Foliations of null hypersurfaces by surfaces of constant spacetime mean curvature near MOTS

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Mean curvature flow

- The *mean curvature flow* (MCF) is a deformation of submanifolds within a semi-Riemannian M^{n+k} ambient space:

$$\partial_t x = \vec{H},$$

where $x\colon [0, T^*) \times \Sigma^n \to M^{n+k}$ and $\vec{H}(t, \cdot)$ is the mean curvature vector of $x(t, \cdot)$.

- For $M^2 = \mathbb{E}^2$ (Euclidean space) and $n = 1$, this flow is called *curve shortening flow*

$$\dot{\gamma} = -\kappa\nu,$$

where ν is the outward pointing normal.

- For a closed embedded initial curve **THIS** happens:

Theorem (Gage/Hamilton, Grayson, mid 80s)

If γ_0 is smooth, the curve shortening flow starting from γ_0 has a unique solution which shrinks to a point.

Solution to the isoperimetric problem in the plane

Theorem

For a smooth domain $\Omega \subset \mathbb{E}^2$ *there holds*

$$\mathcal{I}(\Omega) = L(\partial\Omega)^2 - 4\pi A(\Omega) \geq 0$$

with equality on balls.

Proof.

- Along curve shortening flow, the variations of length and area are (a)

$$\partial_t L(\partial\Omega_t) = -\int_{\gamma_t} \kappa^2 \quad \text{and} \quad \partial_t A(\Omega_t) = -\int_{\gamma_t} \kappa = -2\pi.$$

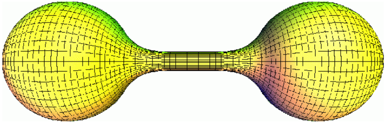
- Gauß-Bonnet and Hölder imply

$$\partial_t (L^2 - 4\pi A) \leq 0,$$

with equality iff $\kappa = \text{const}$.

Mean curvature flow of hypersurfaces

- Similarly, if $M^{n+k} = \mathbb{E}^{n+1}$ we can write the mean curvature flow of hypersurfaces as
$$\dot{x} = -H\nu,$$
where $H = \mathrm{tr}(A)$ is the trace of the Weingarten operator.
- Monotonicity: For $n = 2$,
$$\mathcal{I}(\Omega) = \mathrm{Area}(\partial\Omega)^{\frac{3}{2}} - 6\sqrt{\pi}\,\mathrm{vol}(\Omega)$$
is decreasing along MCF.
- Convergence is more complicated than for $n = 1$.
- Main issue is **Nontrivial singularity formation**:



(www.math.utah.edu/mayer/math/MCF/dumbbell2_js.html)

Convex case

Theorem (Huisken 1984)

Let x_0 be the embedding of the boundary of a convex body in $\mathbb{E}^{n+1}$, then MCF converges to a point and after rescaling to a round sphere.

- Many similar results exists for flow speeds depending on nonlinear functions of the Weingarten operator, e.g.
  - Gauss curvature flow $\dot{x} = -K\nu$ (Andrews in case $n = 2$, Brendle/Choi/Daskalopoulos for $n > 2$)
  - Inverse mean curvature flow $\dot{x} = H^{-1}\nu$ (Gerhardt, Huisken/Ilmanen, Urbas)
  - other general speeds $\dot{x} = \mathcal{F}(x, \nu, A)\nu$ (a lot by Andrews, Langford and many others)
- with many applications to geometry, e.g.
  - Riemannian Penrose inequality (Huisken/Ilmanen)
  - Alexandrov-Fenchel inequalities (Guan/Li, Wang/Xia etc.)

Lorentz manifolds I

- A Lorentz manifold (M, g) is a smooth manifold with a non-degenerate metric tensor of signature one.
- Note that in a Lorentzian manifold, hypersurfaces can be
  1) Riemannian (then called *spacelike*)
  2) Lorentzian (i.e. induced metric non-degenerate but not Riemannian)
  3) null (induced metric degenerate at every point).
- A Riemannian hypersurface of $\mathbb{M}^{n+1}_1$ must be a graph over (a subset of) $\mathbb{E}^n$.
  - With a pretty well-controlled gradient.

Lorentz manifolds II

- Lorentzian (sub)-manifolds in many ways behave like Riemannian manifolds. For example:
  - There is a Levi-Connection ∇ with Christoffel symbols given by

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma})$$

  - If N is a non-degenerate hypersurface of a Lorentzian manifold $(\bar{M}, \bar{g})$, then we can decompose:

$$\bar{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y),$$

    where II is the second fundamental form of N and ∇ its Levi-Civita connection.
  - The 3-tensor $\mathrm{II}(X, Y)$ can be written as

$$\mathrm{II}\,(X, Y) = -g(\nu, \nu) h(X, Y) \nu,$$

    where $h \in T^{0,2}(N)$ and ν is a unit normal vector field along N.

MCF in Lorentz manifolds

- Hence MCF in Lorentz manifolds looks familiar:

$$\dot{x} = H\nu,$$

where $x\colon [0, T^*) \times \Sigma^n \to \bar{M}$ is a family of SPACELIKE embeddings ("spacelike" to ensure parabolicity of the equation).

  - Nice feature: The natural "spacelike" condition gives graphicality, and the gradient estimates (if available) endure that spacelikeness is preserved.

- Previous results for spacelike mean curvature flow in Lorentz spaces for example by Ecker, Ecker/Huisken, Gerhardt, Lambert/Lotay.

MCF in Lorentz manifolds II

Theorem (Lambert/Lotay, 2021)

If M_0 is an entire spacelike graph in the Minkowski space M_m^{n+m} (i.e. m timelike directions), then there exists a smooth spacelike solution to MCF, which exists for all $t > 0$.

Null hypersurfaces

- A hypersurface $\mathcal{N}$ of a Lorentz manifold is called *null*, if its induced metric g is everywhere degenerate.
  - By definition, at every point $x \in \mathcal{N}$:

    $$\exists 0 \neq \bar{L} \in T_x\mathcal{N} \colon \ \ker g(\bar{L}, \cdot) = T_x\mathcal{N},$$

    i.e. $\bar{L}$ annihilates the whole tangent space and hence:
  - Any normal vector field to $\mathcal{N}$ is a tangent to $\mathcal{N}$ and
  - $g(\bar{L}, \bar{L}) = 0$ implies that there are no **unit** normals to $\mathcal{N}$.
- **Observation 1**: $\mathcal{N}$ does not have an induced Levi-Civita connection and hence there is no Gaussian formula for a surface $\Sigma \subset \mathcal{N}$.
- **Observation 2:** For every hypersurface $\Sigma \subset \mathcal{N}$, $\bar{L}$ annihilates $T_x\Sigma$ for every $x \in \Sigma$. Hence normals to Σ only depend on the position $x \in \Sigma$, not on the slope.
- **Observation 3:** No way to decompose the mean curvature vector of a spacelike surface in M in the form $\vec{H} = H\nu$ for some unit vector ν.

Mean curvature flow in null hypersurfaces

- What is a good way to define MCF in a null hypersurface?
- Here is an idea from the Riemannian case:
  - Suppose we have a surface $(\Sigma^n, \tilde{g})$, isometrically sitting in a hypersurface (N^{n+1}, g) of some Riemannian ambient space $(M^{n+2}, \bar{g})$.
  - Taking the Gaussian formula twice gives for $X, Y \in T_x\Sigma$:

$$\begin{aligned}\bar{D}_X Y = D_X Y + \bar{h}(X, Y)\bar{\nu} &= \nabla_X Y + h(X, Y)\nu + \bar{h}(X, Y)\bar{\nu} \\ &\equiv \nabla_X Y + \mathrm{II}_{\Sigma \subset M}.\end{aligned}$$

  - From here we see

$$\mathrm{II}_{\Sigma \subset N} = \mathrm{pr}_N(\mathrm{II}_{\Sigma \subset M}),$$

    where pr is just the standard orthogonal projection.

Mean curvature vector in null hypersurfaces

- Let $\Sigma^n \subset \mathcal{N}^{n+1} \subset M^{n+2}$, where M is Lorentzian and $\mathcal{N}$ is a null hypersurface, be spacelike with induced (from (M, g)) Levi-Civita connection ∇:
  - $D_X Y = \nabla_X Y + \mathrm{II}(X, Y)$.
- For our global null vector field $\bar{L} \in T\mathcal{N} \subset T\Sigma^\perp$, let us define a *null partner* L_Σ with the properties

$$T\Sigma^\perp = \mathrm{span}(\bar{L}, L_\Sigma), \quad g(\bar{L}, L_\Sigma) = 2, \quad g(L_\Sigma, L_\Sigma) = 0.$$

  - $D_X Y = \nabla_X Y + \frac{1}{2} g(\mathrm{II}(X, Y), \bar{L}) L_\Sigma + \frac{1}{2} g(\mathrm{II}(X, Y), L_\Sigma) \bar{L}$.
- A reasonable definition of the *mean curvature vector of* $\Sigma \subset \mathcal{N}$ is

$$\vec{H}_{\Sigma \subset \mathcal{N}} = \frac{1}{2} g(\vec{H}, L_\Sigma) \bar{L}$$

  - This definition does not depend on the choice of $\bar{L}$, as any rescaling will also adjust L_Σ.

Mean curvature flow in null hypersurfaces

Definition

A family of embeddings $x\colon (0,T)\times\Sigma\to\mathcal{N}$, where $\mathcal{N}$ is a null hypersurface sitting in a Lorentzian manifold (M,g) is said to move by mean curvature flow, if

$$\partial_t x = \frac{1}{2} g(\vec{H}, L_\Sigma)\bar{L}.$$

- Comparison to the *null mean curvature flow* of Theodora Bourni and Kristen Moore:
  - Here Σ is sitting in a spacelike hypersurface N (initial data set) with normal ν.
  - They consider the flow

$$\partial_t x = g(\vec{H}, L_\Sigma)\nu$$

within the Riemannian submanifold N.

Comparison to the Riemann/Lorentz case

- Crucial feature of MCF in null geometries: The flow **direction** only depends on the position!
  - If the flowing surfaces are given as graphs over Σ_0, $\Sigma_t = \{(\omega(t,z), z)\colon z \in \Sigma_0\}$, then MCF reads

$$\partial_t \omega = -\frac{1}{2} \operatorname{tr} h,$$

    where h is the second fundamental form of Σ_t, i.e.

$$h(X,Y) = -g(\mathrm{II}(X,Y), L_{\Sigma_t}) = g(D_X L_{\Sigma_t}, Y)$$

  - Compare to graphical MCF in non-degenerate spaces:

$$\partial_t \omega = -Hv, \quad v^2 = 1 \pm |\nabla \omega|^2.$$

  - This helps on the PDE side.
- On the geometry side, things are more complicated as we pick up torsion from the codimension 2 ambient space.

MOTS

- In the above situation, a spacelike surface $\Sigma^n \subset \mathcal{N}$ is called *outer trapped*, if

$$g(L_\Sigma, \vec{H}) > 0$$

  and *outer untrapped*, if the reverse inequality holds.
- Σ is called *marginally outer trapped surface* (MOTS), if

$$g(L_\Sigma, \vec{H}) = 0.$$

- **Caution**: These definitions apply more generally to spacelike surfaces, without any reference to a null hypersurface.
  - In this more general setting, a MOTS is simply a multiple of *one* of the vectors in the null unit pair.

Some more notation

- G is the Einstein tensor of (M, g), $G = \mathrm{Rc} - \frac{1}{2}\,\mathrm{R}\, g$,
- $\bar{\chi}$ is the second fundamental form of the null cone $\mathcal{N}$,

$$\bar{\chi}(X, Y) = -g(\mathrm{II}(X, Y), \bar{L}) = g(D_X \bar{L}, Y).$$

  - Note that for all $\lambda \in \mathbb{R}$ we have

$$\bar{\chi}(X + \lambda\bar{L}, Y) = \bar{\chi}(X, Y) \quad \text{and} \quad g(X + \lambda\bar{L}, Y) = g(X, Y)$$

    and hence $\bar{\chi}$ is determined the restriction to any spacelike $\Sigma \subset \mathcal{N}$.
  - Thus the function $\mathrm{tr}\,\bar{\chi} := \mathrm{tr}_\Sigma\,\bar{\chi}$ is independent of any spacelike $\Sigma \subset \mathcal{N}$.
- $\hat{\bar{\chi}}$ is the traceless part with respect to the foliation by s-slices,

$$\hat{\bar{\chi}} = \bar{\chi} - \frac{1}{2}\,\mathrm{tr}\,\bar{\chi}\gamma_s,$$

  - Here s is the ODE-flow parameter of the vector field $\bar{L}$.
- $\alpha(X, Y) = g(R_{\bar{L}X}\bar{L}, Y)$.

Convergence to a MOTS

Theorem (with Henri Roesch, 2021)

Let (M, g) be a 4-dimensional, time-oriented Lorentzian manifold, $\Sigma_0 \subset M$ a weakly outer trapped two-sphere with respect to a future directed null normal section L_{Σ_0}, and let $\mathcal{N}$ be the null hypersurface generated by the past directed null partner $\bar{\ell}$ of L_{Σ_0}. Now consider $\bar{L} = a\bar{\ell}$, for $a \in C^\infty(\mathcal{N})$, $a > 0$, satisfying the gauge condition:

$$G(\bar{L}, \bar{L}) - d(2\kappa - \operatorname{tr}\bar{\chi})(\bar{L}) \geq |(\operatorname{tr}\bar{\chi} - 4\kappa)\hat{\bar{\chi}}| + 2|\hat{\bar{\alpha}}| + \frac{5}{2}|\hat{\bar{\chi}}|^2,$$

where $\kappa = da(\bar{\ell})$. Then, if the null hypersurface $\Omega \subset \mathcal{N}$ generated by $\bar{L}$ and Σ_0 admits an outer un-trapped cross-section Σ_{ω_0}, the MCF

$$\dot{x} = \frac{1}{2} g(\vec{H}, L_{\omega_t})\bar{L}$$

from $\Sigma_{\omega_0} \subset \Omega$ exists for all times and converges smoothly to a MOTS.

Proof elements - Idea standard, estimates complicated

- As we have seen earlier, our graphical MCF is

$$\partial_t\omega = -\frac{1}{2}\operatorname{tr} h = \Delta\omega + 2\zeta(\nabla\omega) - \frac{1}{2}\operatorname{tr}\chi - (\frac{1}{2}\operatorname{tr}\bar{\chi} + \kappa)|\nabla\omega|^2.$$

  - We observe a term ζ coming from the torsion.
  - χ is the 2^{nd} fundamental form of the background s-slices.
- C^0-estimates/Barriers are easy: As on Σ_0 we have, by assumption, $\operatorname{tr}\chi \leq 0$ and Σ_{ω_t} has $\operatorname{tr}\chi_{\omega_t} > 0$, we obtain monotonicity of the flow.
- C^1-estimates are complicated, because the evolution of $u := \frac{1}{2}|\nabla\omega|^2$ is a mess.
  - That's where most of our assumptions on the ambient geometry come into play.
  - The resulting highest order term has a $\kappa = da(\bar{\ell})$ coefficient, which we can force to be negative, however
  - possibly only in small regions around the lower barrier.

Elements of the proof - Now standard again

- With the C^1-estimates and the quasi-linear parabolic equation, we get C^k-estimates for any k by standard quasi-linear PDE theory.

Theorem

Under the assumptions of the main theorem, the mean curvature flow starting from any embedded spacelike outer un-trapped surface Σ_{ω_0} exists for all times and satisfies uniform estimates in any $C^k(\Sigma_0)$ norm.

- Due to the monotonicity of the graphs, the pointwise limit exists,

$$\omega_\infty = \lim_{t\to\infty} \omega(t, \cdot).$$

Hence the flow speed must be integrable in time over $[0, \infty)$ and thus converges to zero, i.e. the limit is a MOTS.

Evolution of induced metric

- If $\mathcal{N}$ is the standard lightcone of Minkowski space, then the contracted Gauss equation is

$$R = |\vec{H}|^2 - g(h, \bar{\chi}) = \frac{1}{2}|\vec{H}|^2,$$

because the lightcone is totally umbilic, $\bar{\chi} = \frac{1}{2}\,\mathrm{tr}\,\bar{\chi}\bar{\gamma} = \frac{1}{2}\,\mathrm{tr}\,\bar{\chi}g$.

- Markus Wolff 2023: The metric evolves by

$$\partial_t g_{ij} = -H\bar{\chi}_{ij} = -Rg_{ij}.$$

  - Lead to a new proof of Hamilton's early 2d-Ricci flow result.

Pre-Foliation

Corollary (of the MOTS theorem)

Under the assumptions of the MOTS theorem, there exists a foliation of a one-sided neighbourhood of the MOTS by outer untrapped surfaces.

Proof.

The null mean curvature flow initiated at the outer un-trapped upper barrier preserves this property and converges monotonically to the MOTS. Hence it foliates a one-sided neighbourhood. □

The aim is now to find conditions under which we can actually foliate by surfaces of **constant** spacetime mean curvature $|\vec{H}|^2$.

Foliations

- Modification of MCF equation to

$$\dot{x} = \left(\frac{c}{\operatorname{tr} \bar{\chi}} - H \right) \bar{L}$$

  leads to
  - the detection of surfaces with

$$|\vec{H}|^2 = H \operatorname{tr} \bar{\chi} = c,$$

  - provided we can make the flow converge.
  - The latter is again possible under barrier and ambient assumptions.
- Then, if the mean curvature is (locally) monotone with respect to graph ordering, one gets **uniqueness and continuity with respect to** c of $(|\vec{H}|^2 = c)$-surfaces. We call this property **monotone mean curvature property.**

Interlude: Foliations of Lorentzian manifolds

- In earlier contexts, the monotone mean curvature property played a similar role.
- Suppose a spacetime is given as a product $[T, \infty) \times \mathcal{S}_0$, where $\mathcal{S}_0$ is compact Riemannian and the metric splits

$$\bar{g} = -d\tau^2 + \sigma_{\mathcal{S}_0}.$$

- It can be computed that the mean curvature of the time slices evolves by

$$\partial_t H = |A|^2 + \overline{\mathrm{Rc}}(\partial_\tau, \partial_\tau) \geq \frac{1}{n} H^2 + \overline{\mathrm{Rc}}(\partial_\tau, \partial_\tau).$$

  - Hence under a natural *timelike convergence condition*,

$$\overline{\mathrm{Rc}}(V, V) \geq -\Lambda \bar{g}(V, V)$$

    for all timelike vectors, it is clear that the mean curvature increases within the region $\{H > \sqrt{n\Lambda}\}$.

Interlude: Foliations of Lorentzian manifolds II

Theorem (Gerhardt, $\sim$ 2000)

Provided that within the Lorentzian manifold

$$\bar{M} = [T, \infty) \times \mathcal{S}_0, \quad \bar{g} = -d\tau^2 + \sigma_{\mathcal{S}_0}$$

there exist spacelike graphs with sufficiently large mean curvature, a future end F can be foliated by surfaces of constant mean curvature H. This gives rise to a function H on F, which can be used as a new time function.

Foliation near MOTS

Theorem (Sketch, with Wilhelm Klingenberg and Ben Lambert, in progress)

Let Σ_0 be a MOTS in a null hypersurface $\mathcal{N}$ and suppose $\mathcal{N}$ has the monotone mean curvature property. Then there exists a neighbourhood Ω of Σ_0, such that Ω is foliated by surfaces of constant $|H|^2$-curvature

Main steps of proof:

- Pick a surface Σ to the future of the MOTS. Then for every

$$0 < c < \inf_{\Sigma} |\vec{H}|^2$$

  run $\dot{x} = (c/\operatorname{tr}\bar{\chi} - H)\bar{L}$ with initial surface Σ.
- This gives a c-parameter family of surface with constant curvature.
- By the continuity of graphs with respect to c, this foliates a region.
- By the implicit function theorem, this foliation is smooth.

Muchas gracias!