L-serie de Dirichlet "real" $\quad q > 2$ primo.

$$(-1)^{v(n)} = \left(\frac{n}{q}\right) = \begin{cases} +1 & n \equiv x^2 \not\equiv 0 \\ 0 & n \equiv 0 \\ -1 & n \not\equiv x^2 \end{cases}$$

$n \equiv g^{2i}$

$n \equiv g^{2i+1}$

$$L(s) = \sum_{n=1}^{\infty} \left(\frac{n}{q}\right) n^{-s}$$

(converge $s > 0$)

(sabemos $L(1) \geq 0$)

Teorema $\quad L(1) \neq 0$

Sumas de Gauss

$\nearrow \left(\frac{n}{q}\right)$ caracter multiplicativo

$\searrow e_q(n) := e^{2\pi i n/q}$ caracter aditivo

$$G(n) := \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \cdot e_q(m \cdot n)$$

Obs: $G(n) = \left(\frac{n}{q}\right) G \quad$ con $G = G(1)$

Teo $\quad G = \begin{cases} q^{1/2} & q \equiv 1 \ (4) \\ i q^{1/2} & q \equiv 3 \ (4) \end{cases}$ $\quad (\neq 0)$ (luego)

$$\left(\frac{n}{q}\right) = \frac{1}{G} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) e_q(mn)$$

$$\rightsquigarrow L(1) = \frac{1}{G} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \underbrace{\sum_{n=1}^{\infty} \frac{1}{n} e_q(mn)}$$

$$L(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\frac{n}{q}\right)$$

Ejercicio $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} e^{in\theta} = -\log\left(2 \operatorname{sen}\left(\tfrac{1}{2}\theta\right)\right) - \frac{1}{2}(\theta-\pi)i$

$0<\theta<2\pi$

$\boxed{\text{Sug: } -\log(1-z) = \displaystyle\sum_{m=1}^{\infty} \frac{1}{m} z^m \quad (\operatorname{Re}(1-z)>0)}$

$z=e^{i\theta}$

$\theta = 2\pi \frac{m}{q} \rightsquigarrow e_q(mn) = e^{in\theta}$

(XXX) $L(1) = -\dfrac{1}{G} \displaystyle\sum_{n=1}^{q-1} \left(\frac{m}{q}\right)\left[\log\left(2\operatorname{sen}\tfrac{1}{2}\theta\right) + \frac{1}{2}(\theta-\pi)i\right]$

$\in \mathbb{R}$

Caso $q \equiv 3\ (4)$ $\quad G = i\, q^{1/2}$

$\left(\sum \left(\frac{m}{q}\right) = 0\right)$

$\rightsquigarrow L(1) = -\dfrac{1}{q^{1/2}} \displaystyle\sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \frac{\pi m}{q}$

$$\boxed{L(1) = \frac{-\pi}{q^{3/2}} \sum_{m=1}^{q-1} m \left(\frac{m}{q}\right)}$$

$\displaystyle\sum m \underbrace{\left(\frac{m}{q}\right)}_{\pm 1} \equiv \sum_{m=1}^{q-1} m = \frac{q(q-1)}{2} \equiv 1 \pmod 2$ $\quad \underline{\neq 0}$

Caso $q \equiv 1\ (4)$ $\quad G = q^{1/2}$

$$L(1) = -\frac{1}{q^{1/2}} \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) \log\left(2\operatorname{sen}\left(\frac{\pi m}{q}\right)\right)$$

$$L(1) = \frac{\log Q}{q^{1/2}} \qquad Q = \frac{\prod_N \operatorname{sen}(\pi N/q)}{\prod_R \operatorname{sen}(\pi R/q)}$$

N = no-residuo cuadrático
R = residuo cuadrático

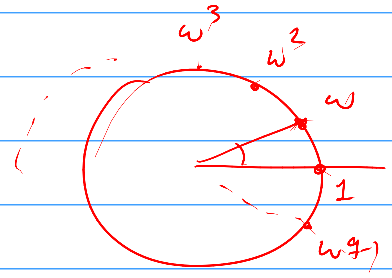
¿#?

Prop

$$\prod_R (x - e_q(R)) = \frac{1}{2}\left[Y(x) - q^{1/2} Z(x)\right]$$

$$\prod_N (x - e_q(N)) = \frac{1}{2}\left[Y(x) + q^{1/2} Z(x)\right]$$

donde $Y(x), Z(x) \in \mathbb{Z}[x]$.



$$\prod_{m=1}^{q-1} (x - e_q(m)) = x^{q-1} + x^{q-2} + \dots + x + 1$$

$e_q(m) = e^{2\pi i m/q} = w^m$ $\qquad w = e^{2\pi i/q}$ es una raíz primitiva q-ésima de 1

$$(w^q = 1)$$

$$(w-1)(w^{q-1} + \dots + 1) = w^q - 1 = 0$$

$$\hookrightarrow \frac{1}{4}\left[Y(x)^2 - q\, Z(x)^2\right] = x^{q-1} + \dots + 1$$

$(x=1)$ $\hookrightarrow$ $Y^2 - qZ^2 = 4q$ $\qquad Y = Y(1)$, $Z = Z(1)$

$\boxed{\text{Obs: } Z \neq 0}$

$$\prod_R \left(1 - e_q(R)\right) = \prod_R e_q(\tfrac{1}{2}R)\left(e_q(-\tfrac{1}{2}R) - e_q(\tfrac{1}{2}R)\right)$$

$$= \prod_R e_q(\tfrac{1}{2}R)\left(-2i \operatorname{sen}\left(\frac{\pi R}{q}\right)\right)$$

$$= \underbrace{(-2i)^{\frac{q-1}{2}} e_q\left(\tfrac{1}{2}\Sigma R\right)}_{\overbrace{\quad}^{?}} \left[\prod_R \operatorname{sen}\left(\frac{\pi R}{q}\right)\right]$$

$$\prod_N \left(1 - e_q(N)\right) = \cdots = (-2i)^{\frac{q-1}{2}} e_q\left(\tfrac{1}{2}\Sigma N\right)\left[\prod_N \operatorname{sen}\left(\frac{\pi N}{q}\right)\right]$$

Ejercicio $\quad \sum R = \sum N = \frac{1}{4} q(q-1) \qquad$ Sug: $\left(\frac{m}{q}\right) = \left(\frac{q-m}{q}\right)$

$$Q = \frac{\prod_N \operatorname{sen}\frac{\pi N}{q}}{\prod_R \operatorname{sen}\frac{\pi R}{q}} = \frac{\prod\left(1 - e_q(N)\right)}{\prod\left(1 - e_q(R)\right)} = \frac{\varphi + q^{1/2} Z}{\varphi - q^{1/2} Z} \neq 1$$

Ejercicio ① Probar $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{q}\right) x^n = \frac{x f(x)}{1 - x^q} \qquad f(x) \in \mathbb{Z}[x]$

② $\displaystyle\Gamma(s) n^{-s} = \int_0^1 x^{n-1} \left(\log x^{-1}\right)^{s-1} dx$

$$\boxed{\Gamma(t) = \int_0^\infty x^t e^{-x} \frac{dx}{x}}$$

1+2 $\rightsquigarrow$ $\displaystyle\Gamma(s) L(s) = -\int_0^1 \frac{f(x)}{x^q - 1} \left(\log x^{-1}\right)^{s-1} dx$

③ Integrar por fracciones parciales, concluir $\cancel{X}\cancel{X}\cancel{X}$

$\boxed{2}$ **Suma de Gauss**

$$G = \sum_{m=1}^{q-1} \left(\frac{m}{q}\right) e_q(m)$$

$q > 2$ primo

Prop $G^2 = \begin{cases} q & q \equiv 1 \ (4) \\ -q & q \equiv 3 \ (4) \end{cases}$ $(\Rightarrow G = \pm q^{1/2})$ $(\Rightarrow G = \pm i q^{1/2})$

$$G^2 = \sum_{m_1=1}^{q-1} \sum_{m_2=1}^{q-1} \left(\frac{m_1 m_2}{q}\right) e_q(m_1 + m_2)$$

$m_2 \equiv m_1 n$

$$= \sum_{m_1=1}^{q-1} \sum_{n=1}^{q-1} \left(\frac{m_1^2 n}{q}\right) e_q\left(m_1 (1+n)\right) =$$

$$\sum_{m_1=1}^{q-1} e_q\left(m_1 (1+n)\right) = \begin{cases} q-1 & n \equiv -1 \ (q) \\ -1 & n \not\equiv -1 \ (q) \end{cases}$$

$$= q\left(\frac{-1}{q}\right) - \underbrace{\sum_{n=1}^{q-1} \left(\frac{n}{q}\right)}_{=0}$$

$$G^2 = q\left(\frac{-1}{q}\right) = \begin{cases} q & q \equiv 1 \ (4) \\ -q & q \equiv 3 \ (4) \end{cases}$$

$(G \neq 0)$

G grupo finito

$\chi : G \to \mathbb{C}^\times$ no trivial

$\Rightarrow \sum \chi(g) = 0$

$\parallel$

$\sum \chi(gh) = \chi(h) \sum \chi(g)$