Why are Zoll metrics interesting?

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IMPA

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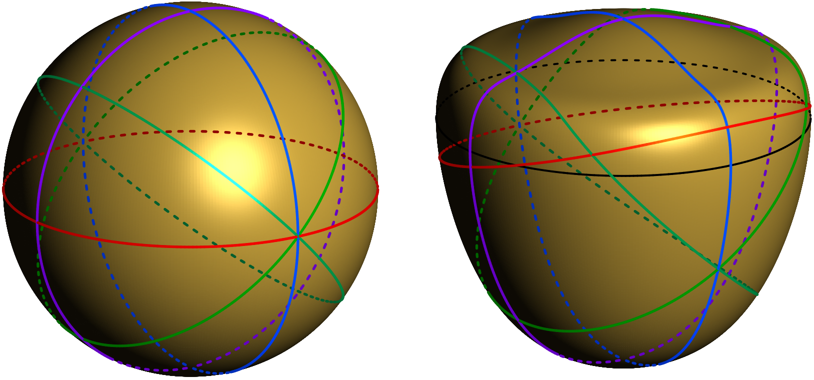
... just like the Euclidean sphere, *all of their unit-speed geodesics are periodic, simple and have the same length!*

Zoll's surfaces



Pictures by Mario Schulz.

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... and still several interesting open problems about them!

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In this talk, we will discuss two of these properties, regarding

- systoles vs area,

- Lusternik-Schnirelmann theory.

And we will see how they can inspire meaningful analogies in other variational theories.

Systoles of Riemannian 2-spheres

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Remark: Best upper bound so far: $4\sqrt{2}$ (R. Rotman, 2006).

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Theorem (Weinstein, 1974)

If g is a Zoll metric on S^2, then $\frac{sys(S^2,g)}{area(S^2,g)^{\frac{1}{2}}} = \sqrt{\pi} \sim 1.772.$

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Theorem (A. Abbondandolo et al., 2018)

If g_z is a Zoll metric on the 2-sphere, then there exists a C^3-neighbourhood $\mathcal{U}$ of g_z such that

$$\frac{sys(S^2,g)}{area(S^2,g)^{\frac{1}{2}}} \leq \sqrt{\pi} \quad \textit{for every} \quad g \in \mathcal{U},$$

and equality holds for $g \in \mathcal{U}$ if and only if g Zoll.

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Theorem (A. Abbondandolo et al., 2021)

If Σ is a sphere of revolution in $\mathbb{R}^3$, then

$$\frac{sys(\Sigma^2, g)}{area(\Sigma^2, g)^{\frac{1}{2}}} \leq \sqrt{\pi} \quad \textit{for every} \quad g \in \mathcal{U}$$

and equality holds for if and only if Σ is a Zoll sphere of revolution in $\mathbb{R}^3$.

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which are detected by a min-max procedure for i-parameter families of embedded circles, $i = 1, 2, 3$,...

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Let (S^2, g) be a Riemannian two-sphere.

i) If $\omega_1(S^2, g) = \omega_2(S^2, g)$ or $\omega_2(S^2, g) = \omega_3(S^2, g)$, then there exists a periodic simple geodesic of length $\omega_2(S^2, g)$ through every point of S^2.

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ii) $\omega_1(S^2, g) = \omega_3(S^2, g)$ *if, and only if, g is a Zoll metric.*

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Embedded minimal spheres
= embedded spheres S^n, $n \geq 2$, with zero mean curvature.
= embedded sphere S^n, $n \geq 2$, that is critical point of the area functional

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It then makes sense to consider the "spherical systole":

$$\mathcal{S}(S^3, g) = \inf \{ \mathit{area}(\Sigma, g) \mid \Sigma \text{ embedded minimal 2-sphere in } (S^3, g) \} > 0.$$

Systolic freedom

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... *even among Berger metrics with sec* > 0.

A sharp inequality

Analogies with eigenvalues suggest to look for estimates inside conformal classes $[g_0] = \{g = e^{2f} g_0 \,|\, f \in C^\infty(S^3)\}$ of metrics.

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If (S^3, g) is conformally flat and has positive Ricci curvature, then

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Proof: study how $\mathcal{S}(S^3, g)$ varies along a the volume-preserving Yamabe Flow $g_t \in [can]$ and use the preserved condition $Ric_{g_t} > 0$ to guarantee $\mathcal{S}(S^3, g_t)$ is realised by an index one minimal sphere.

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$$\int_{S^3} f\, dV_g = \lim_{k \to +\infty} \frac{1}{k} \sum_{i=1}^{k} \int_{\Sigma_i} f\, dV_g \text{ for all } f \in C^0(S^3).$$

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Examples??? All homogeneous metrics on S^3 satisfy the conclusion of the above theorem...

Further questions to be investigated

-) Could analogues of Zoll metrics play a role in the study of

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-) Are there analogues of Zoll metrics in the theory of minimal $(n-1)$-spheres in Riemannian n-spheres, that are as abundant and interesting as Zoll metrics on two-spheres?

Zoll families of $(n-1)$-spheres - I

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Higher dimensional model: (S^n, can) and family of totally geodesic equators

$$\Sigma_\sigma = \sigma^\perp \cap S^n, \quad \sigma \in \mathbb{RP}^n.$$

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Assumption 1:
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Assumption 2:
The assignment $(p,\pi) \mapsto \Sigma_\sigma$ is smooth (in graphical sense).

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$$\vec{\mathcal{H}}(g,\{\Sigma_\sigma\}) \ : \ (p,\pi)\in Gr_{n-1}(S^n) \ \mapsto \ \vec{H}_g^{\Sigma_\sigma}(p)\in T_pS^n.$$

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Remark: If $\vec{H}(g,\{\Sigma_\sigma\})\equiv 0$, then all Σ_σ have the same area.

A new Zoll-like condition and a new problem

Find and understand geometry of solutions to

$$\vec{\mathcal{H}}(g, \{\Sigma_\sigma\}) \equiv 0.$$

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  (Generalises Gullemin's result on $n = 2$).
  Remark: possible to perturb inside $[can]$.
- Some examples have trivial isometry group, arbitrarily close to *can*, and inside $[can]$!
  (Answer to a question by Yau).

Back to Otto Zoll's original construction?

Are there n-spheres of revolution in $\mathbb{R}^{n+1}$ that contain Zoll families of minimal $(n-1)$-spheres, for all $n \geq 3$?

Summary

Riemannian 3-spheres with Zoll families of minimal 2-spheres do not necessarily maximise

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... they are very good candidates, and also abundant, curious geometric objects that deserve to be investigated further.

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Consider the bounded functional

$$\rho : \{x, y\} \in \mathcal{P}/\Delta \mapsto d_g(x, y) \in [0, +\infty).$$

LS theory for distance functions - I

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If ρ is smooth (away from $[\Delta]$), LS theory finds two critical values for ρ,

$$0 < \mathcal{S}(\Gamma) \leq diam(\Gamma),$$

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Remark: in this case non-trivial critical point of ρ if and only if the minimising geodesic joining them is orthogonal to Γ.

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iii) There are arcs $C_t \subset \Gamma$ with $C_0 = \{p_0\}$, $C_1 = \Gamma$ and C_t with extremities $\{p_t, q_t\}$ such that $t \mapsto C_t$ is continuous.

LS theory for distance functions - II

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If $\Gamma = \partial\Omega$, Ω totally convex embedded 2-disc, and natural geometric extra assumptions, we can also characterise minimising geodesics with extremities at a critical point of ρ at the level $\mathcal{S}(\Gamma)$.

Regular and critical points of ρ

Definition

A point $\{p, q\} \in \mathcal{P}$ is a regular point of ρ when there exists

$$(v, w) \in T_p\Gamma \times T_q\Gamma$$

such that, for every minimising geodesic γ joining p and q,

$$\langle w, \nu_\gamma(q)\rangle + \langle v, \nu_\gamma(p)\rangle < 0.$$

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Motivation: first variation formula of length for γ.

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... then $\mathcal{S}(\Gamma) = diam(\Gamma) \Leftrightarrow$ Γ is a curve of constant width.

This theory thus suggests a meaningful generalisation of this classical notion to arbitrary geometries.

Thank you!