Uniqueness of semigraphical translators

(joint with F. Martín and R. Tsiamis)

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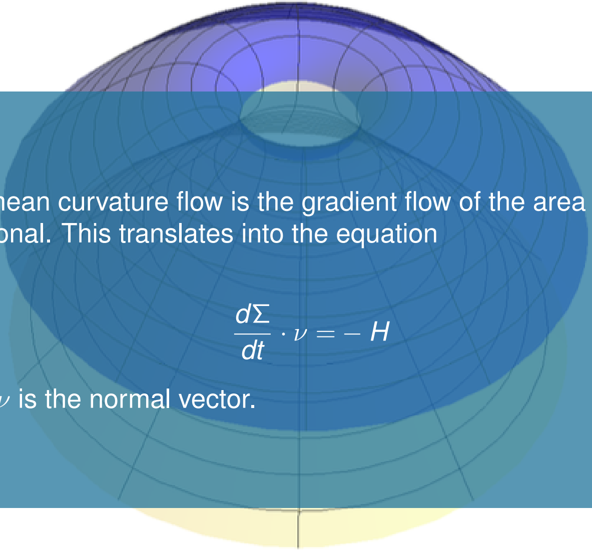
INTRODUCTION



- The mean curvature flow is the gradient flow of the area functional. This translates into the equation

$$\frac{d\Sigma}{dt} \cdot \nu = -H$$

Here ν is the normal vector.

- Translators are a special kind of solution of the form

$$\Sigma(\cdot, t) = \Sigma(\cdot, 0) + t\omega,$$

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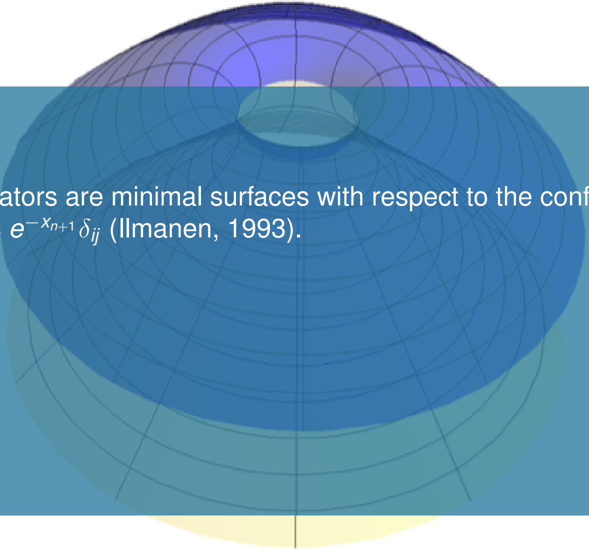
- Translators may appear as singularity models.
- Huisken-Sinestrari proved that under the assumption of mean-convexity, we get translating solitons as singularity models for type 2 singularities.



- Translators are minimal surfaces with respect to the conformal metric $e^{-x_{n+1}}\delta_{ij}$ (Ilmanen, 1993).

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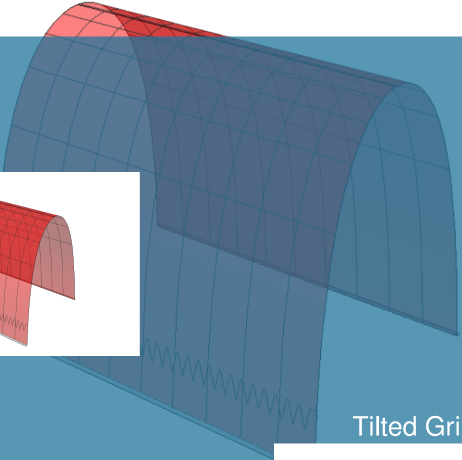
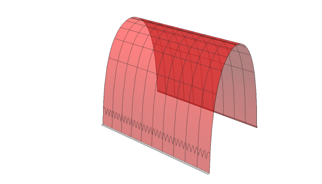
- This allows us to apply $g-$minimal surface theory:
    1. compactness theorems,
    2. curvature estimates,
    3. maximum and tangency principles,
    4. monotonicity results.

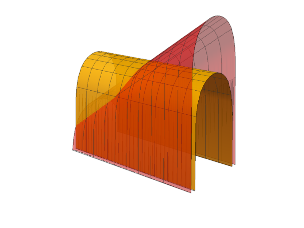
Examples



- Grim Reaper

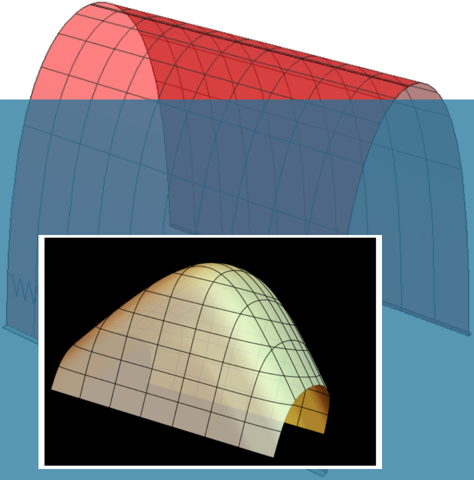


Tilted Grim Reaper



Examples

- Δ-wing



Examples

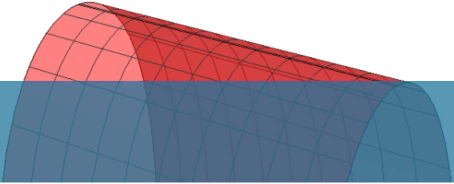
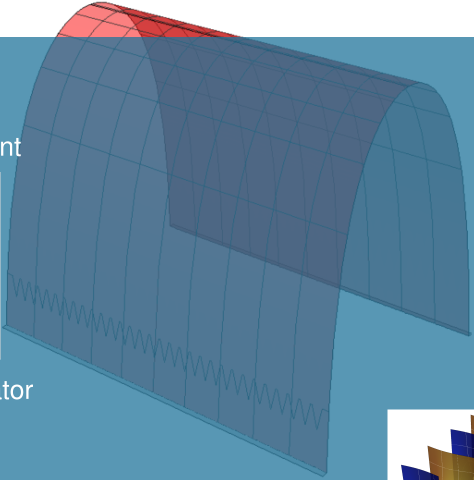
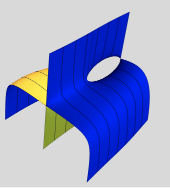


FIGURE 5. The bowl soliton in $\mathbf{R}^3$ and the translating catenoid for $\lambda = 2$.

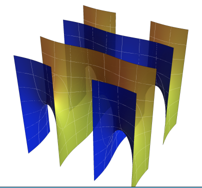
Examples



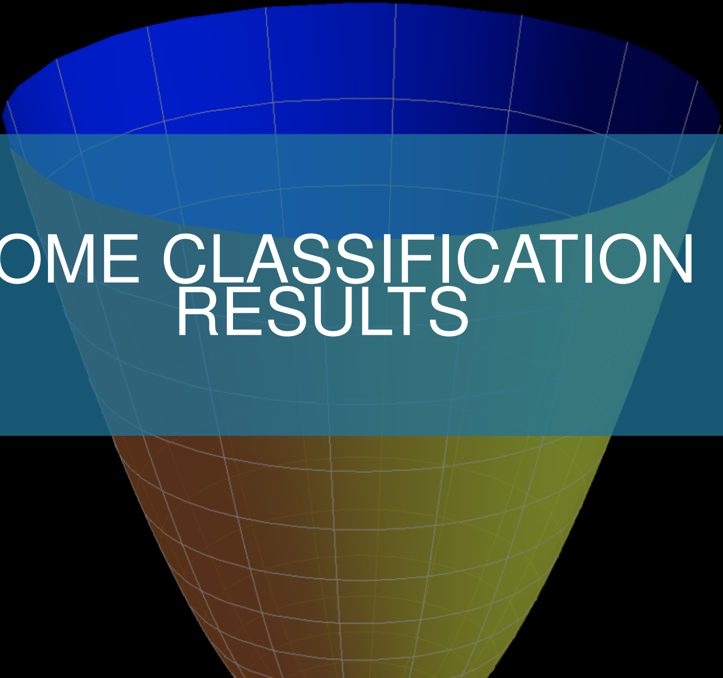
- Nguyen's trident



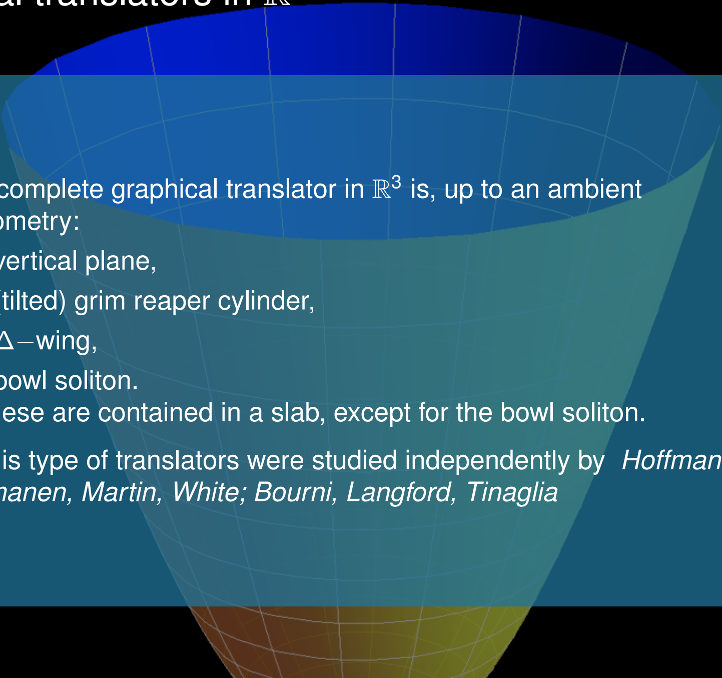
- Scherk-translator



SOME CLASSIFICATION RESULTS



Graphical translators in $\mathbb{R}^3$



A complete graphical translator in $\mathbb{R}^3$ is, up to an ambient isometry:

- a vertical plane,
- a (tilted) grim reaper cylinder,
- a $\Delta-$wing,
- a bowl soliton.

These are contained in a slab, except for the bowl soliton.

This type of translators were studied independently by *Hoffman, Ilmanen, Martin, White; Bourni, Langford, Tinaglia*

Semi-graphical translators in $\mathbb{R}^3$

- A translator M is called *semigraphical* if
  1. M is a smooth, connected, properly embedded submanifold (without boundary) in $\mathbb{R}^3$,
  2. M contains a non-empty, discrete collection of vertical lines $\{L_i\}$.
  3. $M \setminus \bigcup_i L_i$ is a graph.

Theorem *(Hoffman, Ilmanen, Martin, White)*

A semigraphical translator in $\mathbb{R}^3$ is one of the following:

- a (doubly periodic) Scherk translator,
- a (singly periodic) Scherkenoid,
- a (singly periodic) helicoid-like translator,
- a pitchfork,
- a (singly periodic) trident,
- (after a rigid motion) a translator containing the $z-$axis such that $M \setminus Z$ is a graph over $\{(x, y) : y \neq 0\}$.

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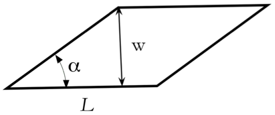


FIGURE 1. The parallelogram with base L, corner angle α, and height w.

Semi-graphical translators in $\mathbb{R}^3$

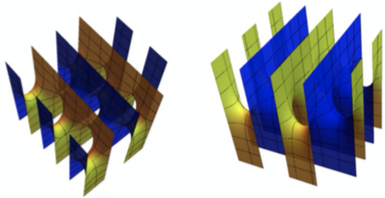


FIGURE 3. The doubly periodic Scherk translator $\mathscr{S}_{\pi/2,\pi/2}$

THE PITCHFORK AND THE HELICOID

Pictures

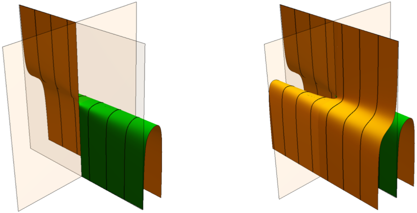


FIGURE 1. From left to right: A fundamental piece of the pitchfork of width π and the whole surface, obtained from a fundamental piece by a 180^o rotation around the z-axis. The asymptotic behavior at the two wings (vertical plane/ grim reaper) is visible here.

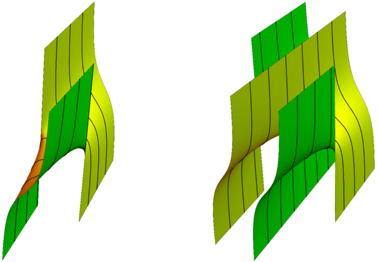
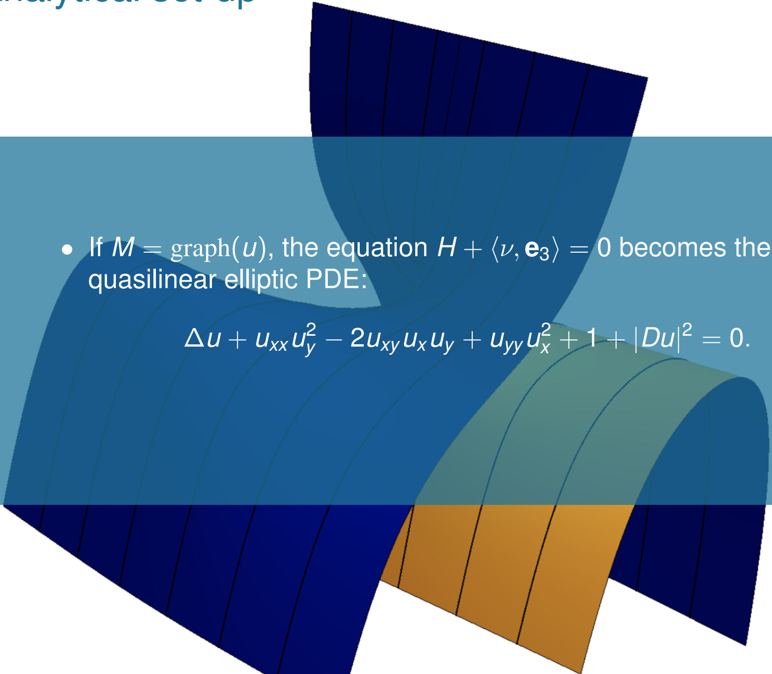


FIGURE 2. From left to right: A fundamental piece of the helicoid of width $\pi/2$; and part of the surface, obtained by repeated reflection along vertical boundary lines.

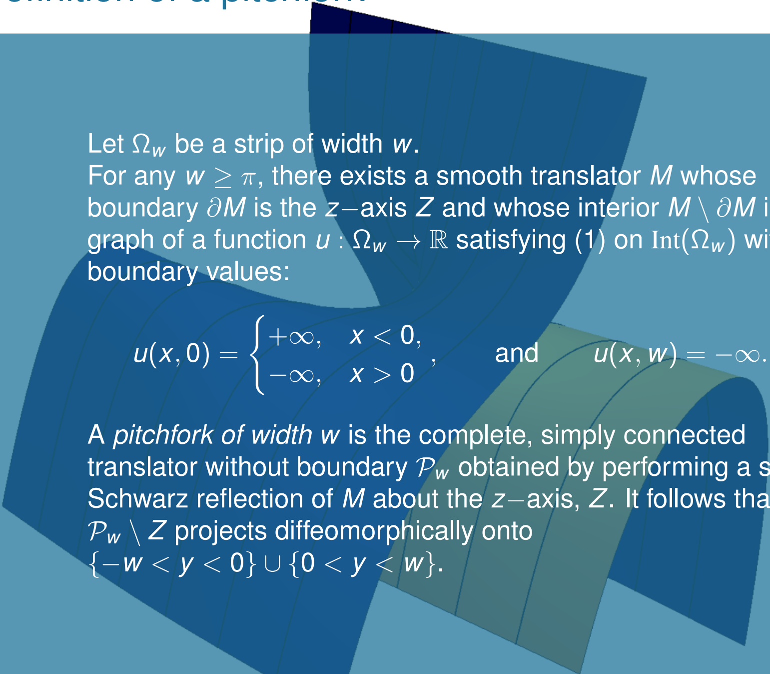
Analytical set-up



- If $M = \mathrm{graph}(u)$, the equation $H + \langle \nu, \mathbf{e}_3 \rangle = 0$ becomes the quasilinear elliptic PDE:

$$\Delta u + u_{xx}u_y^2 - 2u_{xy}u_x u_y + u_{yy}u_x^2 + 1 + |Du|^2 = 0. \qquad (1)$$

Definition of a pitchfork



Let Ω_w be a strip of width w.
For any $w \geq \pi$, there exists a smooth translator M whose boundary ∂M is the $z-$axis Z and whose interior $M \setminus \partial M$ is the graph of a function $u : \Omega_w \to \mathbb{R}$ satisfying (1) on $\mathrm{Int}(\Omega_w)$ with boundary values:

$$u(x,0) = \begin{cases} +\infty, & x < 0, \\ -\infty, & x > 0 \end{cases}, \qquad \text{and} \qquad u(x,w) = -\infty. \tag{2}$$

A *pitchfork of width w* is the complete, simply connected translator without boundary $\mathcal{P}_w$ obtained by performing a single Schwarz reflection of M about the $z-$axis, Z. It follows that $\mathcal{P}_w \setminus Z$ projects diffeomorphically onto $\{-w < y < 0\} \cup \{0 < y < w\}$.

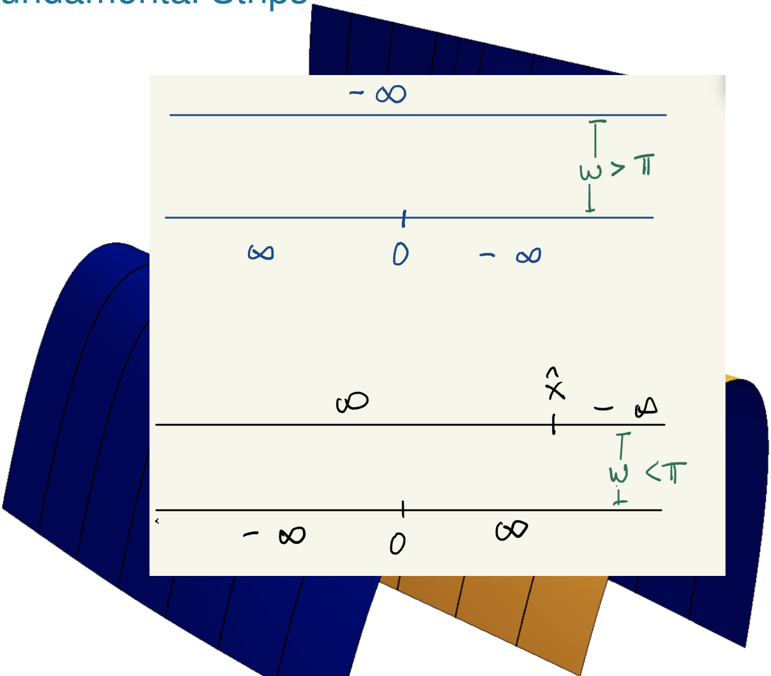
Definition of a Helicoid

For any $w < \pi$, there exists a smooth translator M whose boundary ∂M consists of two vertical lines, the $z-$axis and the line $\{x = \hat{x}, y = w\}$ for some $\hat{x} > 0$, and whose interior $M \setminus \partial M$ is the graph of a function $u : \Omega_w \to \mathbb{R}$ satisfying (1) on $\mathrm{Int}(\Omega_w)$ with boundary values:

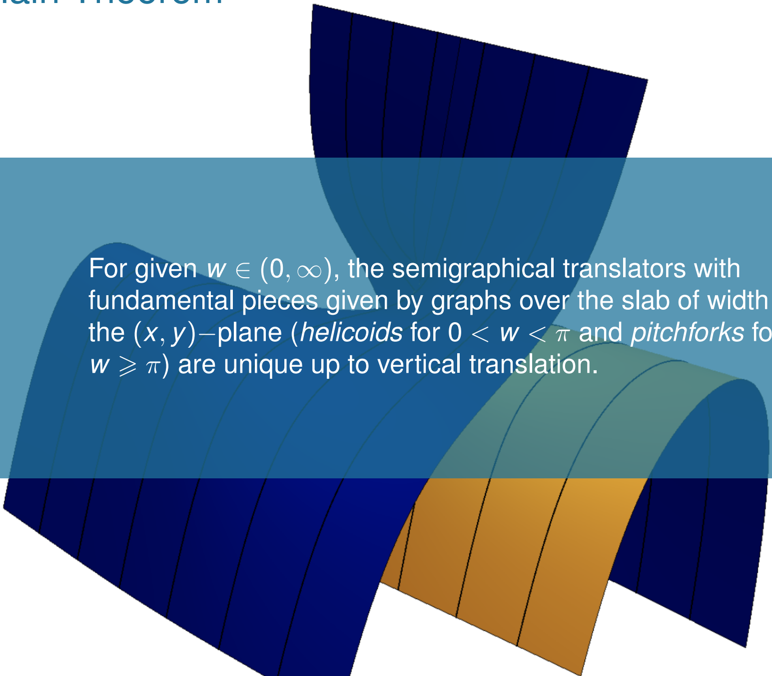
$$u(x,0) = \begin{cases} +\infty, & x < 0, \\ -\infty, & x > 0 \end{cases}, \qquad \text{and} \qquad u(x,w) = \begin{cases} -\infty, & x < \hat{x}, \\ +\infty, & x > \hat{x} \end{cases}. \tag{3}$$

A *helicoid of width w* is the complete, simply connected translator without boundary $\mathcal{H}_w$ obtained from M by performing countably many repeated Schwarz reflection about these axes. It follows that $\mathcal{H}_w$ contains the vertical lines L_n through the points $n(\hat{x}, w)$ for $n \in \mathbb{Z}$ and $\mathcal{H}_w \setminus \bigcup_n L_n$ projects diffeomorphically onto the strip cover $\cup_{n\in\mathbb{Z}}\{nw < y < (n+1)w\}$.

Fundamental Strips



Main Theorem

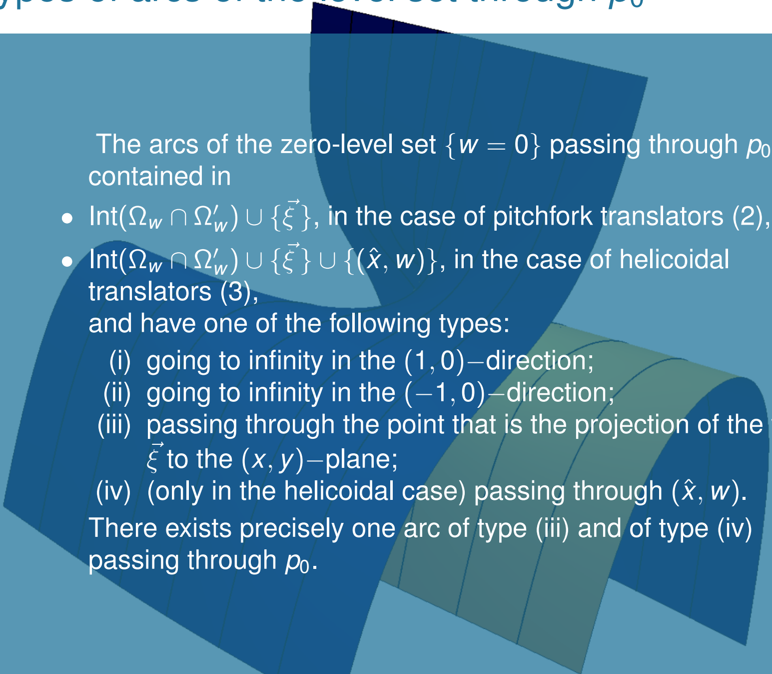


For given $w \in (0, \infty)$, the semigraphical translators with fundamental pieces given by graphs over the slab of width w in the $(x, y)-$plane (*helicoids* for $0 < w < \pi$ and *pitchforks* for $w \geqslant \pi$) are unique up to vertical translation.

Strategy of the proof

- Assume that we have two distinct solutions u_1, u_2. Then there is p_0 such that $D(u_1 - u_2)(p_0) \neq 0$.
- Argue that there is a q_0 such that $Du_1(q_0) = Du_1(p_0)$. Define $\xi = q_0 - p_0 = (\xi_1, \xi_2)$.
- Argue that if it is not possible to pick $\xi_2 \neq 0$, then $u_1 = u_2 + c$.
- Define $u_1'(p) = u_1(p + \xi)$, $w(p) = u_1'(p) - u_2(p)$ and assume that $w(p_0) = 0$.
- Study the zero-level set of w and show the following:

Types of arcs of the level set through p_0



The arcs of the zero-level set $\{w = 0\}$ passing through p_0 are contained in

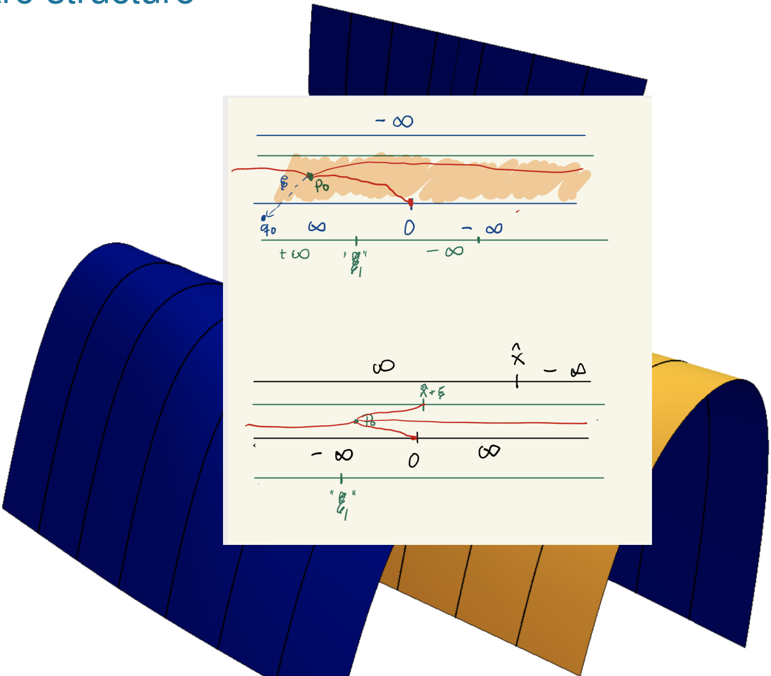
- $\mathrm{Int}(\Omega_w \cap \Omega'_w) \cup \{\vec{\xi}\}$, in the case of pitchfork translators (2),
- $\mathrm{Int}(\Omega_w \cap \Omega'_w) \cup \{\vec{\xi}\} \cup \{(\hat{x}, w)\}$, in the case of helicoidal translators (3),

and have one of the following types:

(i) going to infinity in the $(1, 0)-$direction;
(ii) going to infinity in the $(-1, 0)-$direction;
(iii) passing through the point that is the projection of the vector $\vec{\xi}$ to the $(x, y)-$plane;
(iv) (only in the helicoidal case) passing through $(\hat{x}, w)$.

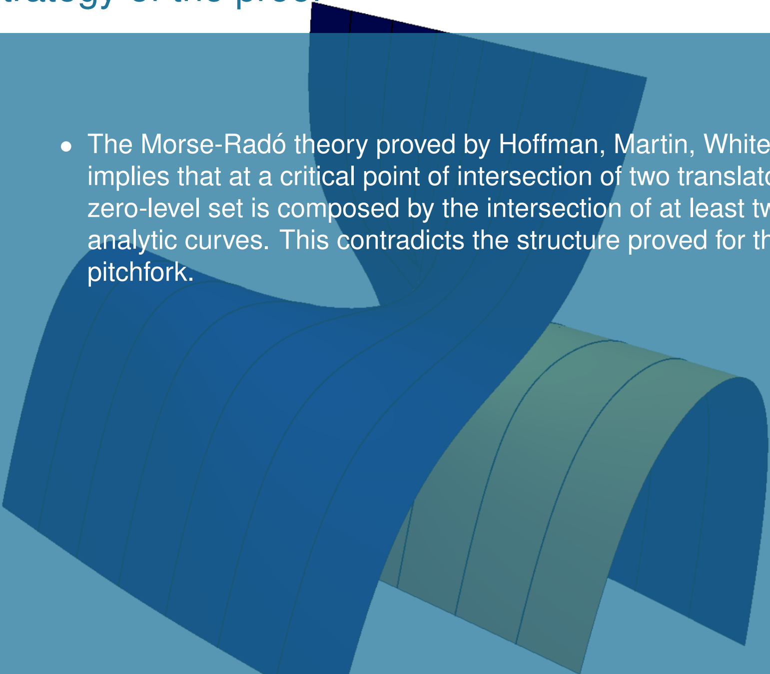
There exists precisely one arc of type (iii) and of type (iv) passing through p_0.

Arc-structure



Strategy of the proof

- The Morse-Radó theory proved by Hoffman, Martin, White implies that at a critical point of intersection of two translators the zero-level set is composed by the intersection of at least two analytic curves. This contradicts the structure proved for the pitchfork.



Strategy of the proof

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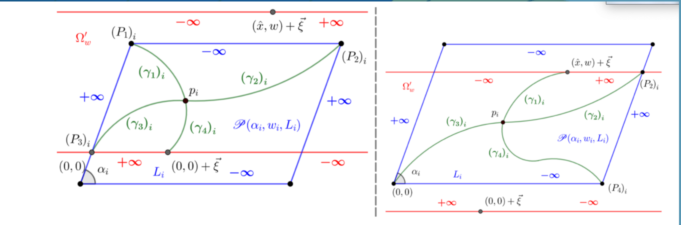
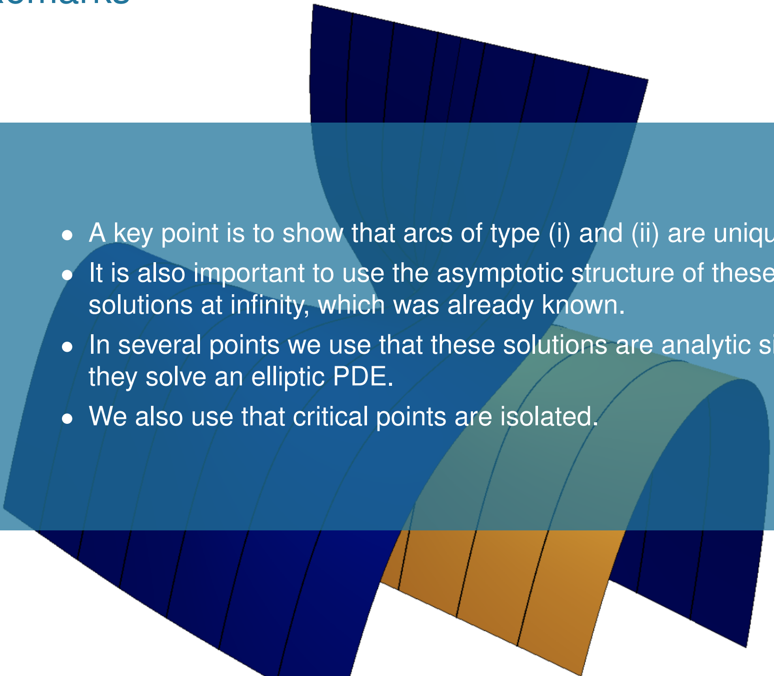
- For the helicoid fix a helicoid constructed as a limit and show that the zero-level set arises as a limit.



- Show that if this is the case, the structure theorem (for the level set) is violated.

Remarks



- A key point is to show that arcs of type (i) and (ii) are unique.
- It is also important to use the asymptotic structure of these solutions at infinity, which was already known.
- In several points we use that these solutions are analytic since they solve an elliptic PDE.
- We also use that critical points are isolated.

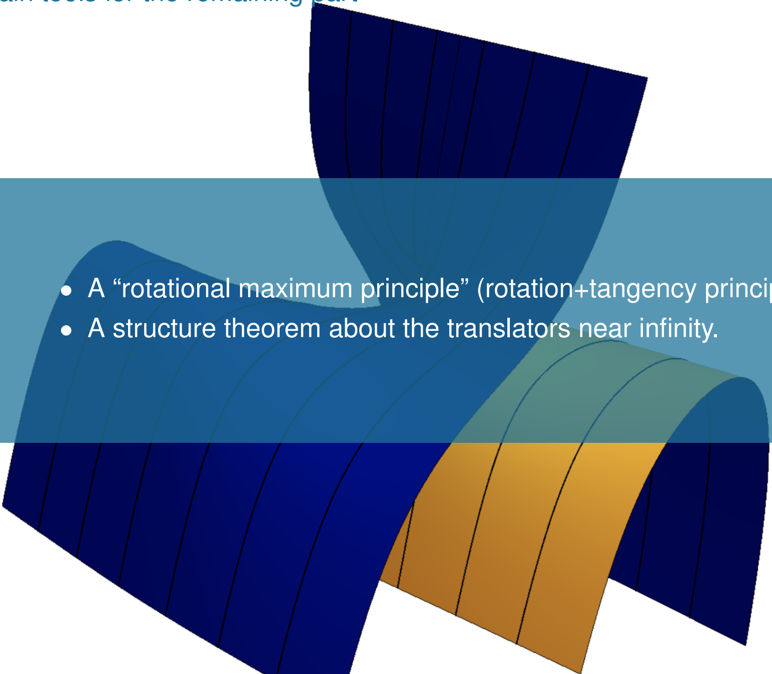
Uniqueness of type (i) for Pitchforks

- Assume there at least two such arcs, consider the semi-infinite region S_0 contained between them.
- Take any sequence of points $\{p_n\} \subset S_0$ with $x-$coordinates tending to $+\infty$.
- Due to White's compactness theorem, for any pitchfork $\mathcal{P}$ constructed from a function u, the sequence of translators formed by

$$\mathcal{P}(n) := \mathcal{P} - (p_n, u(p_n))$$

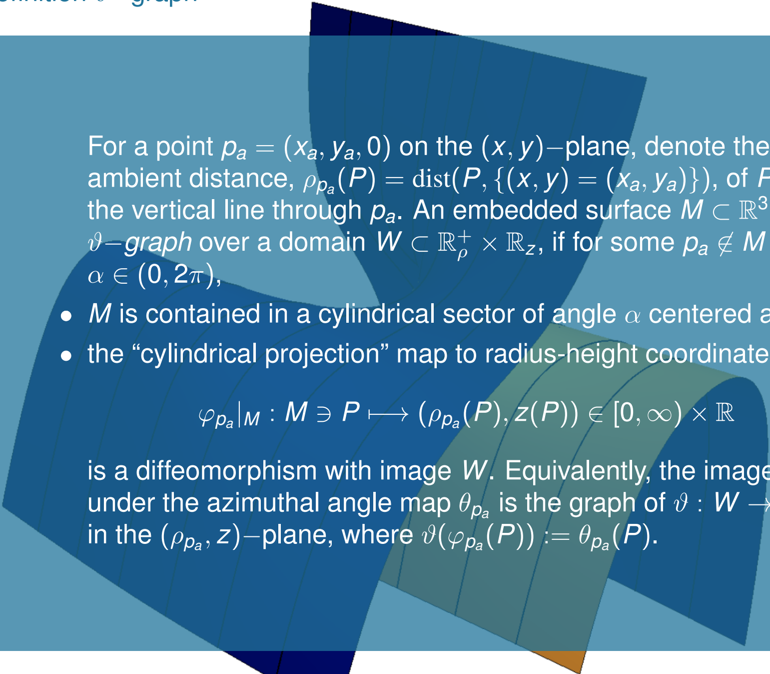
  has a smooth convergent subsequence in the uniform compact topology.
- Work of HMW shows that this limit is a tilted grim reaper G_w over Ω_w. Hence we get two tilted grim reapers of the same slope over different domain (Ω_w and Ω'_w) that intersect along S_0.
- This is a contradiction since two grim reapers cannot have this type of intersection.

Main tools for the remaining part



- A “rotational maximum principle” (rotation+tangency principle).
- A structure theorem about the translators near infinity.

Definition $\vartheta-$graph



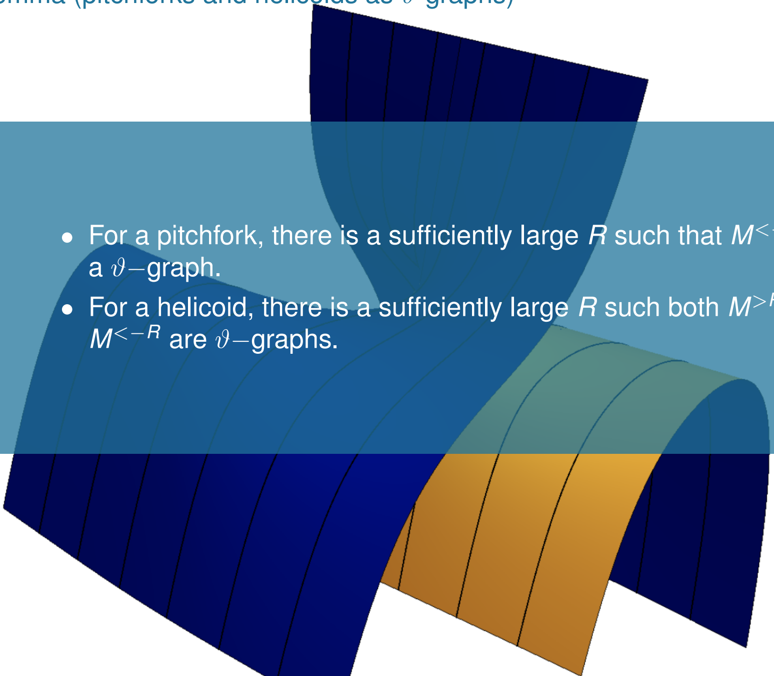
For a point $p_a = (x_a, y_a, 0)$ on the $(x, y)-$plane, denote the ambient distance, $\rho_{p_a}(P) = \mathrm{dist}(P, \{(x, y) = (x_a, y_a)\})$, of P from the vertical line through p_a. An embedded surface $M \subset \mathbb{R}^3$ is a $\vartheta-$*graph* over a domain $W \subset \mathbb{R}^+_\rho \times \mathbb{R}_z$, if for some $p_a \notin M$ and $\alpha \in (0, 2\pi)$,

- M is contained in a cylindrical sector of angle α centered at p_a,
- the "cylindrical projection" map to radius-height coordinates

$$\varphi_{p_a}|_M : M \ni P \longmapsto (\rho_{p_a}(P), z(P)) \in [0, \infty) \times \mathbb{R} \qquad (4)$$

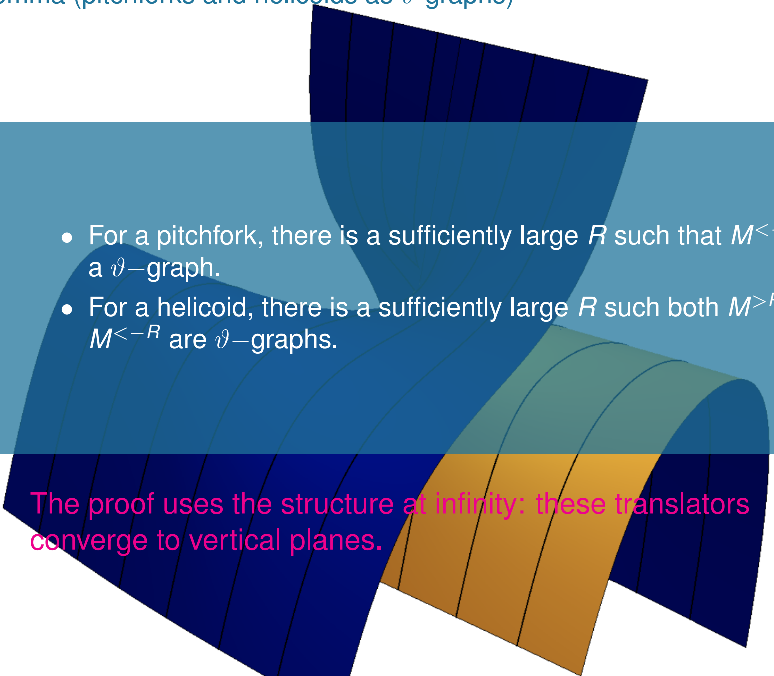
is a diffeomorphism with image W. Equivalently, the image of M under the azimuthal angle map θ_{p_a} is the graph of $\vartheta : W \to (0, \alpha)$ in the $(\rho_{p_a}, z)-$plane, where $\vartheta(\varphi_{p_a}(P)) := \theta_{p_a}(P)$.

Lemma (pitchforks and helicoids as ϑ-graphs)



- For a pitchfork, there is a sufficiently large R such that $M^{<-R}$ is a $\vartheta-$graph.
- For a helicoid, there is a sufficiently large R such both $M^{>R}$ and $M^{<-R}$ are $\vartheta-$graphs.

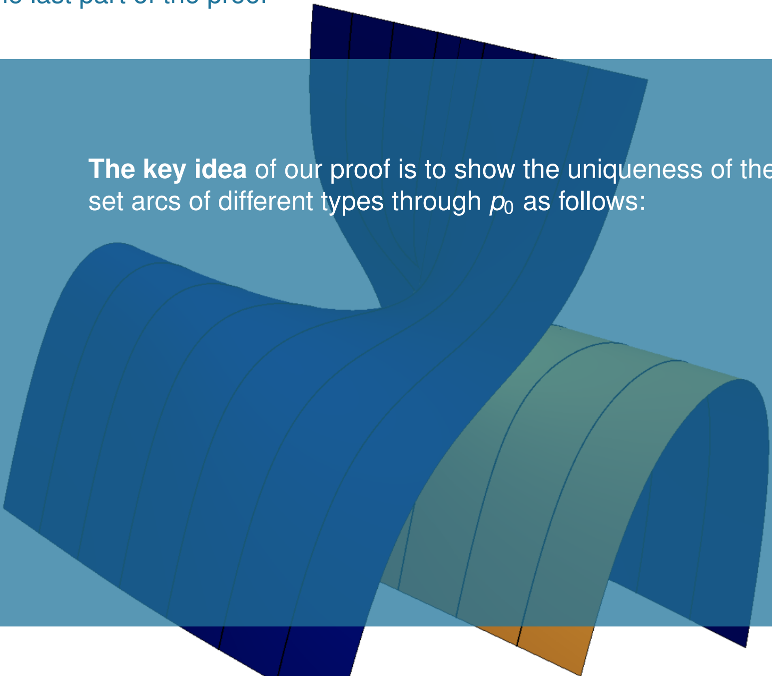
Lemma (pitchforks and helicoids as ϑ-graphs)



- For a pitchfork, there is a sufficiently large R such that $M^{<-R}$ is a $\vartheta-$graph.
- For a helicoid, there is a sufficiently large R such both $M^{>R}$ and $M^{<-R}$ are $\vartheta-$graphs.

The proof uses the structure at infinity: these translators converge to vertical planes.

The last part of the proof



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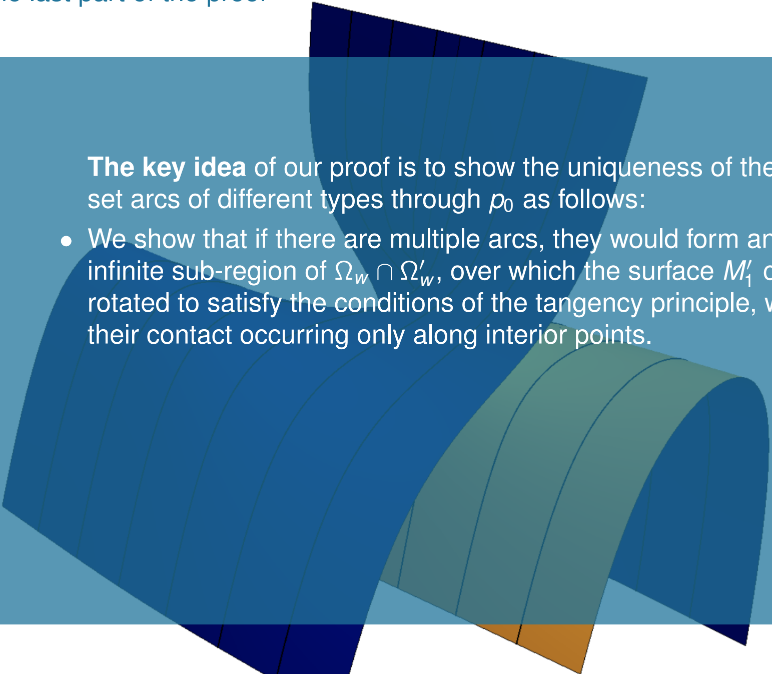

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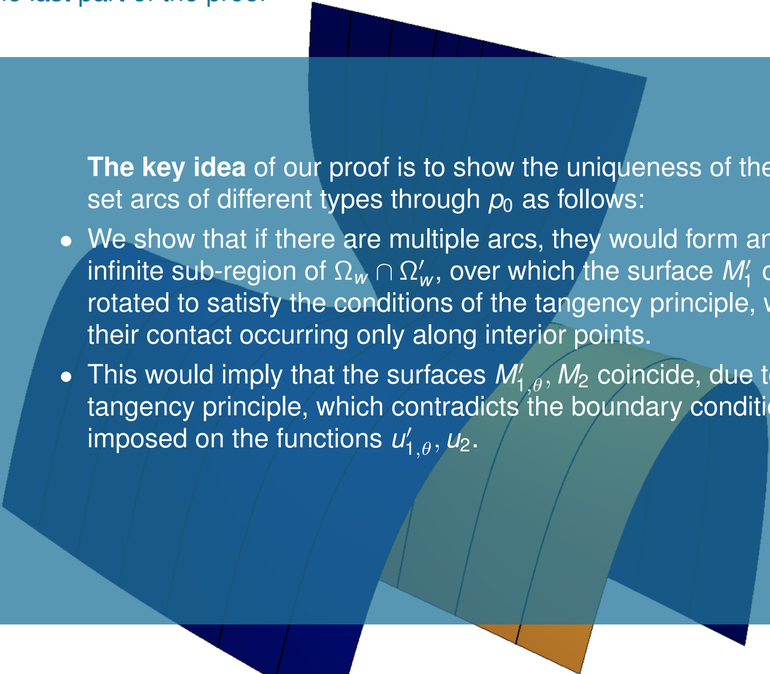
- We show that if there are multiple arcs, they would form an infinite sub-region of $\Omega_w \cap \Omega'_w$, over which the surface M'_1 can be rotated to satisfy the conditions of the tangency principle, with their contact occurring only along interior points.

The last part of the proof



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- This would imply that the surfaces $M'_{1,\theta}, M_2$ coincide, due to the tangency principle, which contradicts the boundary conditions imposed on the functions $u'_{1,\theta}, u_2$.

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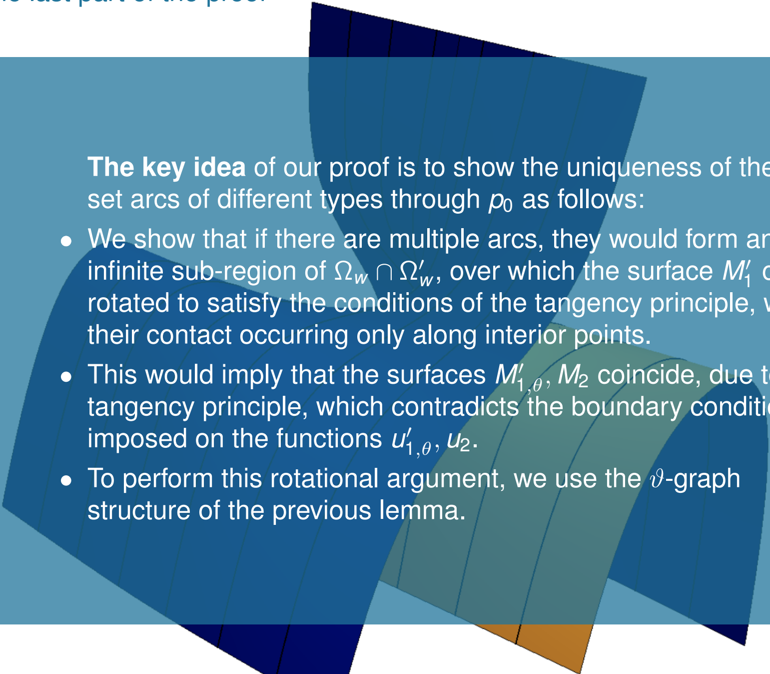

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- This would imply that the surfaces $M'_{1,\theta}, M_2$ coincide, due to the tangency principle, which contradicts the boundary conditions imposed on the functions $u'_{1,\theta}, u_2$.
- To perform this rotational argument, we use the ϑ-graph structure of the previous lemma.

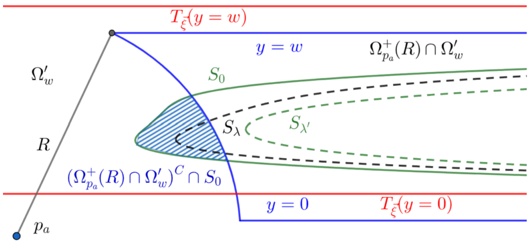

FIGURE 4. Evolution of the level set region. The strips Ω_w, Ω'_w are between the blue and red lines respectively; $\Omega^+_{p_a}(R)$ is contained between the blue curves as before. The shaded region is the one used in the proof; the desired level set here is $S_{\lambda'}$.

The rotation process

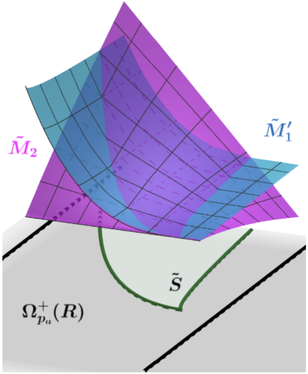

Figure 4: The process of rotating the surface $\tilde{M}'_1$ over the region $\tilde{S}$ until it becomes tangent to $\tilde{M}_2$ at an interior point T.

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¡GRACIAS!