Cap 1 Davenport

$\boxed{q > 2 \text{ primo} \quad q \nmid a}$

Euler $\zeta(s) = \sum n^{-s} = \prod_p (1-p^{-s})^{-1}$

$f(n)$ $\qquad f(p)$

log ... $\rightsquigarrow \sum \frac{1}{p}$ diverge.

IDEA
$$\sum_{p \equiv a \, (q)} \frac{1}{p}$$

Caracteres de Dirichlet

$\to f : \mathbb{N} \to \mathbb{C}$
- periódica, periodo q
- multiplicativa

$\to$ la función característica de $\{a+qx\}$ es CL de caracteres de Dirichlet

Sea g una raíz primitiva módulo q

i.e. $\forall n : q \nmid n \rightsquigarrow \exists \nu(n) \in \mathbb{Z} : \quad n \equiv g^{\nu(n)} \ (q)$

elegimos $\omega \in \mathbb{C}$ tq $\omega^{q-1} = 1$ (hay $q-1$ opciones)

$$f_\omega(n) = \begin{cases} \omega^{\nu(n)} & \text{si } q \nmid n \\ 0 & \text{si } q \mid n \end{cases}$$

ejercicio f_w es un carácter de Dirichlet (mód q)

único tq $f(g) = w$

hay $q-1$ C.D. mód q.

Af ① $\sum_w w^k = \begin{cases} q-1 & \text{si } q-1 \mid k \\ 0 & \text{si } q-1 \nmid k \end{cases}$

② $\frac{1}{q-1} \sum_w \underbrace{\overline{f_w(a)}}_{w^{-\nu(a)}} f_w(n) = \begin{cases} 1 & \text{si } a \equiv n\,(q) \\ 0 & \text{si } a \not\equiv n\,(q) \end{cases}$

Ejercicio

$$L_w(s) := \sum_{\substack{n=1 \\ (q \nmid n)}}^{\infty} f_w(n)\, n^{-s}$$

conv. abs. $s > 1$

Euler

$$= \prod_{\substack{p \\ (p \neq q)}} \left(1 - f_w(p)\, p^{-s}\right)^{-1}$$

$$\log L_w(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} f_w(p^m)\, p^{-ms}$$

② $$\frac{1}{q-1} \sum_w w^{-\nu(a)} \log L_w(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} \underbrace{\left(\frac{1}{q-1} \sum_w w^{-\nu(a)} f_w(p^m)\right)}_{\begin{cases} 1 & p^m \equiv a \\ 0 & p^m \not\equiv a \end{cases}} p^{-ms}$$

$$\frac{1}{q-1}\sum_{\omega}\overline{\omega^{-\nu(a)}}\,\underbrace{\log L_\omega(s)}_{**} = \sum_{p}\sum_{\substack{m=1\\ p^m\equiv a(q)}}^{\infty} m^{-1}p^{-ms} =$$

$$= \underbrace{\sum_{p\equiv a(q)} p^{-s}}\; + \underbrace{\left(\sum_{\substack{p,\,m\geq 2\\ p^m\equiv a}} \cdots\right)}_{=O(1)}$$

BASTA PROBAR $(**) \to +\infty$

$\boxed{A}$ $\omega=1$ $\quad L_1(s) = \sum_{\substack{n=1\\ q\nmid n}}^{\infty} n^{-s} = (1-q^{-s})\,\zeta(s) \longrightarrow +\infty$

$\boxed{B}$ $\omega\neq 1 \rightsquigarrow \log L_\omega(s)$ acotado $(s\to 1^+)$?

Si probamos (B) ✓

$$\overline{L_\omega(s) = \sum_n f_\omega(n)\cdot n^{-s} \qquad (\omega\neq 1)}$$

① $n^{-s} \longrightarrow 0 \qquad (s>0)$

② $\sum_{n=1}^{x} f_\omega(n)$ acotada

$$\left(\sum_{n=1}^{q-1}\omega^{\nu(n)} = \sum_{m=0}^{q-2}\omega^m = 0\right)$$

Test de Dirichlet

$\Longrightarrow L_\omega(s)$ converge $\quad s>0$

uniforme $\quad s \geq \delta > 0$

Si $s \to 1^+$ $\quad L_\omega(s) \longrightarrow L_\omega(1) \in \mathbb{C}$

Precisamos $\boxed{L_\omega(1) \neq 0} \quad \forall\, \omega \neq 1$

$\omega \in \mathbb{R} \longrightarrow \omega = -1$

$\omega \notin \mathbb{R} \longrightarrow \omega \neq \overline{\omega}$

Caso 1 $\quad \omega \notin \mathbb{R}$

$$(**)\quad (a=1)\quad \frac{1}{q-1}\sum_\omega \log L_\omega(s) = \sum_p \sum_{\substack{m \\ p^m \equiv 1\,(q)}} m^{-1} p^{-ms} \geq 0$$

$$\boxed{\prod_\omega L_\omega(s) \geq 1 \quad \forall\, s>1}$$

Sup $L_\omega(1) = 0 \Rightarrow L_{\overline{\omega}}(1) = 0 \Big\} \longrightarrow \prod L_\omega \longrightarrow 0 \;\lightning$

probar

$1 < s < 2$

(i) $L_1(s) = (1-q^{-s})\zeta(s) < (1-q^{-2})\zeta(s)$

(ii) $\zeta(s) = \sum n^{-s} < 1 + \int_1^\infty x^{-s}\,dx = \dfrac{s}{s-1}$

$$\boxed{L_1(s) < \frac{A}{s-1}}$$

(iii) $L_w(1) = 0 \Rightarrow$ $\qquad 1 < s_1 < s$

$$L_w(s) = L_w(s) - L_w(1) = (s-1)\,L'_w(s_1)$$

$$L'_w(s_1) = -\sum_{n=1}^{\infty} \underbrace{f_w(n)}_{\text{sumas acotadas}}\ \underbrace{\log n \cdot n^{-s_1}}_{\to 0}$$

$\hookrightarrow$ Dirichlet $\quad L'_w(s)$ converge unif $\quad s \geq \delta > 0$

$\Rightarrow |L'_w(s_1)|$ está acotada

$\Rightarrow |L_w(s)| < A_1(s-1)$

$$|L_1(s)\,L_w(s)\,L_{\bar{w}}(s)| < A_2\,(s-1)$$

$$\Rightarrow \Big| \underbrace{\prod_{w} L_w(s)}_{\geq 1} \Big| < A_3 (s-1) \to 0 \quad \text{↯}$$

Caso 2 $\quad w = -1$

Símbolo de Legendre $\quad n \equiv g^{v(n)}$

$$f_w(n) = \begin{cases} (-1)^{v(n)} \\ 0 \end{cases} = \left(\frac{n}{q}\right) = \begin{cases} +1 & n \equiv \square \quad (\not\equiv 0) \\ -1 & n \not\equiv \square \\ 0 & n \equiv 0 \end{cases}$$

$$\boxed{L(s) = \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) n^{-s}}$$

si $s > 1 \Rightarrow L(s) > 0$ (ejercicio)

$\rightsquigarrow L(1) \geq 0$

Obs: si $L(1) = 0 \overset{\log}{\rightsquigarrow} \sum_{p} \left(\frac{p}{q}\right) p^{-s} \to -\infty$

$\rightsquigarrow$ gran preponderancia de primos $\not\equiv \square \ (q)$

Prop $\quad q \equiv 3 \ (4) \Rightarrow$

$$L(1) = -\frac{\pi}{q^{3/2}} \underbrace{\sum_{m=1}^{q-1} m \left(\frac{m}{q}\right)}_{\text{impar}}$$

Eg
$q = 23$ $\quad \sum = 1 + 2 + 3 + 4 - 5 + 6 - 7 \cdots - 22 = -69$

$\Rightarrow L(1) = \frac{3\pi}{\sqrt{23}} \neq 0$

En general $\quad \sum_{n=1}^{q-1} n \left(\frac{n}{q}\right) \equiv \sum_{n=1}^{q-1} n = \frac{q(q-1)}{2} \equiv 1 \pmod 2$

Como $q \equiv 3 \ (4)$