(3) $Q_j \leq 2Q_j(\overline{x}_j, \overline{t}_j)$ in $B_{\frac{j}{Q_j(\overline{x}_j,\overline{t}_j)}}(\overline{x}_j,\overline{t}_j) \times (\overline{t}_j - \frac{j^2}{Q_j^2(\overline{x}_j,\overline{t}_j)} \overline{t}_j]$.

Set $\overline{r}_j \doteqdot Q_j^{-1}(\overline{x}_j, \overline{t}_j)$. After parabolically rescaling by $\overline{r}_j^{-1}$, we obtain a sequence of pointed Ricci flows with curvature bounded by two on $B_j(\overline{x}_j, 0) \times (-j^2, 0]$ and $\mathrm{volume}(B_j(\overline{x}_j,0),0) \geq \kappa$. By the compactness theorem (Theorem 2.19), a subsequence converges to a complete ancient Ricci flow with non-negative curvature operator and curvature bounded from above by two, which has *positive asymptotic volume ratio*,

$$\mathcal{V}(M_\infty^n, g_0) \doteqdot \frac{\mathrm{volume}(B_r(x_\infty,0),0)}{r^n} > 0\,.$$

It turns out that this final condition is incompatible with the others. (See Theorem 6.7 in Lecture 6.) □

In the proof , we used the following "point picking" argument.

Lemma 2.22 (Point picking lemma)**.** *Let $(M^n \times I, g)$ be a Ricci flow and $f : M^n \times I \to (0,\infty)$ a continuous function. Given $(x,t) \in M^n \times I$ and any $d > 0$ such that $B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t) \Subset M^n \times I$ there exists $(y,s) \in B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t)$ such that $f(y,s) \geq f(x,t)$ and $f \leq 4f(y,s)$ in $B_{\frac{d}{\sqrt{f(y,s)}}}(y,s) \times (s - \frac{d^2}{f(y,s)}, s)$.*

Proof. Set $(y_0, s_0) \doteqdot (x,t)$. If $(y,s) = (y_0,s_0)$ satisfies the conclusion, we are done. Else there exists $(y_1,s_1) \in (B_{\frac{d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{d^2}{f(x,t)}, t)$ such that $f(y_1,s_1) > 4f(y_0,s_0)$. If (y_1,s_1) satisfies the conclusion, we're done. Else, we continue choosing points (y_j,s_j) in the same way. Since the radii form a geometric series, the points (y_j,s_j) never leave the ball $B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t)$. Since f admits some finite bound within $B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t)$, the process must terminate after finitely many steps. □

While the estimates for derivatives of curvature (under the assumption of bounded curvature) rely entirely on the maxiumum principle, inspired by a classical argument of Bernstein, the estimate of the Theorem 2.21 (due to Perelman) requires a number of new ideas. We will touch on some of these ideas in Lecture 5.

2.6 Exercises.

Exercise 2.1. Let (M^n, g) be a Riemannian manifold equipped with its Levi-Civita connection ∇. Assuming $f \in C^2(M^n)$ attains a local maximum at $x_0 \in M^n$, show that

$$0 = \nabla f(x_0) \ \text{ and } \ \nabla^2 f(x_0) \leq 0\,.$$

Exercise 2.2. Show that any ETERNAL Ricci flow $(M^n \times (-\infty,\infty), g)$ on a compact manifold M^n is Ricci flat.

Lecture 3. Pinching estimates and their applications

We have seen that positivity of scalar curvature is preserved under the Ricci flow, by applying the (scalar) maximum principle to the reaction-diffusion equation for the scalar curvature. The reaction terms in the evolution equation for the Riemann tensor enjoy a far richer and more complicated algebraic structure. Understanding this structure (in relation to the tensor and vector bundle maximum principles) is a crucial step in understanding the long term behaviour of the Ricci flow. We will explore this paradigm in this lecture.

3.1 Three-manifolds with positive Ricci curvature. In three dimensions, the trace-free part of the Riemann curvature tensor (the Weyl tensor) vanishes, so the curvature is entirely determined by the Ricci tensor. In particular, the inequality $\mathrm{Rc} \geq 0$ implies the inequality $\mathrm{Rm} \geq 0$.

Proposition 3.1. *Let $(M^n \times [0,T), g)$ be Ricci flow on a compact three manifold M^3. If $\mathrm{Rc}\,|_{t=0} \geq 0$, then either*

(1) $\mathrm{Rc} > 0$ *for all* $t > 0$*,*
(2) (M^3, g) *is flat, or*
(3) $(M^3 \times I, g)$ *is a quotient of* $(S^2 \times \mathbb{R} \times I, h + dr^2)$ *for some Ricci flow* $(S^2 \times I, h)$*.*

Proof. Recall that

$$(\nabla_t - \Delta)\,\mathrm{Rc}_{ij} = \mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{k\ell}\ .$$

With the tensor maximum principle in mind, consider, for any non-negative definite symmetric two-tensor S, the reaction term $N(S)_{ij} \doteqdot \mathrm{Rm}_{ikj\ell}\,S^{k\ell}$. Given any null eigenvector v of S, we have

$$N(S)(v,v) = \mathrm{Rm}_{ikj\ell}\,S^{k\ell}v^iv^j \geq 0\,.$$

So the tensor maximum principle implies that $\mathrm{Rc} \geq 0$. In fact, the strong maximum principle implies that either $\mathrm{Rc} > 0$ or

$$\min_{|v|=1} \mathrm{Rc}(v,v) \equiv 0\,,$$

and hence Rc admits a null eigenvectorfield v (at every point). Now, starting at some point (x,t), parallel transport v along radial geodesics to form a vector field, and then extend this vector field in time to form a time dependent vector field V by solving $\nabla_t V = 0$. Note that this vectorfield will satisfy $\nabla V = 0$ and $\Delta V = 0$ at the point (x,t). Thus, since $\mathrm{Rc}(V,V) \geq 0$ with equality at (x,t), we find at (x,t) that

$$\nabla\,\mathrm{Rc}(V,V) = \nabla(\mathrm{Rc}(V,V)) = 0\,,$$
$$\nabla^2\,\mathrm{Rc}(V,V) = \nabla^2(\mathrm{Rc}(V,V)) \geq 0\,,$$

and

$$0 = \partial_t(\mathrm{Rc}(V,V)) = \nabla_t\,\mathrm{Rc}(V,V) = \Delta\,\mathrm{Rc}(V,V) + Q(\mathrm{Rc})\,,$$

and hence both terms on the right (being each non-negative) must vanish. In an eigenframe $\{e_1, e_2, e_3\}$ for Rc at (x,t) with $e_1 = v$,

$$0 = Q(\mathrm{Rc}) = \sec_{ij}\,\rho_i\rho_j = (\rho_i + \rho_j - \tfrac{1}{2}\,\mathrm{R})\rho_i\rho_j = (\rho_3 - \rho_2)^2$$

and hence $\rho_2 = \rho_3$. So Rc has eigenvalues $\{0, \rho(x,t), \rho(x,t)\}$ at each point. If ρ vanishes at some (x,t), then so does R, and the strong maximum principle implies that $2\rho = \mathrm{R} \equiv 0$, and hence $\mathrm{Rc} \equiv 0$. So we may assume that $\rho > 0$ everywhere. This guarantees that there is a smooth null eigenvector field, U. Computing as above, we find that

$$0 \geq \mathrm{Rc}(\nabla U, \nabla U) = \rho^2 |\nabla U|^2$$

and hence U is parallel in space. Moreover,

$$\nabla_t U = [\partial_t, U]$$

The claim now follows from the Frobenius theorem (consider the distribution $\mathcal{U} \doteqdot \ker \mathrm{Rc}$). □

Proposition 3.2. *Let $(M^n \times [0,T), g)$ be Ricci flow on a compact three manifold M^3 with positive Ricci curvature.*

$$\min_{M \times \{t\}} \tfrac{1}{3} \mathrm{R} \leq \frac{1}{2(T-t)} \leq \max_{M \times \{t\}} \mathrm{R} \,.$$

Proof. Since $\mathrm{Rc} > 0$, we may estimate $|\mathrm{Rc}|^2 \leq \mathrm{R}^2$, and hence

$$\tfrac{2}{3} \mathrm{R}^2 \leq (\partial_t - \Delta) \mathrm{R} \leq 2 \mathrm{R}^2 \,.$$

Since $\limsup_{t \to T} \max_{M^3 \times \{t\}} \mathrm{R} = \infty$, the ODE comparison principle yields the claims. □

Proposition 3.3 (Pinching is preserved)**.** *Let $(M^3 \times [0,T), g)$ be a Ricci flow on a compact manifold M^3 such that* $\mathrm{Rc} > 0$ *at the initial time. There exists $\alpha > 0$ such that*

$$\mathrm{Rc} \geq \alpha \, \mathrm{R} \, g > 0$$

at all times.

Proof. Since M^3 is compact and $\mathrm{Rc} > 0$, a constant $\alpha > 0$ may be found such that the inequality holds at the initial time. Given such a constant, consider the tensor $S \doteqdot \mathrm{Rc} - \alpha \, \mathrm{R} \, g$. Observe that

$$\begin{aligned}(\nabla_t - \Delta) S_{ij} &= (\nabla_t - \Delta) \mathrm{Rc}_{ij} - \alpha (\partial_t - \Delta) \mathrm{R} \, g_{ij} \\ &= 2 \, \mathrm{Rm}_{ikj\ell} \mathrm{Rc}^{k\ell} - 2\alpha |\mathrm{Rc}|^2 g_{ij} \,.\end{aligned}$$

If v is a null eigenvector of S, then v is an eigenvector of Rc with eigenvalue $\rho_1 = \alpha \, \mathrm{R}$. Consider an orthonormal basis $\{e_1 = v, e_2, e_3\}$ which diagonalizes Rc. With respect to this basis,

$$\begin{aligned}\left(\mathrm{Rm}_{ikj\ell} \mathrm{Rc}^{k\ell} - \alpha |\mathrm{Rc}|^2 g_{ij}\right) v_i v_j &= \left(\mathrm{Rm}_{1k1\ell} - \alpha \, \mathrm{Rc}_{kl}\right) \mathrm{Rc}^{k\ell} \\ &= \left(\sigma_{1k} - \alpha \rho_k\right) \rho_k \\ &= -\alpha \rho_1^2 + (\sigma_{12} - \alpha \rho_2) \rho_2 + (\sigma_{13} - \alpha \rho_3) \rho_3 \,,\end{aligned}$$

where $\sigma_{ij} = \sec(e_i \wedge e_j)$ $(= \rho_i + \rho_j - \frac{1}{2} \mathrm{R})$. Since

$$\sigma_{12} - \alpha \rho_2 + \sigma_{13} - \alpha \rho_3 = \rho_1 - \alpha(\rho_2 + \rho_3) = \alpha(\mathrm{R} - \rho_2 - \rho_3) = \alpha \rho_1 > 0 \,,$$

we have

$$\max\{\sigma_{12} - \alpha \rho_2, \sigma_{13} - \alpha \rho_3\} > 0$$

and hence

$$\begin{aligned}(\sigma_{12} - \alpha \rho_2) \rho_2 + (\sigma_{13} - \alpha \rho_3) \rho_3 &\geq \big((\sigma_{12} - \alpha \rho_2) + (\sigma_{13} - \alpha \rho_3)\big) \min\{\rho_2, \rho_3\} \\ &= \alpha \rho_1 \min\{\rho_2, \rho_3\} \\ &\geq \alpha \rho_1^2 \,.\end{aligned}$$

So the claim follows from the tensor maximum principle. □

Proposition 3.4 (Pinching improves)**.** *Let $(M^3 \times [0,T), g)$ be a Ricci flow on a compact manifold M^3 such that* $\mathrm{Rc} > 0$ *at the initial time. For every $\varepsilon > 0$, there exists $C_{\varepsilon<\infty}$ such that*

$$|\mathring{\mathrm{Rc}}|^2 \le \varepsilon \,\mathrm{R}^2 + C_\varepsilon$$

at all times.

Proof. Given σ, consider the function $\mathrm{R}^\sigma \frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}$. We aim to show, using the maximum principle, that an initial upper bound for this function is preserved, for some $\sigma > 0$ (which will depend on the preserved pinching constant α). The claim then follows from Young's inequality.

First observe that

$$\begin{aligned}
(\partial_t - \Delta)\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} &= (\partial_t - \Delta)\frac{|\mathrm{Rc}|^2}{\mathrm{R}^2} \\
&= 2g\left((\nabla_t - \Delta)\frac{\mathrm{Rc}}{\mathrm{R}}, \frac{\mathrm{Rc}}{\mathrm{R}}\right) - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 \\
&= 2g\left(\frac{(\nabla_t - \Delta)\,\mathrm{Rc}}{\mathrm{R}} - (\partial_t - \Delta)\,\mathrm{R}\,\frac{\mathrm{Rc}}{\mathrm{R}^2} + 2\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\frac{\mathrm{Rc}}{\mathrm{R}}, \frac{\mathrm{Rc}}{\mathrm{R}}\right) - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 \\
&= 4\frac{\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{ij}\,\mathrm{Rc}^{k\ell}}{\mathrm{R}^2} - 4\frac{|\mathrm{Rc}|^4}{\mathrm{R}^3} + 2\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left|\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 .
\end{aligned}$$

Thus,

$$\begin{aligned}
(\partial_t - \Delta)\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right) &= \mathrm{R}^\sigma(\partial_t - \Delta)\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} + \frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}(\partial_t - \Delta)\,\mathrm{R}^\sigma - 2g\left(\nabla \mathrm{R}^\sigma, \nabla\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right) \\
&= \mathrm{R}^\sigma(\partial_t - \Delta)\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} + \sigma\,\mathrm{R}^\sigma \frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\left[\frac{(\partial_t - \Delta)\,\mathrm{R}}{\mathrm{R}} + (1-\sigma)\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\right] \\
&\quad - 2\sigma\,\mathrm{R}^\sigma\, g\left(\frac{\nabla \mathrm{R}}{\mathrm{R}}, \nabla\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right) \\
&= \mathrm{R}^\sigma\left[4\frac{\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{ij}\,\mathrm{Rc}^{k\ell}}{\mathrm{R}^2} + 2\sigma\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\frac{|\mathrm{Rc}|^2}{\mathrm{R}} - 4\frac{|\mathrm{Rc}|^4}{\mathrm{R}^3}\right. \\
&\quad \left. + 2(1-\sigma)\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 + \sigma(1-\sigma)\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\right] .
\end{aligned}$$

Since

$$\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right) = \sigma\,\mathrm{R}^\sigma \frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} + \mathrm{R}^\sigma\,\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} ,$$

we arrive at

$$\begin{aligned}(\partial_t-\Delta)\left(\mathrm{R}^\sigma\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right) &= \mathrm{R}^\sigma\left[4\frac{\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{ij}\,\mathrm{Rc}^{k\ell}}{\mathrm{R}^2}+2\sigma\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\frac{|\mathrm{Rc}|^2}{\mathrm{R}}-4\frac{|\mathrm{Rc}|^4}{\mathrm{R}^3}\right.\\ &\qquad\left.-2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2-\sigma(1-\sigma)\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\frac{|\nabla\mathrm{R}|^2}{\mathrm{R}^2}\right]+2(1-\sigma)\nabla_{\frac{\nabla\mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right)\\ &\le 4\,\mathrm{R}^{1+\sigma}\left(Z+\frac{\sigma}{2}\frac{|\mathrm{Rc}|^2}{\mathrm{R}^2}\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right)+2(1-\sigma)\nabla_{\frac{\nabla\mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right),\end{aligned}$$

where

$$Z\doteqdot\frac{\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{ij}\,\mathrm{Rc}^{k\ell}}{\mathrm{R}^3}-\frac{|\mathrm{Rc}|^4}{\mathrm{R}^4}.$$

Observe that, with respect to an eigenframe for Rc,

$$\begin{aligned}\mathrm{R}\,\mathrm{Rm}_{ikj\ell}\,\mathrm{Rc}^{ij}\,\mathrm{Rc}^{k\ell}-|\mathrm{Rc}|^4 &= \sum_{i,k}\mathrm{R}\sec(e_i\wedge e_k)\rho_i\rho_k-\left(\sum_i\rho_i^2\right)^2\\ &= \sum_{i\neq k}\mathrm{R}(\rho_i+\rho_k-\tfrac{1}{2}\,\mathrm{R})\rho_i\rho_k-\left(\sum_i\rho_i^2\right)^2.\end{aligned}$$

By Exercise 3.1, this expression is non-positive, with equality only if at least one of the eigenvalues is zero. It follows that the homogeneous expression Z (as an algebraic function of the Ricci eigenvalues) takes a negative maximum, $-\zeta_\alpha$, on the cone described by the condition $\{\mathrm{Rc}\ge\alpha\,\mathrm{R}\}$. Since $|\mathring{\mathrm{Rc}}|\le|\mathrm{Rc}|\le\mathrm{R}$ on the positive cone, we may take σ to be twice ζ_α to obtain

$$(\partial_t-\Delta)\left(\mathrm{R}^\sigma\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right)\le 2(1-\sigma)\nabla_{\frac{\nabla\mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}\right),$$

at which point the claim follows from the maximum principle. □

Proposition 3.4 ensures that the metric is becoming round at any point where the curvature is becoming large, in the sense that the scale invariant ratio $|\mathring{\mathrm{Rc}}|/\mathrm{R}$ is becoming small. We already know that $\max\mathrm{R}\ge\frac{1}{2(T-t)}$ is blowing up at the final time. We thus need to show that $\min\mathrm{R}$ blows up at the same rate. We will be able to establish this from the following gradient estimate in conjunction with Myers' theorem.

Proposition 3.5. *Let $(M^3\times[0,T),g)$ be a Ricci flow on a compact manifold M^3 such that $\mathrm{Rc}\ge\alpha\,\mathrm{R}$ at the initial time. For any $\varepsilon>0$, there exists $C_\varepsilon<\infty$ such that*

$$|\nabla\,\mathrm{Rc}|^2\le\varepsilon\,\mathrm{R}^3+C_\varepsilon\,.$$

at all times.

Proof. Given $\varepsilon>0$, choose C_ε (as permitted by Proposition 3.4) so that

$$|\mathring{\mathrm{Rc}}|^2\le\varepsilon\,\mathrm{R}^2+C_\varepsilon$$

and consider, for suitable $C_\varepsilon<\infty$, the function

$$G_\varepsilon\doteqdot 2C_\varepsilon+\varepsilon\,\mathrm{R}^2-|\mathring{\mathrm{Rc}}|^2\ge C_\varepsilon>0\,.$$

Estimating $Z \geq 0$, $|\mathrm{Rc}| \leq \mathrm{R}$, and (see Exercise 3.2) $|\nabla \mathrm{Rc}|^2 \geq \frac{7}{20}|\nabla \mathrm{R}|^2$, we find that

$$(\partial_t - \Delta)|\nabla \mathrm{Rc}|^2 \leq c|\mathrm{Rc}||\nabla \mathrm{Rc}|^2 ,$$

$$\begin{aligned}(\partial_t - \Delta)G_\varepsilon &= 2(|\nabla \mathrm{Rc}|^2 - \left(\tfrac{1}{3} - \varepsilon\right)|\nabla \mathrm{R}|^2) + 4(\left(\tfrac{1}{3} - \varepsilon\right)\mathrm{R}\,|\mathrm{Rc}|^2 - \mathrm{Rm}(\mathrm{Rc},\mathrm{Rc}))\\ &\geq 4\frac{|\mathrm{Rc}|^2}{\mathrm{R}}(G_\varepsilon - 2C_\varepsilon) + \kappa|\nabla \mathrm{Rc}|^2\\ &\geq -4|\mathrm{Rc}|G_\varepsilon + \kappa|\nabla \mathrm{Rc}|^2 .\end{aligned}$$

so long as $\varepsilon \leq ..$, where $\kappa \doteqdot ...$

We aim to preserve upper bounds for the function $\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}\,G_\varepsilon}$. So consider

$$\begin{aligned}(\partial_t - \Delta)\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} = {}& \frac{(\partial_t - \Delta)|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} - \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\left(\frac{(\partial_t - \Delta)\mathrm{R}}{\mathrm{R}} - \frac{(\partial_t - \Delta)G_\varepsilon}{G_\varepsilon}\right)\\ &+ 2g\left(\nabla\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}, \nabla\log(\mathrm{R}G_\varepsilon)\right) + 2\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}\,G_\varepsilon}g\left(\frac{\nabla \mathrm{R}}{\mathrm{R}}, \frac{\nabla G_\varepsilon}{G_\varepsilon}\right) .\end{aligned}$$

We estimate the terms on the first line as above. To control the terms on the second line, observe that, at a new local maximum of $\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}$,

$$0 = \nabla_k\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} = 2\frac{g(\nabla_k\nabla \mathrm{Rc}, \nabla \mathrm{Rc})}{\mathrm{R}G_\varepsilon} - \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\left(\frac{\nabla_k \mathrm{R}}{\mathrm{R}} + \frac{\nabla_k G_\varepsilon}{G_\varepsilon}\right)$$

and hence

$$4\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}g\left(\frac{\nabla \mathrm{R}}{\mathrm{R}}, \frac{\nabla G_\varepsilon}{G_\varepsilon}\right) \leq \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\left|\frac{\nabla \mathrm{R}}{\mathrm{R}} + \frac{\nabla G_\varepsilon}{G_\varepsilon}\right|^2 \leq 4\frac{|\nabla^2 \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}$$

Thus, at such a point,

$$\begin{aligned}0 &\leq (\partial_t - \Delta)\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\\ &\leq \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\left((c+8)|\mathrm{Rc}| - \kappa\frac{|\nabla \mathrm{Rc}|^2}{G_\varepsilon}\right)\\ &= \frac{|\nabla \mathrm{Rc}|^2}{G_\varepsilon}\left((c+6)\frac{|\mathrm{Rc}|}{\mathrm{R}} - \kappa\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\right)\end{aligned}$$

and hence

$$\kappa\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} \leq (c+8)\frac{|\mathrm{Rc}|}{\mathrm{R}} \leq \frac{15(c+8)}{2} .$$

We conclude that

$$\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} \leq C \doteqdot \max\left\{\frac{15(c+8)}{2\kappa}, \max_{M^3\times\{0\}}\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\right\} .$$

The claim then follows from Young's inequality. $\square$

Proposition 3.6. *Let $(M^3 \times [0,T), g)$ be the maximal Ricci flow of a compact Riemannian three-manifold (M^3, g) with positive Ricci curvature.*

$$\frac{\mathrm{R}_{\max}}{\mathrm{R}_{\min}} \to 1 \text{ and } \operatorname{diam}(M^3, g_{(\cdot,t)}) \to 0 \text{ as } t \to T , \tag{3.1}$$

where $\mathrm{R}_{\max} \doteqdot \max\limits_{M^3\times\{t\}} \mathrm{R}$, $\mathrm{R}_{\min} \doteqdot \min\limits_{M^3\times\{t\}} \mathrm{R}$.

Proof. By the gradient estimate (Proposition 3.5), for every $\eta > 0$ there is a constant $C_\eta < \infty$ such that

$$|\nabla \mathrm{R}| \leq \frac{1}{2}\eta^2 \mathrm{R}^{\frac{3}{2}} + C_\eta .$$

Since $\mathrm{R}_{\max}(t) \to \infty$ as $t \to T$, there is, for every $\eta > 0$, some point $(x_\eta, t_\eta) \in M^3 \times [0,T)$ such that

$$\mathrm{R}_\eta^{\frac{3}{2}} \doteqdot \mathrm{R}^{\frac{3}{2}}(x_\eta, t_\eta) = \mathrm{R}_{\max}^{\frac{3}{2}}(t_\eta) \geq 8C_\eta/\eta^2$$

and hence

$$|\nabla \mathrm{R}|(x, t_\eta) \leq \eta^2 \mathrm{R}^{\frac{3}{2}}(x_\eta, t_\eta)$$

for all $x \in M$. Now let γ be a unit speed $g_{(\cdot,t_\eta)}$-geodesic through $\gamma(0) = x_\eta$. For each $s \leq L \doteqdot \eta^{-1} \mathrm{R}_\eta^{-\frac{1}{2}}$, the mean value theorem provides some $s_0 \in (0,s)$ such that

$$\mathrm{R}(\gamma(s), t_\eta) = \mathrm{R}_\eta + s\nabla_{\gamma'(s_0)} \mathrm{R}(\gamma(s_0), t_\eta) \geq \mathrm{R}_\eta(1-\eta) . \tag{3.2}$$

Applying the preserved pinching estimate $\mathrm{Rc} \geq \alpha \mathrm{R}\, g$, we may estimate

$$\mathrm{Rc}(\gamma', \gamma') \geq \alpha \mathrm{R} \geq \alpha \mathrm{R}_\eta(1-\eta)$$

for $s \leq L$. If $\eta < \frac{1}{2}$, then

$$\mathrm{Rc}(\gamma', \gamma') \geq 2Kg ,$$

where $K \doteqdot \frac{\alpha}{4} \mathrm{R}_\eta$. Choosing further $\eta \leq \frac{\alpha}{4\pi}$, we obtain $L \geq \pi K^{-1}$. Myers' theorem then implies that every point of M^3 is reached by a $g_{(\cdot,t_\eta)}$-geodesic of length at most L and we conclude from (3.2) that

$$\mathrm{R}_{\min}(t_\eta) \geq (1-\eta)\, \mathrm{R}_{\max}(t_\eta) .$$

Since $\mathrm{R}_{\min}$ is non-decreasing, we then have

$$\mathrm{R}^2_{\max}(t) \geq (1-\eta)^2\, \mathrm{R}^2_{\max}(t_\eta) \geq \frac{1}{4}\mathrm{R}_\eta^2 \quad \text{for all} \quad t \geq t_\eta ,$$

so that the above arguments hold for all $t \geq t_\eta$. We now conclude that, given any $\eta \leq \min\{\frac{\alpha}{4\pi}, \frac{1}{2}\}$, there is some time $t_\eta \in [0,T)$ such that

$$\mathrm{diam}(M, g_{(\cdot,t)}) \leq \frac{1}{\eta\, \mathrm{R}_{\max}(t)} \quad \text{and} \quad \mathrm{R}_{\min}(t) \geq (1-\eta)\, \mathrm{R}_{\max}(t)$$

for all $t > t_\eta$. The proposition follows since $\mathrm{R}_{\max}(t) \geq \frac{1}{2(T-t)}$. □

It follows that the diameter of the rescaled metrics $\frac{1}{2(n-t)} g_{(\cdot,t)}$ remains bounded, and their scalar curvature converges uniformly to a constant as $t \to T$. Bootstrapping arguments then yield smooth convergence to a round metric.

Theorem 3.7 (Hamilton [**18**])**.** *Let (M^3, g_0) be a compact Riemannian three manifold with positive Ricci curvature. The maximal Ricci flow $(M^3 \times [0,T), g)$ of (M^3, g_0) satisfies*

$$\frac{1}{2(T-t)} g_{(\cdot,t)} \to \bar{g}$$

uniformly in the smooth topology, where $\bar{g}$ is a metric of constant sectional curvature 1. In particular, M^3 is a quotient of S^3.

3.2 Manifolds with positive curvature operator. In higher dimensions, Böhm and Wilking [**6**] were able to exploit the algebraic structure of the reaction terms in the evolution equation for the curvature tensor to prove (using the vector bundle maximum principle) that pinching of the curvature operator is preserved and improves under Ricci flow in all dimensions. As a result, they obtained the following higher dimensional analogue of Hamilton's theorem.

Theorem 3.8. *Let (M^n, g_0) be a compact Riemannian manifold with positive curvature operator. The maximal Ricci flow $(M^n \times [0,T), g)$ of (M^n, g_0) satisfies*

$$\frac{1}{2(T-t)} g_{(\cdot,t)} \to \bar{g}$$

uniformly in the smooth topology, where $\bar{g}$ is a metric of constant sectional curvature 1. In particular, M^n is a quotient of S^n.

3.3 Positive isotropic curvature and the 1/4-pinched differentiable sphere theorem. The quarter pinched sphere theorem of Rauch–Klingenberg–Berger states that a simply connected, complete Riemannian manifold whose sectional curvature satisfies $\frac{1}{4} < \sec \leq 1$ must be homeomorphic to a sphere.[5]

Micallef and Moore [**24**] later showed that manifolds of POSITIVE ISOTROPIC CURVATURE are homeomorphic to spheres. This condition states that the curvature operator takes only positive values when acting on totally isotropic two planes. Since, by Berger's lemma, any manifold whose curvature is pointwise quarter pinched has positive isotropic curvature, this generalizes the quarter pinched sphere theorem.

It is natural to expect that these results also holds within the smooth category (i.e. such a manifold should be *diffeomorphic* to the sphere) but attempts to prove this failed for almost fifty years, with the problem becoming known as the quarter pinched differentiable sphere conjecture. The conjecture was finally resolved (positively) in 2009 by Brendle and Schoen [**7**] using the Ricci flow.

Theorem 3.9 (Quarter pinched differentiable sphere theorem)**.** *Let (M^n, g) be a compact Riemannian manifold. If $\frac{1}{4} < \sec \leq 1$, then M^n is diffeomorphic to S^n.*

The key ingredient was the discovery that non-negative isotropic curvature is preserved by Ricci flow (established independently by Nguyen [**26**]).

3.4 Pinched manifolds are compact. By establishing local versions of Hamilton's arguments, it becomes possible to apply them in the *non-compact* setting.

Theorem 3.10 (Ricci pinched three-manifolds are compact [**1, 22, 23**])**.** *Let (M^3, g) be a complete three-manifold. If*

$$\mathrm{Rc} \geq \alpha\, \mathrm{R}$$

for some $\alpha > 0$, then M^3 is compact.

[5] The hypothesis is optimal since the sectional curvatures of the Fubini–Study metric on $\mathbb{C}P^n$ take values between 1/4 and 1 inclusive.

The idea is to flow the metric by Ricci flow, preserving and improving the pinching condition until it converges to a round point.

Theorem 3.10 should be compared with the Bonnet–Myers theorem.

There are higher dimensional versions which hold under stronger conditions [**21**, **27**].

3.5 Exercises.

Exercise 3.1. Given non-negative numbers ρ_1, ρ_2 and ρ_3, show that

$$\sum_{i\neq k} \mathrm{R}(\rho_i + \rho_k - \tfrac{1}{2}\,\mathrm{R})\rho_i\rho_k - \left(\sum_i \rho_i^2\right)^2 \leq 0$$

with equality only if at least one of the numbers ρ_i vanish, where $\mathrm{R} \doteqdot \rho_1 + \rho_2 + \rho_3$.

Exercise 3.2.) Show that, on any Riemannian three-manifold (M^3, g),

$$|\nabla \mathrm{Rc}|^2 \geq \tfrac{7}{20}|\nabla \mathrm{R}|^2 \,.$$

HINT: *Split* $\nabla \mathrm{Rc}$ *into its trace and trace-free components.*

Lecture 4. Ricci flow on surfaces

A key step in the proof of Hamilton's theorem on the convergence of three manifolds of positive Ricci curvature (and its higher dimensional analogues) was the improvement of pinching of the eigenvalues of the Ricci curvature. No such estimate is possible in the two-dimensional setting, as, in that case, the Ricci tensor has only one eigenvalue. Fortunately, in two-dimensions, the Ricci flow enjoys some additional structure, which actually allows us to prove something far stronger.

4.1 Special properties of the Ricci flow in two-dimensions. Since in two dimensions the Ricci tensor is in proportion to the metric, the Ricci flow takes the form

$$\partial_t g = -2\,\mathrm{K}\,g\,, \tag{4.1}$$

where K is the Gauss curvature. This equation is also the two-dimensional special case of a number of other higher dimensional flows (e.g. the Kähler Ricci flow, the Q-curvature flow, the Yamabe flow, and conformal flows by functions of the Schouten tensor). With this in mind, it is perhaps not surprising that (4.1) displays properties of these higher dimensional flows that are not necessarily shared by the Ricci flow in general in higher dimensions.

4.1.1 The logarithmic fast diffusion equation and conformal invariance. Two dimensional Ricci flow $(M^2 \times I, g)$ of a compact manifold M^2 is actually a CONFORMAL FLOW; that is, we can find a function $u \in C^\infty(M^2 \times I)$ such that

$$g_{(x,t)} = \mathrm{e}^{2u(x,t)} g_{(x,0)}\,. \tag{4.2}$$

To prove this, observe that a time-dependent metric of the form (4.2) satisfies Ricci flow if and only if

$$\partial_t u g = \frac{1}{2}\mathcal{L}_{\partial_t} g = -\,\mathrm{Rc} = -\,\mathrm{K}\,g\,.$$

That is,

$$\partial_t u = -\,\mathrm{K}\,.$$

By Exercise 4.1,

$$\mathrm{K}(x,t) = \mathrm{e}^{-2u(x,t)}(\Delta_0 u(x,t) + \mathrm{K}_0(x))\,,$$

where Δ_0 and K_0 are the Laplace–Beltrami operator and sectional curvature of g_0, so we conclude that $\mathrm{e}^{2u} g_0$ satisfies Ricci flow if and only if

$$\partial_t u = \mathrm{e}^{-2u}(\Delta_0 u + \mathrm{K}_0)\,. \tag{4.3}$$

But this is a parabolic equation, and hence admits a (unique) solution u for a short time, given the initial condition $u_0 = 0$. By uniqueness of solutions to Ricci flow on compact manifolds, $g = \mathrm{e}^{2u}$ must be the unique Ricci flow starting from g_0.

4.1.2 Preservation of negative curvature. Since $\mathrm{Rc} = \mathrm{K}\,g$, the Gauss curvature (which is half the scalar curvature) evolves according to

$$(\partial_t - \Delta)\,\mathrm{K} = 2\,\mathrm{K}^2\,. \tag{4.4}$$

This means that negativity of curvature is also preserved in two dimensions. We also obtain an analogue of 2.11:

Proposition 4.1. *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a compact two-manifold M^2.*

(1) If $\max_{M^2\times\{\alpha\}} \mathrm{K} = 0$ then either $\mathrm{K} \equiv 0$ or $\mathrm{K} < 0$ for $t \in (\alpha, \omega)$.

(2) *If* $\max_{M^2\times\{\alpha\}} \mathrm{K} = -r^{-2} < 0$, *then*

$$\max_{M^2\times\{t\}} \mathrm{K} \le -\frac{1}{r^2+2(t-\alpha)}$$

for $t \in (\alpha,\omega)$.

(3) *If* $\max_{M^2\times\{\alpha\}} \mathrm{K} = r^{-2} > 0$, *then*

$$\max_{M^2\times\{t\}} \mathrm{K} \le \frac{1}{r^2-2(t-\alpha)}$$

for $t \in (\alpha,\omega)$.

In fact, we can do better by making use of the Gauss–Bonnet theorem.

4.1.3 Constant rate of change of area. By the Gauss–Bonnet theorem and the first variation of area, the area of a two-dimensional Ricci flow changes at a precise rate:

$$\frac{d}{dt}\,\mathrm{area}(t) = -2\int_{M^2} \mathrm{K}\; d\mu = -4\pi\chi(M^2)\,, \tag{4.5}$$

where $\chi(M^2)$ is the Euler characteristic of M^2. Integrating yields

$$\mathrm{area}(M^2,t) = \mathrm{area}(M^2,0) - 4\pi\chi(M^2)t\,, \tag{4.6}$$

a remarkably simple (and useful) formula. Indeed, consider the average Gauss curvature

$$\kappa(t) \doteqdot \frac{\int_{M^2}\mathrm{K}\,d\mu}{\int_{M^2} d\mu} = \frac{2\pi\chi(M^2)}{\mathrm{area}(M^2,t)} = \frac{2\pi\chi(M^2)}{\mathrm{area}(M^2,0)-4\pi\chi(M^2)t}\,.$$

By (4.5) (or (4.6)),

$$\frac{d}{dt}\kappa = -\frac{2\pi\chi(M^2)}{\mathrm{area}^2(M^2,t)}\frac{d}{dt}\,\mathrm{area}(M^2,t) = 2\kappa^2\,.$$

Recalling (4.4), we thus find that

$$(\partial_t - \Delta)(\mathrm{K}-\kappa) = 2(\mathrm{K}-\kappa)\left(\mathrm{K}-\kappa + \frac{4\pi\chi(M^2)}{\mathrm{area}(M^2,0)-4\pi\chi(M^2)t}\right)$$

and hence, if we normalize so that $\mathrm{area}(M^2,0) = 4\pi$,

$$\min_{M^2\times\{t\}} \mathrm{K} \ge \kappa + \phi \tag{4.7}$$

for $t \in [0,T)$, where ϕ is the solution to the problem

$$\begin{cases} \dfrac{d\phi}{dt} = 2\phi\left(\phi + \dfrac{\chi(M^2)}{1-\chi(M^2)t}\right) \\ \phi(0) = \phi_0 \doteqdot \min\limits_{M^2\times\{0\}}(\mathrm{K}-\kappa)\,; \end{cases}$$

that is (note that $\phi_0 \le 0$),

$$\phi(t) = \frac{\phi_0}{(1-\chi(M^2)t)(1-\chi(M^2)t-2\phi_0 t)} \sim \begin{cases} -\dfrac{1}{t^2} \;\text{ as }\; t\to\infty \;\text{ if }\; \chi(M^2)<0 \\ -\dfrac{1}{t} \;\text{ as }\; t\to\infty \;\text{ if }\; \chi(M^2)=0 \\ \dfrac{-1}{1-\chi(M^2)t} \;\text{ as }\; t\to\dfrac{1}{\chi(M^2)} \;\text{ if }\; \chi(M^2)>0\,. \end{cases}$$

In particular, $T \leq \frac{1}{\chi(M^2)}$ if $\chi(M^2) > 0$.

There is, of course, a similar comparison from above for $\max_{M^2\times\{t\}} \mathrm{K}$, but that estimate is of little utility. We will obtain a congruous estimate from above by a different argument, which is strongly informed by the soliton setting.

4.2 Self-similar solutions. Recall that a metric g on a two-manifold M^2 generates a self-similarly expanding, steady or shrinking Ricci flow, respectively, if there are a constant $\lambda \in \mathbb{R}$ and a vector field V such that

$$\mathrm{Rc} = \lambda g + \tfrac{1}{2}\mathcal{L}_V g\,. \tag{4.8a}$$

An important special class of solutions are those with $V = \operatorname{grad} f$ for some POTENTIAL FUNCTION f (e.g. the cigar soliton). In that case,

$$\mathrm{Rc} = \lambda g + \nabla^2 f\,. \tag{4.8b}$$

Taking the trace of (4.8a) yields (note that, for any vector field V, $\frac{1}{2}\mathcal{L}_V g$ is equal to the symmetric part of ∇V)

$$\mathrm{K} = \lambda + \tfrac{1}{2}\operatorname{div} V\,,$$

from which we see that (4.8a) is equivalent to

$$\mathcal{L}_V g - \operatorname{div} V g = 0\,. \tag{4.9a}$$

On a gradient Ricci soliton, (4.8b) this becomes

$$\nabla^2 f - \tfrac{1}{2}\Delta f g = 0\,. \tag{4.9b}$$

Moreover, in case M^2 is compact,

$$0 = \int_{M^2} \operatorname{div} V\, d\mu = 2\int_{M^2} (\mathrm{K} - \lambda)\; d\mu$$

and hence

$$\lambda = \frac{\int_{M^2} \mathrm{K}\; d\mu}{\int_{M^2} d\mu}\,.$$

Taking the divergence of (4.9a), we find that

$$\Delta V + \mathrm{Rc}(V) = 0 \tag{4.10a}$$

which, on a gradient Ricci soliton becomes

$$\nabla \mathrm{K} + \mathrm{K}\,\nabla f = 0\,. \tag{4.10b}$$

Taking the divergence of the latter yields

$$\Delta \mathrm{K} + \nabla_{\nabla f} \mathrm{K} + 2\,\mathrm{K}(\mathrm{K} - \lambda) = 0\,. \tag{4.11}$$

We may also rewrite (4.10b), using (4.9b), as

$$\begin{aligned} 0 &= \nabla \mathrm{K} + (\mathrm{K} - \kappa)\nabla f + \kappa\nabla f \\ &= \nabla \mathrm{K} + \frac{1}{2}\Delta f \nabla f + \kappa \nabla f \\ &= \nabla \mathrm{K} + \nabla_{\nabla f}\nabla f + \kappa\nabla f \\ &= \nabla\left(\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 + \kappa f\right)\,. \end{aligned} \tag{4.12}$$

Theorem 4.2. *Every compact, two-dimensional Ricci soliton has constant curvature.*

Proof. Let (M^2, g, f) be a gradient Ricci soliton on a compact two-manifold. By Exercise 4.2, the vector field $K \doteqdot \mathrm{J}(\nabla f)$ is Killing. Since M^2 is compact, there must be some $o \in M^2$ such that $\nabla f(o) = 0$ and hence $K(o) = 0$. It follows that K generates rotations, and hence we can find coordinates $(r, \theta) \in (0, R) \times \mathbb{R}/2\pi\mathbb{Z}$ such that $g = dr^2 + \psi^2(r)d\theta^2$. The claim now follows from the result of Exercise 1.4. □

Essentially the same argument yields the following (recall Example 1.2).

Theorem 4.3. *The cigar is the only steady two-dimensional gradient Ricci soliton with positive curvature.*

Proof. By Theorem 4.2, M^2 cannot be compact. It follows from Theorem 2.21 (though indirectly) that $\mathrm{K} \to 0$ as the distance to any fixed point x of M^2 goes to infinity. But then K attains a (positive) maximum at some point, at which $\nabla f = \nabla \mathrm{K} / \mathrm{K} = 0$. The claim now follows as in the previous theorem and Example 1.2. □

4.3 The differential Harnack inequality. The heat equation satisfies a remarkable property, known as the "differential Harnack inequality", which states that any positive solution $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ must satisfy

$$\nabla^2 \log u + \frac{\mathrm{I}}{2t} \geq 0\,.$$

In fact, the inequality must be strict, unless u is a constant multiple of the (self-similar) fundamental solution, $\rho(x,t) \doteqdot (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-x_0|^2}{4t}}$ for some x_0. For an ancient solution $u : \mathbb{R}^n \times (-\infty, \infty) \to \mathbb{R}$, performing a series of time-translations yields the stronger inequality

$$\nabla^2 \log u \geq 0\,.$$

Again, we have strict inequality, except in the exceptional circumstance that $\nabla^2 \log u = 0$; that is, u is a constant multiple of the travelling wave solution, $u(x,t) = \mathrm{e}^{(x+tv)\cdot v}$ for some $v \in \mathbb{R}^n$.

Observe that, by (4.10b) and (4.11), a two-dimensional expanding gradient self-similar Ricci flow must satisfy

$$\partial_t \mathrm{K} = \Delta \mathrm{K} + 2\mathrm{K}^2 = \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}} - \frac{\mathrm{K}}{t}\,,$$

while a two-dimensional steady gradient self-similar Ricci flow must satisfy

$$\partial_t \mathrm{K} = \Delta \mathrm{K} + 2\mathrm{K}^2 = \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}}\,.$$

Theorem 4.4 (Differential Harnack inequality for two-dimensional Ricci flow)**.** *Along any Ricci flow $(M^2 \times [0,T), g)$ with positive curvature on a compact two-manifold,*

$$\frac{\partial_t \mathrm{K}}{\mathrm{K}} - \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}^2} + \frac{1}{t} \geq 0 \tag{4.13}$$

for $t \in (0,T)$. In fact, the inequality is strict, unless $(M^2 \times [0,T), g)$ is a self-similarly expanding solution.

On any non-flat ancient two-dimensional Ricci flow $(M^2 \times (-\infty, T), g)$,

$$\frac{\partial_t \mathrm{K}}{\mathrm{K}} - \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}^2} \geq 0\,. \tag{4.14}$$

In fact, the inequality is strict, unless $(M^2 \times (-\infty, T), g)$ is a steady self-similar solution.

Proof. Consider the functions

$$Q \doteqdot \partial_t \log \mathrm{K} - |\nabla \log \mathrm{K}|^2$$

and

$$P \doteqdot t(\partial_t \log \mathrm{K} - |\nabla \log \mathrm{K}|^2) + 1\,.$$

It turns out that $P \equiv 0$ if and only if $(M^n \times I, g)$ is an expanding self-similar solution and $Q \equiv 0$ if and only if $(M^n \times I, g)$ is a steady self-similar solution.

Now, after some relatively straightforward calculation, we find that

$$(\partial_t - \Delta)P \geq 2g(\nabla \log \mathrm{K}, \nabla P) + QP\,.$$

Since $P|_{t=0} = 1 > 0$, the maximum principle implies that $P \geq 0$ for positive times, and either $P > 0$ or $P \equiv 0$. The claims follow. □

Note that, by continuity, smooth limits of Ricci flows on compact surfaces satisfy the differential Harnack inequality (and hence also the rigidity case by the strong maximum principle).

Corollary 4.5 ((Integral) Harnack inequality for two-dimensional Ricci flow)**.** *Along any Ricci flow $(M^2 \times [0, T), g)$ with positive curvature on a compact two-manifold,*

$$\frac{\mathrm{K}(x_2, t_2)}{\mathrm{K}(x_1, t_1)} \geq \left[\frac{t_2}{t_1} \exp\left(\frac{d^2(x_1, x_2, t_1)}{4(t_2 - t_1)}\right)\right]^{-1}$$

for any $x_1, x_2 \in M^2$ and any $0 < t_1 < t_2 < T$.

Proof. Integrate the differential Harnack inequality along "spacetime geodesics" $t \mapsto (t, \gamma(t))$.
□ □

In fact, Theorem 4.4 is the trace version of the following more general "matrix Harnack inequality".

Theorem 4.6 (Matrix differential Harnack inequality for two-dimensional Ricci flow)**.** *Along any Ricci flow $(M^2 \times [0, T), g)$ with positive curvature on a compact two-manifold M^2,*

$$\left(\partial_t \mathrm{K} - \mathrm{K}^2 + \frac{1}{t}\mathrm{K}\right)|W|^2 - \nabla_W \nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}\,|U|^2 \geq 0 \tag{4.15}$$

for every time-dependent vector field W and two-form U. In fact, the inequality is strict, unless $(M^2 \times [0, T), g)$ is a self-similarly expanding solution.

Along any ancient Ricci flow $(M^2 \times (-\infty, T), g)$ with positive curvature on a compact two-manifold M^2,

$$\left(\partial_t \mathrm{K} - \mathrm{K}^2\right)|W|^2 - \nabla_W \nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}\,|U|^2 \geq 0 \tag{4.16}$$

for every time-dependent vector field W and two-form U. In fact, the inequality is strict, unless $(M^2 \times [0, T), g)$ is a steady self-similar solution.

Proof. Motivated by various identities which hold on expanding (and steady) solitons, one considers the forms

$$Q(U,W) \doteqdot \left(\partial_t \mathrm{K} - \mathrm{K}^2\right) g(W,W) - \nabla_W \nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}\, g(U,U)$$

and

$$P(U,W) \doteqdot tQ(U,W) + \mathrm{K}\, g(W,W)\,.$$

After some arduous computations (motivated by various identities which hold on solitons), it is possible to obtain a suitable differential inequality for P. □

4.4 Uniformization of surfaces by Ricci flow. Recall the lower curvature bound

$$\mathrm{K} - \kappa \gtrsim \begin{cases} -\dfrac{1}{t^2} \text{ as } t \to \infty \text{ if } \chi(M^2) < 0 \\ -\dfrac{1}{t} \text{ as } t \to \infty \text{ if } \chi(M^2) = 0 \\ -\dfrac{1}{1-\chi(M^2)t} \text{ as } t \to \dfrac{1}{\chi(M^2)} \text{ if } \chi(M^2) > 0\,. \end{cases}$$

from (4.7). We shall obtain a complimentary upper bound by seeking an estimate which is saturated by soliton solutions. Recall that, on a gradient Ricci soliton, the potential function f satisfies

$$\Delta f = 2(\mathrm{K} - \kappa)\,. \tag{4.17}$$

On the other hand, since its right hand side has zero average, the equation (4.17) admits a solution f on *any* compact 2-d Ricci flow. Moreover, by the maximum principle, the solution f is unique up to the addition of a function of time.

Lemma 4.7. *Every Ricci flow $(M^2 \times [0,T), g)$ on a compact two-manifold M^2 admits a curvature potential function satisfying*

$$(\partial_t - \Delta) f = 2\kappa f$$

and hence, assuming $\mathrm{area}(M^2, 0) = 4\pi$,

$$\frac{\min_{M^2\times\{0\}} f}{1-\chi(M^2)t} \le f \le \frac{\max_{M^2\times\{0\}} f}{1-\chi(M^2)t}\,. \tag{4.18}$$

Proof. Since, for any function u,

$$\partial_t \Delta u = \Delta \partial_t u + 2\,\mathrm{K}\,\Delta u\,,$$

we find that

$$\begin{aligned} \Delta \partial_t f &= \partial_t \Delta f - 2\,\mathrm{K}\,\Delta f \\ &= 2\partial_t(\mathrm{K} - \kappa) - 4\,\mathrm{K}(\mathrm{K} - \kappa) \\ &= 2\Delta(\mathrm{K} - \kappa) + 4(\mathrm{K}^2 - \kappa^2) - 4\,\mathrm{K}(\mathrm{K} - \kappa) \\ &= \Delta\Delta f + 2\kappa\Delta f \\ &= \Delta(\Delta f + 2\kappa f)\,. \end{aligned}$$

That is,

$$\Delta(\partial_t f - \Delta f - 2\kappa f) = 0\,.$$

So $\partial_t f - \Delta f - 2\kappa f$ is a function of t only. By exploiting the freedom to add a function of t to f, we can guarantee that

$$(\partial_t - \Delta)f - 2\kappa f = 0$$

as claimed. The second claim then follows from the maximum principle, since, under the area normalization, $\kappa = \frac{\chi(M^2)}{1-\chi(M^2)t}$ □

Recall from (4.12) that, on a two-dimensional Ricci soliton,

$$0 = \nabla\left(\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 + \kappa f\right) .$$

That is, $\mathrm{K} + \frac{1}{2}|\nabla f|^2 + \kappa f$ is a function of time only. Consider then, on a general (compact) two dimensional Ricci flow, the function

$$F \doteqdot \mathrm{K} + \tfrac{1}{2}|\nabla f|^2 + \kappa f$$

where f is a curvature potential satisfying Lemma 4.7.

Proposition 4.8. *The function F satisfies*

$$(\partial_t - \Delta)F = 2\kappa F - 2\left|\nabla^2 f - \tfrac{1}{2}\Delta f g\right|^2 \tag{4.19}$$

and hence

$$F \leq \frac{\max_{M^2\times\{0\}} F}{1-\chi(M^2)t} \tag{4.20}$$

with strict inequality unless $(M^2 \times I, g)$ is a soliton.

Proof. We leave the verification of (4.19) as an exercise. The inequality (4.20) follows from the maximum principle, with strict inequality unless it holds identically. But in that case (4.19) implies that $\nabla^2 f - \frac{1}{2}\Delta f g = 0$. The final claim follows. □

This is an extremely useful estimate. For instance, we immediately obtain precise control on the maximal time of existence.

Corollary 4.9. *Let $(M^2 \times [0,T), g)$ be the maximal Ricci flow of a compact Riemannian surface (M^2, g_0). If $\chi(M^2) \leq 0$, then $T = \infty$. If $\chi(M^2) > 0$, then $T = \frac{1}{\chi(M^2)}$.*

Proof. By (4.18) and (4.20), there is a constant $C < \infty$ such that

$$\mathrm{K} \leq \frac{C}{1-\chi(M^2)t}\left(1 - \frac{\chi}{1-\chi(M^2)t}\right) .$$

So the claim follows from the long time existence theorem (Theorem 2.16). □

In fact, the estimate (4.20) in conjunction with the lower bound (4.7) will be sufficient to establish infinite time existence and convergence of the flow in case $\chi(M^2) \leq 0$. The case $\chi(M^2) > 0$ is somewhat trickier due to the finite time singularity. In that case, we analyze the singularity by rescaling and applying Theorem 2.19. The rescaling normalizes the curvature, but we still need to establish lower bounds for the injectivity radius. Note that, in the elliptic case, $\chi(M^2) > 0$, the universal cover is S^2 (which is compact), so it suffices to work on S^2.

The ISOPERIMETRIC CONSTANT of a Riemannian two-sphere $(M^2 \cong S^2, g)$ is defined by

$$\mathrm{C}(M^2, g) \doteqdot \inf_{\Gamma}\left(\frac{\mathrm{length}^2(\Gamma)}{\mathrm{area}(\Omega_1)} + \frac{\mathrm{length}^2(\Gamma)}{\mathrm{area}(\Omega_2)}\right)$$

where the infimum is taken over all regular Jordan curves $\Gamma \subset M^2$ which (necessarily, by the Schoenflies theorem) separate M^2 into two topological disks, Ω_1 and Ω_2. By considering small loops, it is clear that

$$\mathrm{C}(M^2, g) \leq 4\pi\,.$$

Hamilton proved that the isoperimetric constant of a Riemannian sphere does not decrease under Ricci flow.

Proposition 4.10. *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a surface $M^2 \cong S^2$.*

$$\frac{d}{dt}\mathrm{C}(M^2, g_t) \geq 0$$

in the sense of forward difference quotients whenever $C_H(M^2, g_t) < 4\pi$.

Combining this with Klingenerg's lemma yields the following lower bound for the injectivity radius.

Corollary 4.11. *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a surface $M^2 \cong S^2$.*

$$\mathrm{inj}^2(M^2, g_t) \geq \frac{\pi}{4\,\mathrm{K}_{\max}(t)}\mathrm{C}(M^2, g_0)\,. \tag{4.21}$$

Theorem 4.12 (Hamilton [**19**], Chow [**10**])**.** *Given a compact Riemannian surface (M^2, g_0), let $(M \times [0,T), g)$ be the maximal Ricci flow starting at (M^2, g_0).*

- *If $\chi(M^2) > 0$, then $T < \infty$ and $\frac{1}{2(T-t)}g_t$ converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = +1$ as $t \to T$.*
- *If $\chi(M^2) = 0$, then $T = \infty$ and g_t converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = 0$ as $t \to T$.*
- *If $\chi(M^2) < 0$, then $T = \infty$ and $\frac{1}{2t}g_t$ converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = -1$ as $t \to \infty$.*

Sketch. Consider first the case $\chi(M^2) = 0$. Since in this case $\kappa = 0$, we find, for any $x \in M^2$ and any $v \in T_xM^2$,

$$\frac{d}{dt}\log g_{(x,t)}(v,v) = -2\,\mathrm{K}(x,t) = -2\Delta f(x,t) = \partial_t f(x,t)\,. \tag{4.22}$$

Thus, recalling (4.18), we can find $C \in (0,\infty)$ such that

$$C^{-1}g_{(x,0)}(v,v) \leq g_{(x,t)}(v,v) \leq Cg_{(x,0)}(v,v)$$

for all $x \in M^2$ and all $v \in T_xM^2$. In particular, the diameter of (M^2, g_t) is bounded uniformly from above and below. Since, by (4.7), $\mathrm{K} \gtrsim -\frac{1}{t}$ and the average of K is zero, we find that $K \to 0$ uniformly as $t \to \infty$. It follows from (4.22) that g is Cauchy in C^0. Bootstrapping arguments yield higher order estimates and convergence.

The hyperbolic case, $\chi(M^2) < 0$, may be treated similarly as the flat case, $\chi(M^2) = 0$. We omit the details.

In the elliptic case, $\chi(M^2) > 0$, we may work on the universal cover: S^2. The lower bound (4.21) for the injectivity radius allows us to blow-up at the final time to obtain an ancient limit Ricci flow. Note that (by the ODE comparison principle) $\max_{M^2\times\{t\}}\mathrm{K} \geq \frac{1}{2(T-t)}$. Assume first that $\max_{M^2\times\{t\}}\mathrm{K}\le C(T-t)^{-1}$ (the expected rate of blow-up). Given any sequence of times $t_j\nearrow T$, choose points $x_j\in M^2$ such that

$$r_j^{-2}\doteqdot\max_{M^2\times\{t_j\}}\mathrm{K}=\mathrm{K}_{(x_j,t_j)}$$

and consider the pointed rescaled Ricci flows $(M^2\times I_j,x_j,g_j)$, where $I_j\doteqdot[-r_j^{-2}t_j,r_j^{-2}(T-t_j))$ and $(g_j)_{(x,t)}\doteqdot r_j^{-2}g(x,r_j^2t+t_j)$. Observe that the curvature K_j of the rescaled Ricci flow satisfies

$$\mathrm{K}_j(x,t)=r_j^2\,\mathrm{K}(x,r_j^2t+t_j)\le\frac{Cr_j^2}{T-t_j-r_j^2t}=\frac{C}{r_j^{-2}(T-t_j)-t}\le\frac{2C}{1-2t}\,.$$

Since, by Corollary 4.11,

$$\mathrm{inj}(M^2,(g_j)_t)\ge\frac{\sqrt{\pi\mathrm{C}(M^2,g_0)}}{2}\,,$$

some subsequence of the pointed rescaled Ricci flows $(M^2\times I_j,x_j,g_j)$ converges locally uniformly in the smooth sense to a limit ancient Ricci flow $(M^2_\infty\times(-\infty,1),g_\infty)$. Since, by Proposition 2.7,

$$\mathrm{diam}(M^2,(g_j)_t)=r_j^{-1}\,\mathrm{diam}(M^2,g_{r_j^2t+t_j})\le 10r_j^{-2}(T-t_j-r_j^2t)\le C(1-2t)\,,$$

the limit is compact, and hence $M^2_\infty=M^2\cong S^2$.

Next, we claim that $\max_{M^2\times\{t\}}F/\kappa$ is constant on the limit flow. Recall that $\max_{M^2\times\{t\}}F/\kappa$ is non-increasing on the original flow since

$$(\partial_t-\Delta)\frac{F}{\kappa}=-2\left|\nabla^2f-\tfrac{1}{2}\Delta fg\right|^2\,.$$

In particular, $\max_{M^2\times\{t\}}F/\kappa$ takes a limit as $t\to T$. Now, since both numerator and denominator scale like curvature, we have, for any $a<b\in(-\infty,1)$,

$$\max_{M^2\times\{b\}}\frac{F_j}{\kappa_j}-\max_{M^2\times\{a\}}\frac{F_j}{\kappa_j}=\max_{M^2\times\{r_j^2b+t_j\}}\frac{F}{\kappa}-\max_{M^2\times\{r_j^2a+t_j\}}\frac{F}{\kappa}$$

for all j sufficiently large. But both $r_j^2a+t_j$ and $r_j^2a+t_j$ tend to T, so the right hand side tends to zero. So $\max_{M^2\times\{t\}}F/\kappa$ is indeed constant on the limit flow. But then $\frac{F}{\kappa}$ must be constant, due to the strong maximum principle. We conclude that

$$\nabla^2f-\tfrac{1}{2}\Delta fg\equiv 0$$

on the limit flow, which must therefore be a gradient Ricci soliton, and hence the shrinking sphere by Theorem 4.2. The theorem now follows from bootstrapping arguments.

It remains to prove that $\mathrm{K}(T-t)$ remains bounded. Suppose then, that

$$\limsup_{t\nearrow T}\max_{M^2\times\{t\}}\mathrm{K}(T-t)=\infty\,.$$

For each j, choose $(x_j,t_j)\in M^2\times[0,T)$ so that

$$(T-j^{-1}-t_j)\,\mathrm{K}(x_j,t_j)=\max_{M^2\times[0,T-j^{-1}]}(T-j^{-1}-t)\,\mathrm{K}$$

and set $r_j^{-2} \doteqdot \mathrm{K}(x_j, t_j)$. Consider the pointed rescaled Ricci flows $(M^2 \times [\alpha_j, \omega_j), x_j, g_j)$, where $\alpha_j \doteqdot -r_j^{-2} t_j$, $\omega_j \doteqdot r_j^{-2}(T - j^{-1} - t_j)$ and $(g_j)_{(x,t)} \doteqdot r_j^{-2} g_{(x, r_j^2 t + t_j)}$. Observe in this case that

$$\alpha_j \to -\infty, \;\; \omega_j \to \infty\,,$$

and

$$\mathrm{K}_j(x,t) = r_j^2\, \mathrm{K}(x, r_j^2 t + t_j) \leq \frac{T - j^{-1} - t_j}{T - j^{-1} - r_j^2 t + t_j} = \frac{\omega_j}{\omega_j - t}\,,$$

which is uniformly bounded on any compact time interval for j sufficiently large. Since, by Proposition 4.11, the injectivity radii remain uniformly bounded from below after rescaling, some subsequence of the pointed, rescaled Ricci flows $(M^2 \times [\alpha_j, \omega_j), x_j, g_j)$ must converge to an eternal limit pointed Ricci flow $(M^2_\infty \times (-\infty, \infty), x_\infty, g_\infty)$. Since this Ricci flow is the limit of compact Ricci flows, it satisfies the differential Harnack inequality. But, by construction,

$$\mathrm{K} \leq \limsup_{j \to \infty} \frac{\omega_j}{\omega_j - t} = 1 = \mathrm{K}(x_\infty, 0)\,.$$

Thus, at $(x_\infty, 0)$, $\partial_t \mathrm{K} = \nabla \mathrm{K} = 0$, so the rigidity case of the differential Harnack implies that $(M^2_\infty \times (-\infty, \infty), g_\infty)$ is a steady soliton, and hence a cigar by Theorem 4.3. But the cigar violates the (scale invariant) lower bound for the isoperimetric constant (which passes to the limit as it is scale invariant and lower semi-continuous under local uniform congvergence). This completes the proof. $\square$

A different proof was later found by Andrews–Bryan [**2**] and Bryan [**8**]. They were able to obtain a very sharp estimate for the isoperimetric profile, sharp enough indeed to control the curvature and thereby obtain the convergence directly (bypassing the blow-up argument).

The arguments of Hamilton, Chow and Andrews–Bryan each make use of the uniformization theorem at some point in the argument. Chen–Lu–Tian [**9**] were able to remove this in the Chow–Hamilton argument and, as a result, provide a new proof of the uniformization theorem.

4.5 Exercises.

Exercise 4.1. Suppose that the two metrics g_2 and g_1 on a surface M^2 are related by $g_2 = \mathrm{e}^{2u} g_1$ for some function u. Show that the respective sectional curvatures K_2 and K_1 are related by

$$\mathrm{K}_2 = \mathrm{e}^{-2u}(\Delta_1 u + \mathrm{K}_1)\,,$$

where Δ_1 is the Laplace–Beltrami operator induced by g_1.

Exercise 4.2. Let (M^2, g, f) be a two-dimensional gradient Ricci soliton. Show that

$$K \doteqdot \mathrm{J}(\nabla f)$$

is a Killing vector field, where $\mathrm{J} : TM^2 \to TM^2$ denotes counterclockwise rotation in the fibres through 90 degrees. HINT: *first show that J is parallel.*

Exercise 4.3. Show that a solution to the heat equation $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ satisfies

$$\nabla^2 \log u + \frac{n}{2t} = 0\,.$$

if and only if it is a fundamental solution.

Exercise 4.4. Prove that

$$\Delta \log u + \frac{1}{2t} \geq 0$$

for any positive periodic solution $u : T^n \times [0, \infty) \to \mathbb{R}$ to the heat equation. HINT: Consider the function $P \doteqdot 2t\Delta \log u + 1$.

Exercise 4.5. Prove that

$$\nabla^2 \log u + \frac{\mathrm{I}}{2t} \geq 0$$

for any positive periodic solution $u : T^n \times [0, \infty) \to \mathbb{R}$ to the heat equation, where I is the Euclidean inner product. HINT: Consider the function $P \doteqdot 2t\nabla_V \nabla_V \log u + I$ for any fixed vector $V \in S^n$.

Exercise 4.6. Set $U = V \wedge W$ in (4.15) and trace with respect to W, and then optimize with respect to V to obtain (4.13).

Lecture 5. Singularities and their analysis

We have seen that finite time singularities will occur under Ricci flow if, for example, the scalar curvature is initially positive. In two dimensions, we were able to deal with finite time singularities by "blowing up" and classifying the possible blow-up limits.

Note the following immediate corollary of Theorem 2.19, which demonstrates the importance of *ancient* Ricci flows in the analysis of singularities.

Lemma 5.1. *Let $(M^n \times [0,T), g)$ be a Ricci flow with $T < \infty$ and $\{(x_k, t_k)\}_{k\in\mathbb{N}}$ a sequence of spacetime points $(x_k, t_k) \in M^n \times [0,T)$ with $t_k \to T$. Suppose that*

(1) $r_k^{-2} \doteqdot |\mathrm{Rm}_{(x_k,t_k)}| \to \infty$ as $k \to \infty$;

(2) for every $A < \infty$ some $C < \infty$ and $k_0 \in \mathbb{N}$ can be found such that $B_{Ar_k}(x_k, t_k) \times (t_k - A^2 r_k^2, t_k) \Subset M^n \times [0,T)$ and

$$\sup_{B_{Ar_k}(x_k,t_k)\times(t_k-A^2r_k^2,t_k)} |\mathrm{Rm}| \leq C r_k^{-2}\,,$$

for every $k \geq k_0$, and

(3) There exists $\omega > 0$ such that

$$\mathrm{volume}(B_{Ar_k}(x_k, t_k), t_k) \geq \kappa r_k^n > 0$$

for every k.

For each k, define the rescaled Ricci flow $(M^n \times [-r_k^{-2} t_k, r_k^{-2}(T - t_k)), g_k)$ by

$$(g_k)_{(x,t)} \doteqdot r_k^{-2} g_{(x, r_k^2 t + t_k))}\,.$$

There exists a complete pointed ancient Ricci flow $(M^n \times (-\infty, \omega), x_\infty, g_\infty)$ such that, after passing to a subsequence, the pointed rescaled Ricci flows $(M^n \times (-r_k^2 t_k, 0], x_k, g_k)$ converge locally uniformly in the smooth topology to $(M^n \times (-\infty, 0], x_\infty, g_\infty)$. That is, there exists an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by precompact open sets U_k satisfying $\overline{U}_k \subset U_{k+1}$ and a sequence of diffeomorphisms $\phi_k : \overline{U}_k \to M$ with $\phi_k(x_\infty) = x_k$ such that $\phi_k^* g \to g_\infty$ uniformly in the smooth topology on any compact subset of $M_\infty \times (-\infty, 0]$.

5.1 Curvature pinches towards positive. Recall that the Ricci flow forces scalar curvature towards the positive. For three dimensional Ricci flow, a similar phenomenon holds for the full curvature operator.

Theorem 5.2. *Let $(M^3 \times [0,T), g)$ be a Ricci flow on a compact three-manifold M^3. Denote by $\lambda_1 \leq \lambda_2 \leq \lambda_3$ the eigenvalues of the curvature operator. If $\lambda_1(\cdot, 0) \geq -r^{-2}$, then*

$$-\lambda_1(\log(-\lambda_1) + \log(r^2 + t) - 3) \leq \mathrm{R} \tag{5.1}$$

for all $t \in [0,T)$ wherever $\lambda_1 < 0$.

Proof. □

Since $\mathrm{R} \geq 0$ on any compact ancient Ricci flow, replacing t by $t - \alpha$ and taking $\alpha \to -\infty$, it follows immediately that

Corollary 5.3. *any ancient Ricci flow $(M^3 \times (-\infty, \omega), g)$ on a compact three-manifold has non-negative curvature operator.*

Observe, moreover, that any sequence of eigenvalues $\lambda_1^j \leq \lambda_3^j \leq \lambda_3^j$ such that

– $\lambda_1^j + \lambda_2^j + \lambda_3^j \geq -C$,
– $\lambda_1^j \to -\infty$, and
– $-\lambda_1^j(\log(-\lambda_1^j) + \log(r^2+T) - 3) \leq 2(\lambda_1^j + \lambda_2^j + \lambda_3^j)$

satisfies

$$-\frac{\lambda_1^j}{\lambda_3^j} \leq \frac{6}{\log(-\lambda_1^j) + \log(r^2+T) - 3} \to 0 \;\text{ as }\; j \to \infty.$$

It therefore follows from Theorem 5.2 that any (not necessarily compact) blow-up limit (i.e. an ancient Ricci flow obtained as in Lemma 5.1) about a finite time singularity of a Ricci flow on a compact three-manifold has non-negative curvature.

5.2 Ricci solitons. Recall that a triple (M^n, g, V) is a Ricci soliton if

$$\mathrm{Rc} = \lambda g - \frac{1}{2}\mathcal{L}_V g \tag{5.2a}$$

for some $\lambda \in \mathbb{R}$. When $V = \nabla f$, the triple (M^n, g, f) is a gradient Ricci soliton, and

$$\mathrm{Rc} = \lambda g - \nabla^2 f\,. \tag{5.2b}$$

As in the two dimensional case (gradient) Ricci solitons satisfy a number of informative identities. Indeed, tracing the soliton equation yields

$$\mathrm{R} = n\lambda - \operatorname{div} V \tag{5.3a}$$

which for a gradient soliton becomes

$$\mathrm{R} = n\lambda - \Delta f\,. \tag{5.3b}$$

Taking the divergence of the soliton equation and applying the contracted second Bianchi identity then yields

$$\Delta V + \mathrm{Rc}(V) = 0\,, \tag{5.4a}$$

which for a gradient soliton becomes

$$\tfrac{1}{2}\nabla\,\mathrm{R} + \mathrm{Rc}(\nabla f) = 0 \tag{5.4b}$$

Contracting the gradient soliton equation with ∇f and applying (5.4b) yields

$$\mathrm{R} + |\nabla f|^2 - 2\lambda f = C\,, \tag{5.5a}$$

where C is constant. Applying (5.3b) then yields

$$-\Delta f + |\nabla f|^2 + n\lambda - 2\lambda f = C \tag{5.5b}$$

Taking the difference between half of (5.5a) and (5.5b) yields

$$\tfrac{1}{2}\,\mathrm{R} + \Delta f - \tfrac{1}{2}|\nabla f|^2 + \lambda f = n\lambda - \tfrac{1}{2}C\,. \tag{5.5c}$$

Observe that, on the self-similarly shrinking Ricci flow $(M^n \times (-\infty, 0), \phi^* g)$, $\frac{d\phi}{dt} = \phi^* \nabla f$, corresponding to a gradient shrinking Ricci soliton (M^n, g, f),

$$\begin{aligned}\partial_t f &= \nabla_{\nabla f} f \\ &= |\nabla f|^2 \\ &= -\Delta f + |\nabla f|^2 - \mathrm{R} + \frac{n}{-2t}\end{aligned}$$

due to (5.3b). Writing $h \doteqdot (-2t)^{-\frac{n}{2}} \mathrm{e}^{-f}$, we find that

$$-(\partial_t + \Delta - \mathrm{R})h = 0\,.$$

That is, h satisfies the CONJUGATE HEAT EQUATION. The name comes from the fact that, along any Ricci flow,

$$\begin{aligned}\frac{d}{dt}\int_{M^2} u\varphi\, d\mu &= \int_{M^2} (\partial_t u\varphi + u\partial_t\varphi - \mathrm{R}\, u\varphi)\, d\mu \\ &= \int_{M^2} (\varphi(\partial_t - \Delta)u + u(\partial_t + \Delta - \mathrm{R})\varphi)\, d\mu\end{aligned}$$

so long as $\varphi(\cdot, t)$ is compactly supported. In particular, a smooth function $u : M^n \times (a, b) \to \mathbb{R}$ satisfies the heat equation if and only if every smooth function $\varphi : M^n \times (a, b) \to \mathbb{R}$ which is compactly supported in $M^n \times (a, b)$ satisfies

$$\int u(\partial_t - \Delta)^* \varphi\, d\mu = 0\,,$$

where

$$(\partial_t - \Delta)^* = -(\partial_t + \Delta - \mathrm{R})$$

is the CONJUGATE HEAT OPERATOR.

Theorem 5.4. *All compact shrinking Ricci solitons are gradient.*

Proof. Let (M^n, g, V) be a compact shrinking Ricci soliton. We seek a solution f to the equation

$$\frac{1}{2}\mathcal{L}_V g = \nabla^2 f\,.$$

(Rather than $\nabla f = V$, which may not be possible since $(M^n, g, V + K)$ is also a shrinking Ricci soliton for any Killing vector field K.) Equivalently, we seek a function f such that the tensor

$$S \doteqdot \mathrm{Rc} + \nabla^2 f - \lambda g$$

vanishes identically. Observe that

$$\operatorname{div} S = \tfrac{1}{2}\nabla \mathrm{R} + \nabla\Delta f + \mathrm{Rc}(\nabla f)$$

and

$$S(\nabla f) = \mathrm{Rc}(\nabla f) + \tfrac{1}{2}\nabla|\nabla f|^2 - \lambda\nabla f$$

and hence

$$\nabla\left(\tfrac{1}{2}\mathrm{R} + \Delta f - \tfrac{1}{2}|\nabla f|^2 + \lambda f\right) = \operatorname{div} S - S(\nabla f)\,,$$

which we may rewrite as

$$\nabla\left(\tfrac{1}{2}\mathrm{R} + \Delta f - \tfrac{1}{2}|\nabla f|^2 + \lambda f\right)\mathrm{e}^{-f} = \operatorname{div}(\mathrm{e}^{-f} S)\,.$$

Thus,

$$\begin{aligned}\int_{M^2} |S|^2 \mathrm{e}^{-f}\, d\mu &= \int_{M^2} g(\nabla(\nabla f - V), \mathrm{e}^{-f} S)\, d\mu \\ &= -\int_{M^2} g(\nabla f - V, \operatorname{div}(\mathrm{e}^{-f} S))\, d\mu \\ &= -\int_{M^2} g\Big(\nabla f - V, \nabla\left(\tfrac12 \mathrm{R} + \Delta f - \tfrac12 |\nabla f|^2 + \lambda f\right)\Big)\mathrm{e}^{-f}\, d\mu\,.\end{aligned}$$

So it suffices to find a constant C and a function f satisfying

$$\tfrac12 \mathrm{R} + \Delta f - \tfrac12 |\nabla f|^2 + \lambda f = C \tag{5.6a}$$

or equivalently, a function $h = \mathrm{e}^{-\frac{f}{2}}$ satisfying

$$\Delta h - \tfrac14 \mathrm{R}\, h + \lambda h \log h = -\tfrac12 C h\,. \tag{5.6b}$$

The equations (5.6a) and (5.6b) are the Euler–Lagrange equations for the constrained functionals

$$F(f) \doteqdot \frac12 \int_{M^2} \left(|\nabla f|^2 + \mathrm{R} + \lambda f\right) \mathrm{e}^{-f}\, d\mu \ \text{ subject to } \int_{M^2} \mathrm{e}^{-f}\, d\mu = \text{const.} \tag{5.7a}$$

and

$$G(h) \doteqdot 2 \int_{M^2} \left(|\nabla h|^2 + \tfrac14 \mathrm{R}\, h^2 - \lambda h^2 \log h\right) d\mu \ \text{ subject to } \int_{M^2} h^2\, d\mu = \text{const.} \tag{5.7b}$$

respectively. We have thus reduced the problem to finding a minimizer for (5.7b). This is fairly classical: first observe that, by interpolation and the Sobolev embedding theorem, we may estimate, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{M^2} h^2 \log h\, d\mu &\le \varepsilon \int_{M^2} h^{2+\delta}\, d\mu + C_\varepsilon \int_{M^2} h^2\, d\mu \\ &\le \varepsilon C \int_{M^2} |\nabla h|^2\, d\mu + C_\varepsilon \int_{M^2} h^2\, d\mu\,.\end{aligned}$$

Choosing $\varepsilon = \frac{1}{2\lambda C_{\mathrm{S}}}$, we find that

$$G(h) \ge \int_{M^2} |\nabla h|^2\, d\mu - C\,.$$

From this we deduce two things: first that G is bounded from below, and second that the H^1 norm is uniformly bounded along any infimizing sequence $\{h_j\}_{j\in\mathbb{N}}$. It follows that a minimizer exists in H^1. □

5.3 The differential Harnack inequality. The differential Harnack inequalities for two-dimensional Ricci flow (Theorems 4.4 and 4.6) have the following higher dimensional analogues.

Theorem 5.5. *Along any Ricci flow $(M^n \times [0,T), g)$ with positive curvature operator on a compact manifold M^n,*

$$M_{ij} W_i W_j + 2 P_{ijk} U_{ij} W_k + \mathrm{Rm}_{ikjl} U_{ik} U_{jl} + \frac{1}{2t} \mathrm{Rc}_{ij} \ge 0 \tag{5.8}$$

for every time-dependent vector field W and two-form U, where

$$M_{ij} \doteqdot \Delta \mathrm{Rc}_{ij} + 2 \mathrm{Rm}_{ikjl} \mathrm{Rc}_{kl} - \tfrac12 \left(\nabla_i \nabla_j \mathrm{R} + 2 \mathrm{Rc}^2_{ij}\right)$$

and

$$P_{ijk} \doteqdot \nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik}\ .$$

The inequality (5.8) *is strict unless* $(M^n \times [0,T), g)$ *is an expanding soliton.*

Along any ancient Ricci flow $(M^n \times (-\infty, 0), g)$ *with positive curvature operator on a compact manifold* M^n,

$$M_{ij}W_iW_j + 2P_{ijk}U_{ij}W_k + \mathrm{Rm}_{ikjl}\, U_{ik}U_{jl} \geq 0 \tag{5.9}$$

for every time-dependent vector field W *and two-form* U, *with strict inequality unless* $(M^n \times [0,T), g)$ *is a steady soliton.*

Sketch. Motivated by various identities which hold on expanding (and steady) solitons, one considers the forms

$$Q(U,W) \doteqdot M(W,W) + 2P(U,W) + \mathrm{Rm}(U,U)$$

and

$$P(U,W) \doteqdot 2tQ(U,W) + \mathrm{Rc}(W,W)\,.$$

After many arduous computations (motivated by various identities which hold on solitons), it is possible to obtain a suitable differential inequality for P. □

Theorem 5.6. *Along any Ricci flow* $(M^n \times [0,T), g)$ *with positive curvature operator on a compact manifold* M^n,

$$\partial_t \mathrm{R} + 2\nabla_V \mathrm{R} + 2\,\mathrm{Rc}(V,V) + \frac{1}{2t}\mathrm{R} \geq 0 \tag{5.10}$$

for every time-dependent vector field V, *with strict inequality unless* $(M^n \times [0,T), g)$ *is an expanding soliton.*

Along any ancient Ricci flow $(M^n \times (-\infty, 0), g)$ *with positive curvature operator on a compact manifold* M^n,

$$\partial_t \mathrm{R} + 2\nabla_V \mathrm{R} + 2\,\mathrm{Rc}(V,V) \geq 0 \tag{5.11}$$

for every time-dependent vector field V, *with strict inequality unless* $(M^n \times [0,T), g)$ *is a steady soliton.*

Proof. Take the trace of (5.8) and (5.9). □

Note that, by continuity, smooth limits of Ricci flows on compact manifolds satisfy the differential Harnack inequality (and hence also the rigidity case by the strong maximum principle).

When $\mathrm{Rc} > 0$, the differential Harnack inequality (5.11) is optimized by the vector field $V = -\frac{1}{2}\mathrm{Rc}^{-1}(\nabla \mathrm{R})$, giving

$$\partial_t \mathrm{R} \geq \tfrac{1}{2}\mathrm{Rc}^{-1}(\nabla \mathrm{R}, \nabla \mathrm{R})\,. \tag{5.12}$$

Equivalently, $\mathrm{R}(\phi^\tau(\cdot,t), t+\tau)$ is pointwise monotone non-decreasing with respect to t for each $\tau < 0$, where ϕ^τ is the solution to

$$\begin{cases} \dfrac{d\phi^\tau}{dt}(x,t) = V(\phi^\tau(x,t), \tau + t) \\ \phi^\tau(x,0) = x\,. \end{cases}$$

Thus, the scalar curvature R^τ of the reparametrized flow $\{g^\tau_t\}_{t\in(-\infty,-\tau)}$, where $g^\tau_t \doteqdot \phi^\tau(\cdot,t)^* g_{t+\tau}$, is uniformly bounded on any time interval of the form $(-\infty, T]$, and hence, in any (pointed) limit as $\tau_j \to -\infty$, we obtain a Ricci flow (plus Lie derivative term) for which R is constant in t — a steady soliton!

Corollary 5.7 (Ancient solutions decompose into steady solutions)**.** *Let $(M^n \times (-\infty, 0), g)$ be an ancient Ricci flow on a compact manifold M^n. Given any point $o \in M^2$ and any sequence of times $t_j \to -\infty$, some subsequence of the pointed Ricci flows $(M^n \times (-\infty, 0), o, g^j)$, where $g^j_{(x,t)} \doteqdot g(x, t+t_j)$, converges locally uniformly in the smooth topology to a steady Ricci flow.*

5.4 The entropy formulae. Given a compact Ricci flow $(M^n \times [0,T), g)$, define the functionals

$$\mathcal{F}(f,g) \doteqdot \int_{M^n} \left(|\nabla f|^2 + \mathrm{R}\right) \mathrm{e}^{-f} d\mu \tag{5.13a}$$

and

$$\mathcal{W}(f,g,\tau) \doteqdot \int_{M^n} \left[\tau\left(|\nabla f|^2 + \mathrm{R}\right) + f - n\right] (4\pi\tau)^{-\frac{n}{2}} \mathrm{e}^{-f} d\mu\,. \tag{5.13b}$$

Observe that the $\mathcal{F}$-FUNCTIONAL is (up to a factor of two) simply an extension to general Ricci flows of the F-functional (5.7a) in the steady case, $\lambda = 0$, while the $\mathcal{W}$-FUNCTIONAL corresponds to F in the shrinking case, $\lambda > 0$ (as τ will correspond to negative time). Indeed, applying the soliton identities derived above, it is not too hard to verify the following:

(1) If along a steady self-similar Ricci flow $(M^n \times (-\infty, \infty), g)$ a function f satisfies

$$(\partial_t + \Delta + \mathrm{R}) f = |\nabla f|^2\,,$$

then (assuming all integrals are finite and all integrations by parts are permissible)

$$\frac{d}{dt}\mathcal{F}(f,g) = 2\int_{M^2} \left|\mathrm{Rc} + \nabla^2 f\right|^2 \mathrm{e}^{-f}\, d\mu\,. \tag{5.14}$$

(2) If along a shrinking self-similar Ricci flow $(M^n \times (-\infty, 0), g)$ a function f satisfies

$$(\partial_t + \Delta + \mathrm{R}) f = |\nabla f|^2 + \tfrac{n}{-2t}\,,$$

then (assuming all integrals are finite and all integrations by parts are permissible)

$$\frac{d}{dt}\mathcal{W}(f,g,-t) = -2t\int_{M^2} \left|\mathrm{Rc} + \nabla^2 f - \tfrac{1}{-2t} g\right|^2 \mathrm{e}^{-f}\, d\mu\,. \tag{5.15}$$

In fact, these identities hold along any Ricci flow on a compact manifold.

Theorem 5.8 (Perelman's monotonicity formulae)**.** *Let $(M^n \times I, g)$ be a Ricci flow on a compact manifold M^n.*

(1) If f satisfies

$$(\partial_t + \Delta + \mathrm{R}) f = |\nabla f|^2\,,$$

then

$$\frac{d}{dt}\mathcal{F}(f,g) = 2\int_{M^2} \left|\mathrm{Rc} + \nabla^2 f\right|^2 \mathrm{e}^{-f}\, d\mu\,. \tag{5.16}$$

(2) If f and τ satisfy

$$\begin{cases} (\partial_t + \Delta + \mathrm{R}) f = |\nabla f|^2 + \tfrac{n}{2\tau}\,, \\ \dfrac{d\tau}{dt} = -1\,, \end{cases}$$

then
$$\frac{d}{dt}\mathcal{W}(f,g,\tau) = 2\tau\int_{M^2}\left|\mathrm{Rc}+\nabla^2 f-\tfrac{1}{2\tau}g\right|^2 \mathrm{e}^{-f}\,d\mu \tag{5.17}$$
so long as $\tau > 0$.

In particular, on a compact manifold M^n*, the functional*
$$\lambda(M^n,g) \doteqdot \inf\left\{\mathcal{F}(g,f) : \int_{M^n}\mathrm{e}^{-f}\,d\mu\right\}$$
is non-decreasing under Ricci flow and the functional
$$\mu(M^n,g,\tau) \doteqdot \inf\left\{\mathcal{W}(g,f,\tau) : \int_{M^n}(4\pi\tau)^{-\frac{n}{2}}\mathrm{e}^{-f}\,d\mu = 1\right\}$$
is non-decreasing under Ricci flow for $\tau > 0$.

5.5 No local collapsing. Consider a Ricci flow $(M^n\times[0,T),g)$. Given $t_0\in[0,T)$, set $\tau = t_0+r^2-t$ and consider the test function $u(\cdot,t_0) = (4\pi r^2)^{-\frac{n}{2}}\mathrm{e}^{-f(\cdot,t_0)}$ with $\mathrm{e}^{-f} = A\chi_{B_r(x_0,t_0)}$. Observe that, in order to satisfy the constraint
$$\int_{M^n}u(\cdot,t_0)d\mu_{t_0} = 1\,,$$
we should take $A\sim\frac{\mathrm{volume}(B_r(x_0,t_0),t_0)}{r^n}$. Monotonicity of the μ-entropy then implies
$$\begin{aligned}\mu(M^n,g_0,t_0+r^2) &\le \mu(M^n,g_{t_0},r^2)\\ &\le \mathcal{W}(g,f(\cdot,t_0),r^2)\\ &\lesssim r^2\max_{B_r(x_0,t_0)}\mathrm{R}(\cdot,t_0)+\ln\frac{\mathrm{volume}(B_r(x_0,t_0),t_0)}{r^n}\,.\end{aligned}$$
Thus, if $\mathrm{R}(\cdot,t_0)\lesssim r^{-2}$, then
$$\mathrm{volume}\,\frac{(B_r(x_0,t_0),t_0)}{r^n} \ge \kappa(M^n,g_0,T)\,!$$
This is not quite rigorous, as the test function is not smooth. To rectify this, we introduce a cut-off function.

Theorem 5.9. *Let* $(M^n\times[0,T),g)$ *be a Ricci flow on a compact manifold* M^n*. Given* $(x,t)\in M^n\times[0,T)$*, if* $|\mathrm{Rm}|^2\le r^{-2}$ *on* $B_r(x,t)$*,* $r\le 1$*, then*
$$\mathrm{volume}(B_r(x,t),t)\ge\kappa r^n\,,$$
where $\kappa = \kappa(M^n,g_0,T)$.

Proof. Set $\tau\doteqdot t_0+r^2-t$ and let $\phi:[0,\infty)\to[0,1]$ be any fixed smooth function satisfying $\phi|_{[0,\frac{1}{2}]} = 1$, $\phi|_{[1,\infty)} = 0$, and $\frac{|\phi'(\xi)|}{\phi(\xi)}\le C$. Define
$$f(x,t_0)\doteqdot A-\log\left(\phi\left(\frac{\mathrm{dist}(x_0,x,t_0)}{r}\right)\right)$$
and
$$\begin{aligned}u(x,t_0) &\doteqdot (4\pi r^2)^{-\frac{n}{2}}\mathrm{e}^{-f}\\ &= (4\pi r^2)^{-\frac{n}{2}}\phi\left(\frac{\mathrm{dist}(x_0,x,t_0)}{r}\right)\mathrm{e}^{-A}\,,\end{aligned}$$

where A is chosen so that

$$\int_{M^n} u(\cdot, t_0)\, d\mu_{t_0} = 1\,.$$

Note that

$$\begin{aligned} A =\ & \log\left((4\pi r^2)^{-\frac{n}{2}} \int_{B_r(x_0,t_0)} \phi\left(\frac{d(x_0,\cdot,t_0)}{r}\right) d\mu_{t_0}\right) \\ \leq\ & \log\left((4\pi)^{-\frac{n}{2}} \frac{\operatorname{volume}(B_r(x_0,t_0),t_0)}{r^n}\right). \end{aligned}$$

Thus, upper bounds for u will imply lower bounds for the volume ratio.

Observe that

$$\begin{aligned} \mathcal{W}(g_{t_0}, f(\cdot,t_0), r^2) &= \int_{B_r(x_0,t_0)} \left(r^2(\mathrm{R}^2 + |\nabla f|^2) + f - n\right) u(\cdot,t_0)\, d\mu_{t_0} \\ &\leq \int_{B_r(x_0,t_0)} \left[r^2\left(\frac{n}{r^2} + \frac{C}{r^2}\right) - \log\left(\phi\left(\frac{\operatorname{dist}(x_0,x,t_0)}{r}\right)\right) + A - n\right] u(\cdot,t_0)\, d\mu_{t_0} \\ &= C + A - \int_{B_r(x_0,t_0)} \log\left(\phi\left(\frac{\operatorname{dist}(x_0,x,t_0)}{r}\right)\right) u(\cdot,t_0)\, d\mu_{t_0} \\ &= C + A - \frac{\int_{B_r(x_0,t_0)} \log\left(\phi\left(\frac{\operatorname{dist}(x_0,x,t_0)}{r}\right)\right) \frac{\phi(\operatorname{dist}(x_0,x,t_0))}{r}\, d\mu_{t_0}}{\int_{B_r(x_0,t_0)} \frac{\phi(\operatorname{dist}(x_0,\cdot,t_0))}{r}\, d\mu_{t_0}} \\ &\leq C + A + C' \frac{\operatorname{volume}(B_r(x_0,t_0),t_0)}{\operatorname{volume}(B_{\frac{r}{2}}(x_0,t_0),t_0)} \\ &\leq C'' + A \end{aligned}$$

due to the Bishop–Gromov inequality. The claim follows since

$$\mu(M^n, g_0, t_0 + r^2) \leq \mu(M^n, g_{t_0}, r^2) \leq C'' + A$$

and $t_0 + r^2 \leq T$. □

5.6 Exercises.

Exercise 5.1. Show that any ancient Ricci flow $(M^3 \times (-\infty,\omega), g)$ on a compact three-manifold has non-negative curvature operator.

Exercise 5.2. Verify the soliton identities (5.4a), (5.4b) and (5.5a).

Exercise 5.3. Show that $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{2}\lambda|x|^2)$ is a shrinking/steady/expanding soliton according to the sign of λ.

Exercise 5.4. Suppose that the data $(M^n \times I, g, f)$ satisfy the system

$$\begin{cases} \mathcal{L}_{\partial_t} g = -2(\mathrm{Rc} + \nabla^2 f) \\ (\partial_t + \Delta + \mathrm{R}) f = 0\,. \end{cases}$$

Set

$$\mathcal{N}(g,f) \doteqdot \int_{M^2} f \mathrm{e}^{-f}\, d\mu\,.$$

Show that

$$\frac{d}{dt}\mathcal{N}(g,f) = -\int_{M^n} \left(|\nabla f|^2 + \mathrm{R}\right)\, d\mu$$

Lecture 6. On the classification of ancient solutions

Let $(M^n \times [0,T), g)$ be a maximal Ricci flow on a compact manifold M^n such that $T < \infty$. If we choose (x_j, t_j) such that

$$Q_j \doteqdot |\mathrm{Rm}_{(x_j,t_j)}| = \max_{M^n \times [0, T-j^{-1}]} |\mathrm{Rm}|\,,$$

then the pointed Ricci flows $(M^n \times I_j, g_j)$, where $(g_j)_{(x,t)} \doteqdot Q_j g_{(x, Q_j^{-1} t + t_j)}$ and $I_j \doteqdot [-Q_j t_j, 0]$ satisfy

(1) $|\mathrm{Rm}| \le 1$,

(2) $|\mathrm{Rm}_{(x_j,t_j)}| = 1$,

(3) If $r \le Q_j^{\frac{1}{2}}$ and $|\mathrm{Rm}| \le r^{-1}$ in $B_r(x,t)$, then $\mathrm{volume}(B_r(x,t),t) \ge \kappa r^n$.

By the compactness theorem, we can find a complete subsequential limit $(M^n \times (-\infty, 0], o, g)$ such that

(1) $|\mathrm{Rm}| \le 1$,

(2) $|\mathrm{Rm}_{(o,0)}| = 1$,

(3) If $|\mathrm{Rm}| \le r^{-1}$ in $B_r(x,t)$, then $\mathrm{volume}(B_r(x,t),t) \ge \kappa r^n$.

Moreover, when $n = 2$ or $n = 3$,

(4) $\mathrm{Rm} \ge 0$.

Note that the limit also satisfies the differential Harnack inequality, by continuity.

Definition 6.1 (κ-solutions)**.** *Given $\kappa > 0$, a complete ancient Ricci flow $(M^n \times (-\infty, 0], g)$ is called a κ-solution if the following properties are satisfied:*

(1) $|\mathrm{Rm}| \le K < \infty$,

(2) $\mathrm{Rm} \ge 0$,

(3) $\mathrm{R} > 0$,

(4) $\partial_t \mathrm{R} + 2\nabla_v \mathrm{R} + 2\,\mathrm{Rc}(v,v) \ge 0$ *for all* $v \in TM$.

(5) If $|\mathrm{Rm}| \le r^{-1}$ *in* $B_r(x,t)$*, then* $\mathrm{volume}(B_r(x,t),t) \ge \kappa r^n$.

The motivation for the definition is clear: a good understanding of κ-solutions will provide a good understanding of singularity formation in three-dimensional Ricci flow on compact manifolds.

Of course, understanding ancient solutions in general is certainly an interesting problem in its own right. However, without additional conditions (such as curvature positivity and/or non-collapsing), it is an exceedingly difficult task in general.

6.1 Ancient solutions in two-dimensions. So far, the only ancient Ricci flows we have seen in two dimensions are solitons. Namely, the static/shrinking plane, the shrinking sphere, and the cigar soliton (modulo quotients). There is a further (non-soliton) example (discovered independently by Fateev [**17**] King [**20**] and Rosenau [**28**]).

Example 6.1 (The Fateev–King–Rosenau solution)**.** The time-dependent metric

$$g = \chi^2\, dr^2 + \psi^2\, d\theta^2\,,$$

$$\begin{aligned}\chi^2(r,t) &\doteqdot \frac{\tanh(-2t)}{1-\sin^2 r\tanh^2(-2t)}\\ \psi^2(r,t) &\doteqdot \cos^2 r\chi^2(r,t)\end{aligned}$$

extends to a (time-dependent) metric on S^2 and evolves by Ricci flow.

Another way to view this metric is as follows: recall that solutions $u : S^2 \times I \to \mathbb{R}$ to the logarithmic fast diffusion equation,

$$u_t = \Delta_{S^2} \log u - 2\,,$$

give rise to Ricci flows via $g = ug_{S^2}$. Consider the PRESSURE FUNCTION $v \doteqdot u^{-1}$. Observe that

$$v_t = v^2(\Delta_{S^2} \log v + 2)\,. \tag{6.1}$$

The shrinking sphere is the similarity solution $v = \frac{1}{-2t}$ to (6.1). The King–Rosenau solutions take the form

$$v(r,\theta,t) = \coth(-2t) - \tanh(-2t)\sin^2 r\,.$$

But this completes the list!

Theorem 6.2 (Classification of ancient Ricci flows in two-dimensions [**13**–**16**])**.** *Every maximal, complete ancient Ricci flow $(M^2 \times (-\infty,\omega), g)$ on a connected surface M^2 is either*

- *a shrinking round sphere,*
- *a static or shrinking flat plane,*
- *a cigar soliton,*
- *a Fateev–King–Rosenau solution,*
- *an isometric quotient of one of the above examples.*

Sketch. We may suppose, without loss of generality, that $(M^2 \times (-\infty,\omega), g)$ has positive curvature. Next note that, if M^2 is not compact, then $(M^2 \times (-\infty,\omega), g)$ must have curvature tending to zero at infinity. Indeed, for any t_0 and any sequence of points x_j such that $d(x_j, o, t_0) \to \infty$, the sequence (M^n, x_j, g_{t_0}) subconverges in the pointed Gromov–Hausdorff sense to a limit space which contains a line, and hence splits off a line. But in two-dimensions, this limit must be locally isometric to $\mathbb{R}^2$. Thus, for j sufficiently large, $B_r(x_j, t_0)$ is close to a Euclidean ball in the Gromov–Hausdorff sense after passing to its universal cover. In particular, its volume (in the universal cover) is close to πr^2. So Perelman's curvature estimate implies that $\mathrm{K}(x_j, t_0) \to 0$.

Bounded curvature at infinity ensures that the differential Harnack inequality holds. By exploiting the differential Harnack inequality and a type-I vs type-II analysis, Chu and Daskalopoulos–Šešum were able to show that the cigar is the only possibility in the non-compact case.

The compact examples were classified by Daskalopoulos–Hamilton–Šešum. The key ideas are a monotonicity formula for the pressure function,

$$\frac{d}{dt}\int_{S^2}\left(\frac{|\nabla^{S^2}v|^2}{v}-4v\right)d\mu_{S^2}\leq 0\,,$$

and an analysis of the backwards limits. □

6.2 Perelman's examples and the structure of noncollapsing ancient solutions. So far, our only examples of ancient solutions are solitons or the King–Rosenau solution (which we obtained by reduction to an ODE). Perelman provided the first truly "parabolic" (in the sense of PDE methods) ancient Ricci flows.

Theorem 6.3 (Perelman's spheroids)**.** *There exists an ancient Ricci flow $(S^3\times(-\infty,0),g)$ which has positive curvature and on which $O(1)\times O(3)$ acts by isometries, but is not the shrinking sphere.*

Sketch. The idea is to take a limit of "very old" solutions constructed by evolving suitable initial data. We begin by evolving a sequence of ($O(1)\times O(3)$-invariant) capped cylinders $C_k=S^2\times[-k,k]$ of radius one and length $2k$. When $k=0$, the solution is just the sphere of radius one, which shrinks to a point ~ 1. For other values of k, C_k still shrinks to a point in time ~ 1 (since $\mathrm{R}\sim 1$ at the initial time), becoming round in the process (in accordance with Hamilton's theorem). After translating time, we can arrange that the final time is $t=0$. The "perigee" and "apogee" take a fixed time to decrease by $1/2$. So we can parabolically rescale so that, for $k\geq 1$, the eccentricity is ~ 2 and the diameter is $\sim 1/2$ at time $t=-1$, and that the initial time α_k goes to $-\infty$ as $k\to\infty$. Since the volumes are uniformly controlled from below, Perelman's curvature estimate and the Bernstein estimates ensure that the curvature and its derivatives are uniformly bounded along the sequence. We can now take a limit using the compactness theorem. Since we ensured that the eccentricity is ~ 2 at time -1, the limit cannot be the shrinking sphere. □

Theorem 6.4. *Let $(M^n\times(-\infty,0],g)$ be a κ-solution. If M^n is non-compact, then the* ASYMPTOTIC CURVATURE RATIO[6]

$$\mathcal{R}(M^n,g_0)\doteqdot\limsup_{\mathrm{dist}(x,x_0,0)\to\infty}\mathrm{R}(x,0)\,\mathrm{dist}^2(x,x_0,0)$$

is infinite.

Sketch. Suppose, contrary to the claim, that $\mathcal{R}(M^n,g_0)<\infty$. Consider the rescaled flow $(M^n\times(-\infty,0],\lambda^2g_{\lambda^{-2}t})$. Note that at time zero, the rescaled metrics (M^n,λ^2g_0) always limit to some metric cone (C,d,o) as $\lambda\searrow 0$ in the Gromov–Hausdorff sense. Due to the curvature bound (and non-collapsing) the limit and the convergence will be smooth away from the tip, o. But since the radial direction must be a null eigenvalue of Rc, we deduce (as before) that the limit splits off a line. But this is only possible if the limit cone is flat, and this violates positive curvature (by Toponogov's theorem). □

Corollary 6.5. *Let $(M^n\times(-\infty,0],g)$ be a κ-solution. If M^n is non-compact, then there are points $x_j\in M^n$ and scales λ_j such that $(M^n\times(-\infty,0],x_j,g_j)$, $(g_j)_{(x,t)}\doteqdot\lambda_j^2g_{(x,\lambda_j^{-2})}$ converges to a κ-solution which splits off a line.*

[6]This number is independent of the choice of point x_0.

Sketch. Since the asymptotic curvature ratio is infinite, we can find points $x_j \in M^n$ such that

$$d_j^2 \doteqdot 10\,\mathrm{R}(x_j,0)\,\mathrm{dist}^2(x_j,x_0,0) \to \infty\,.$$

In particular, $\mathrm{dist}^2(x_j,x_0,0) \to \infty$. By point-picking, we can find $y_j \in B_{2d_j/\sqrt{\mathrm{R}(x_j,0)}}(x_j,0)$ that $\mathrm{R}(y_j,0) \geq \mathrm{R}(x_j,0)$ and $\mathrm{R} \leq 2\,\mathrm{R}(x_j,0)$ in $B_{d_j/\sqrt{\mathrm{R}(y_j,0)}}(y_j,0)$. Since $d_j \to \infty$, the rescaled flows $(M^n \times (-\infty,0], y_j, Q_j g_{(\cdot,Q_j^{-1}t)})$ converge locally smoothly to a limit κ-solution. But (since $y_j \to \infty$) this solution must contain a line, and hence split off a line. $\square$

Corollary 6.6. *All two-dimensional κ-solutions are compact.*

Theorem 6.7. *Let $(M^n \times (-\infty,0], g)$ be a κ-solution. The* ASYMPTOTIC VOLUME RATIO[7]

$$\mathcal{V}(M^n,g_0) \doteqdot \limsup_{r\to\infty} \frac{\mathrm{volume}(B_r(x_0,0))}{r^n}$$

is zero.

Sketch. If $n=2$, then M^n compact, and the claim is true. So suppose the claim is true for some dimension $n \geq 2$ and let $(M^{n+1} \times (-\infty,0], g)$ be a non-compact κ-solution. By (6.5), $(M^{n+1} \times (-\infty,0], g)$ splits off a line at infinity after rescaling. The claim then follows from the inductive hypothesis, since, by the Bishop–Gromov volume comparison theorem, $\mathrm{volume}(B_r(x)) \geq \mathcal{V}r^n$, which is invariant under rescaling, and hence passes to the limit. $\square$

6.3 Noncollapsing ancient solutions in three-dimensions. Perelman's machinery yields the following chatracterization of κ-solutions in three dimensions.

Theorem 6.8 (Non-collapsing ancient solutions in three dimensions)**.** *Every connected oriented three-dimensional κ-solution is one of the following.*

(1) A shrinking round spherical space form;

(2) A shrinking round cylinder or finite quotient;

(3) A C-component: an S^3 or $\mathbb{R}P^3$ whose diameter, curvature and volume are all bounded uniformly (between C^{-1} and C) after rescaling to normalize any one of them;

(4) A C-capped ε-tube (after removing one C-cap and rescaling, it is ε close to a unit round cylinder of length ε^{-1}); or

(5) A doubly C-capped ε-tube.

Sketch. A key step is to show that the space of pointed three-dimensional curvature normalized κ-solutions is compact.

After blowing down (taking the limit of $\lambda^2 g_{(\cdot,\lambda^{-2}t)}$ as $\lambda \searrow 0$) we see an "asymptotic shrinker".

Since the only asymptotic shrinking solitons are finite quotients of shrinking round spheres or cylinders, every solution of sufficiently large normalized diameter is made up of ε-tubes and regions of uniformly bounded diameter.

Any example which is not a shrinking cylinder or quotient must satisfy $\mathrm{Rm} > 0$. Thus, if it is non-compact, then the soul theorem implies that it is diffeomorphic to $\mathbb{R}^3$. So a

[7]This number is independent of the choice of point x_0.

noncompact example with $\mathrm{Rm} > 0$ must be C-capped (the existence of such a C comes from the compactness of the space of κ-solutions).

A similar argument shows that a compact example either has uniformly bounded diameter, or is a doubly-capped ε-tube. In every case $\mathrm{Rm} > 0$, so Hamilton's theorem implies that the manifold is diffeomorphic to a spherical space form. The uniformly bounded diameter components are either round or C-components, where again C comes from the compactness of the space of κ-solutions. $\square$

In fact, there is a complete list of such solutions.

Theorem 6.9 (Angenent–Daskalopoulos–Šešum, Brendle)**.** *The only κ-solutions in three dimensions are:*

(1) The (static/shrinking) flat $\mathbb{R}^3$.
(2) The shrinking sphere.
(3) The shrinking cylinder.
(4) Bryant's soliton.
(5) Perelman's example.
(6) Isometric quotients of the above.

6.4 Many examples in higher dimensions. There are a great many further examples of ancient Ricci flows of positive curvature in higher dimensions.

Example 6.2 (Perelman's examples)**.** Perelman's construction generalizes to spheres S^n of any dimension $n \geq 3$ and any bisymmetry class $O(k) \times O(n+1-k)$, $k = 3, \ldots, n$. These examples have positive curvature and their volume does not collapse at any scale as $t \to -\infty$.

Example 6.3 (Fateev's example, [**17**])**.** The time-dependent metric

$$g = \chi^2\, dr^2 + \psi^2\, d\theta^2 + \varphi^2\, d\omega^2\,,$$

$$\begin{aligned}
\chi^2(r,t) &\doteqdot \frac{\cosh(-2t)\sinh(-2t)}{2(\cos^2 r + \sin^2 r\cosh(-2t))(\sin^2 r + \cos^2 r\sinh(-2t))}\\
\psi^2(r,t) &\doteqdot \frac{\cos^2 r\sinh(-2t)}{2(\sin^2 r + \cos^2 r\cosh(-2t))}\\
\varphi^2(r,t) &\doteqdot \frac{\sin^2 r\sinh(-2t)}{2(\cos^2 r + \sin^2 r\cosh(-2t))}
\end{aligned}$$

on $(-\frac{\pi}{2}, \frac{\pi}{2}) \times S^1 \times S^1 \times (-\infty, 0)$ extends to S^3 and satisfies Ricci flow. This example has positive curvature but its volume collapses (relative to the scale of the curvature) as $t \to -\infty$.

This example is related to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. It generalizes to a large class of explicit Ricci flows on odd dimensional spheres [**4**].

These examples in some sense generalize the Fateev–King–Rosenau solution on S^2. So does the following.

Example 6.4. There is an $O(2) \times O(n-1)$ invariant ancient Ricci flow on S^n which has positive curvature and is not the shrinking sphere. Indeed, its contains a closed geodesic Γ

whose length satisfies $\text{length}(\gamma, t) \nearrow 2\pi$ as $t \to -\infty$ (and hence is volume collapsing at the scale of the curvature as $t \to -\infty$).

There are also a great many steady soliton examples. In the positive curvature setting, we have the following.

Example 6.5 (Lai's flying wings I)**.** For each $\theta \in (0, \frac{\pi}{2})$, there is an $O(2) \times O(n-2)$ invariant steady soliton on $\mathbb{R}^n$, $n \geq 3$, which has positive curvature and is not the bowl soliton. Indeed, its metric cone at infinity is the round cone of dimension $n-1$ with opening angle θ. It is asymptotic to an S^{n-2} family of cigar planes.

Example 6.6 (Lai's flying wings II)**.** For each $\theta \in (0, \frac{\pi}{2})$, there is an $O(3) \times O(n-3)$ invariant steady soliton on $\mathbb{R}^n$, $n \geq 4$, which has positive curvature and is not the bowl soliton. Indeed, it is asymptotic to an S^{n-3} family of three dimensional bowl solitons.

The full classification of ancient solutions is thus a very difficult question in general. The three dimensional case may be within reach, however.

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