Eigenvalue Problems and free boundary minimal surfaces in spherical caps

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Definition

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- Φ is a *free boundary minimal immersion* if:

  (i) $\vec{H} = 0$;

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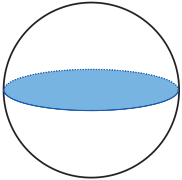

Figure: $(\mathcal{N}, \mathbf{g})$ = Euclidean 3-ball and $\Phi(\Sigma)$ = equatorial disc.

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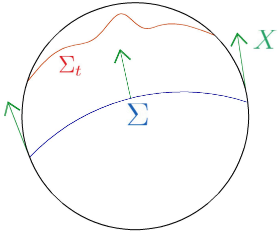
- Denote $X = \frac{\partial \Phi}{\partial t}$ ($X|_{\partial\Sigma}$ is tangent to $\partial\mathcal{N}$).



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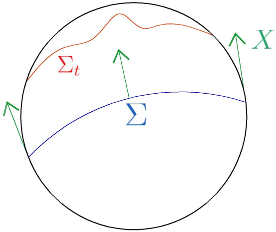

The first variation formula gives:

$$\frac{d}{dt}\bigg|_{t=0} |\Phi_t(\Sigma)| = -\int_\Sigma \langle \vec{H}, X\rangle \, dA + \int_{\partial\Sigma} \langle \nu, X\rangle \, dL.$$

$$\text{Critical point} \iff \vec{H} = 0 \text{ and } \nu \perp \partial\mathcal{N}.$$

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i.e., ϕ_j is a eigenfunction of $-\Delta_g$ with eigenvalue 2.

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  The maximizer induces a harmonic map $\phi : (M, \mathcal{C}) \to \mathbb{S}^n$.
- (Karpukhin-Kusner-Mcgrath-Stern, new preprint-2024): Let M_γ be the closed orientable surface of genus γ. Then $\lambda_1^*(M_\gamma)$ or $\lambda_1^*(M_{\gamma+1})$ admits a maximizing metric, for each γ.

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- Orientable surface of genus 2 (Nayatani-Shoda, 2019): induced by a certain branched cover $M \to \mathbb{S}^2$, $\lambda_1^* = 16\pi$.

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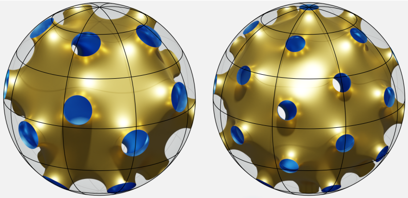

Figure: Picture by M. Schulz.

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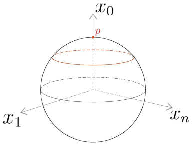

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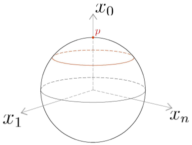

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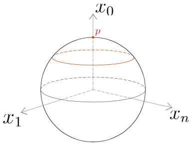

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- $\sigma = 2$ is not an eigenvalue of $-\Delta_g$ with Dirichlet boundary condition:

$$\begin{cases} -\Delta_g w = 2w, & \text{in } \Sigma, \\ w = 0, & \text{on } \partial\Sigma, \end{cases} \quad \Rightarrow \quad w \equiv 0.$$

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$$\begin{aligned} \mathcal{D}_\alpha : C^\infty(\partial\Sigma) &\to C^\infty(\partial\Sigma) \\ \mathcal{D}_\alpha \phi &= \frac{\partial \widehat{\phi}}{\partial \nu}. \end{aligned}$$

- The spectrum of $\mathcal{D}_\alpha$ is discrete (Steklov eigenvalues with frequency α)

$$\sigma_0(g,\alpha) < \sigma_1(g,\alpha) \leq \sigma_2(g,\alpha) \leq \cdots \to +\infty.$$

Steklov eigenvalue problem with frequency

- Fix $\alpha \in \mathbb{R}$ which is not on the spectrum of $-\Delta_g$ with Dirichlet boundary condition;
- given $u \in C^\infty(\partial\Sigma)$, there is a unique $\widehat{u} \in C^\infty(\Sigma)$, such that

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- The case $\alpha = 0$ corresponds to the usual Steklov spectrum.

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The eigenvalue $\sigma_0(g,\alpha)$ is simple and is given by

$$\sigma_0(g,\alpha) = \inf\left\{ \frac{\displaystyle\int_\Sigma |\nabla^g \widehat{u}|_g^2 \, dA_g - \alpha \int_\Sigma \widehat{u}^2 \, dA_g}{\displaystyle\int_{\partial\Sigma} u^2 \, dL_g};\ u \in \mathrm{dom}(\mathcal{D}_\alpha) \setminus \{0\} \right\}.$$

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Denote by ϕ_0 a first eigenfunction, which we can choose to be positive. Then,

$$\begin{aligned} \sigma_1(g,\alpha) &= \inf \left\{ \frac{\int_\Sigma |\nabla^g \widehat{u}|_g^2 \, dA_g - \alpha \int_\Sigma \widehat{u}^2 \, dA_g}{\int_{\partial\Sigma} u^2 \, dL_g} ; \, u \in \mathrm{dom}(\mathcal{D}_\alpha) \setminus \{0\} \right. \\ &\qquad \left. \text{and} \int_{\partial\Sigma} u\phi_0 \, dL_g = 0 \right\}. \end{aligned}$$

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- By using conformal diffeomorphisms of $\mathbb{B}^2_r$, we can assume

$$\int_{\partial\Sigma} u_j \phi_0 \, dL = 0, \quad j = 1, 2,$$

where ϕ_0 is a positive eigenfunction associated to $\sigma_0(g, 2)$.

By the variational characterization of the eigenvalues, we have

$$\begin{aligned}\sigma_0(g,2)\int_{\partial\Sigma} u_0^2\, dL_g &\le \int_\Sigma |\nabla^g \widehat{u}_0|_g^2\, dA_g - 2\int_\Sigma \widehat{u}_0^2\, dA_g\\ &\le \int_\Sigma |\nabla^g u_0|_g^2\, dA_g - 2\int_\Sigma u_0^2\, dA_g,\end{aligned}$$

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$$\left(\sigma_0(g,2)\cos^2 r + \sigma_1(g,2)\sin^2 r\right)|\partial\Sigma|_g + 2|\Sigma|_g \le 4\pi(1-\cos r)(\gamma+\ell).$$

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We define

$$\Theta_r(\Sigma,g) = \left(\sigma_0(g,2)\cos^2 r + \sigma_1(g,2)\sin^2 r\right)|\partial\Sigma|_g + 2|\Sigma|_g.$$

Main results I

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Theorem A (L., Menezes, 2023)

Let Σ be a compact orientable surface of genus γ and ℓ boundary components. Then, for any $g \in \mathcal{M}(\Sigma)$, we have

$$\Theta_r(\Sigma, g) \leq 4\pi(1 - \cos r)(\gamma + \ell).$$

Moreover, if Σ is a disk, the equality holds if, and only if, (Σ, g) is isometric to $\mathbb{B}^2_r$.

Therefore $\Theta^*_r(\Sigma) = \sup_{g \in \mathcal{M}(\Sigma)} \Theta_r(\Sigma, g)$ is finite.

Free Boundary Minimal Surfaces
Eigenvalue shape optimization
Main results

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Theorem B (L., Menezes, 2023)

Let Σ be a compact surface with boundary. If $g \in \mathcal{M}(\Sigma)$ satisfies $\Theta_r(\Sigma, g) = \Theta^*_r(\Sigma)$, then there exist a $\sigma_0(g, 2)$-eigenfunction ϕ_0 and independent $\sigma_1(g, 2)$-eigenfunctions $\phi_1, \ldots, \phi_n$, which induce a free boundary minimal isometric immersion $\Phi = (\phi_0, \phi_1, \ldots, \phi_n) : (\Sigma, g) \to \mathbb{B}^n_r$.

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where $-\frac{1}{2} < a \leq 0$ is a constant and $\varphi(s)$ is given by

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Proposition (Li-Xiong, 2018): For any $0 < r \leq \frac{\pi}{2}$, there exist $-\frac{1}{2} < a \leq 0$ and $s_0 \in \mathbb{R}$ such that $\Phi_a : [-s_0, s_0] \times \mathbb{S}^1 \to \mathbb{B}^3_r$ is a free boundary minimal immersion.

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Let Σ be an annulus and consider a free boundary minimal immersion $\Phi = (\phi_0, \ldots, \phi_n) : (\Sigma, g) \to \mathbb{B}^n_r$. Suppose ϕ_j is a $\sigma_1(g, 2)$-eigenfunction, for $j = 1, \ldots, n$. Then $n = 3$ and Φ is one of the rotational immersions described previously.

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Remark 3: Inspired by our work, Medvedev (2023) obtained analogous results for geodesics balls in $\mathbb{H}^n$.

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Recall that

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Since $\Sigma \simeq [0,1] \times \mathbb{S}^1$, then $g = \lambda g_{\mathrm{cyl}}$, for some positive function $\lambda = \lambda(s,\theta)$. In particular, $\Delta_g = \lambda^{-1}\Delta_{\mathrm{cyl}}$ and $\nu_g = \lambda^{-\frac{1}{2}}\nu_{\mathrm{cyl}}$.

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$$\begin{aligned} \Delta_g \phi_i + 2\phi_i = 0 \quad \Rightarrow \quad & \Delta_{\mathrm{cyl}} \phi_i + 2\lambda\phi_i = 0 \\ \Rightarrow\ & \Delta_{\mathrm{cyl}} \frac{\partial \phi_i}{\partial \theta} + 2\frac{\partial \lambda}{\partial \theta}\phi_i + 2\lambda\frac{\partial \phi_i}{\partial \theta} = 0 \\ \Rightarrow\ & \Delta_g \frac{\partial \Phi}{\partial \theta} + 2\lambda^{-1}\frac{\partial \lambda}{\partial \theta}\Phi + 2\frac{\partial \Phi}{\partial \theta} = 0. \end{aligned}$$

Claim: The condition $\sigma_0(g,2) = -\tan r$ and $\sigma_1(g,2) = \cot r$ implies

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- An O.D.E analysis implies that Φ is rotational in the sense of do Carmo-Dajczer-Otsuki, so $\Phi(\Sigma)$ is one of the annuli described before.

Thank you!