Teorema de Dirichlet caso general $q > 1$

Caracteres de Dirichlet $\chi : \mathbb{Z} \to \mathbb{C}$ mód q

$\{0 \leq n < q : (n,q) = 1\} = (\mathbb{Z}/q)^\times$ $\qquad \chi(n) = 0 \quad (n,q) > 1$

$\overset{\shortparallel}{\left(\chi : U(q) \to \mathbb{C}^\times\right.},$ multiplicativa

① $q = p^\alpha,\ p > 2 \qquad U(q)$ cíclico de orden $\varphi(q) = p^{\alpha-1}(p-1)$

Fijo g raíz primitiva $\quad \omega \in \mathbb{C}^\times \quad \omega^{\varphi(q)} = 1$

$\rightsquigarrow \chi(n) = \omega^{\nu(n)} \qquad (n,q) > 1$

hay $\varphi(q)$ caracteres.

② $q = 2^\alpha \qquad \alpha = 1 \qquad U(2) = \{1\} \qquad \chi(n) = 1$ (n impar)

$\alpha = 2 \qquad U(4) = \{1, 3\} \to \chi(n) = 1 \quad \forall n$ impar.

$\to \chi(n) = \begin{cases} +1 & n \equiv 1\ (4) \\ -1 & n \equiv 3\ (4) \end{cases}$

$C(m) = (\mathbb{Z}/m, +)$

$\alpha \geq 3 \quad U(2^\alpha)$ no es cíclico $\qquad \left(U(8) = \{1, 3, 5, 7\}\right)$

$C(2) \times C(2^{\alpha-2}) \xrightarrow{\sim} U(2^\alpha)$

$\nu \times \nu' \longmapsto (-1)^{\nu} \cdot 5^{\nu'} \equiv n \ (2^\alpha)$

$\left.\begin{array}{l} \omega^2 = 1 \\ (\omega')^{2^{\alpha-2}} = 1 \end{array}\right\} \rightsquigarrow \chi(n) = \omega^{\nu} \cdot (\omega')^{\nu'}$

$\rightsquigarrow$ hay $2 \times 2^{\alpha-2} = \varphi(2^\alpha)$ caracteres.

③ $q = 2^{\alpha} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

$$\left\{\begin{array}{ll} \chi_0 & \text{mod } 2^{\alpha} \\ \chi_1 & \text{mod } p_1^{\alpha_1} \\ \vdots & \\ \chi_r & \text{mod } p_r^{\alpha_r} \end{array}\right.$$

$\hookrightarrow \chi(n) = \chi_0(n)\,\chi_1(n) \cdots \chi_r(n)$

$\rightsquigarrow \# = \varphi(2^{\alpha})\,\varphi(p_1^{\alpha_1}) \cdots \varphi(p_r^{\alpha_r})$

$= \varphi(q) \; . = \# U(q)$.

Obs Forman un grupo abeliano!

neutro $\chi_0(n) = \begin{cases} 1 & (n,q)=1 \\ 0 & \text{———} \end{cases}$

caracter principal.

(G grupo abeliano finito $\quad \hat{G} = \mathrm{Hom}(G, \mathbb{C}^{\times}) = \{\chi : G \to \mathbb{C}^{\times} \text{ mult}\}$

$\Rightarrow \quad \hat{G} \simeq G \quad$ NO canónico)

$w_0^2 = 1 \rightsquigarrow w_0 = e(m_0/2)$

$w_0'^{\,2^{\alpha-2}} = 1 \rightsquigarrow w_0' = e(m_0'/2^{\alpha-2})$

$w_i^{\varphi(p_i^{\alpha_i})} = 1 \longrightarrow w_i = e(m_i/\varphi(p_i^{\alpha_i}))$

$n \equiv (-1)^{\nu} \cdot 5^{\nu'} \quad (2^{\alpha})$

$n \equiv g_i^{\nu_i} \quad (p_i^{\alpha_i})$

$$\hookrightarrow \chi(n) = e\left(\frac{m_0 \nu_0}{2} + \frac{m_0' \nu_0'}{2^{\alpha-2}} + \frac{m_1 \nu_1}{\varphi(p_1^{\alpha_1})} + \cdots + \frac{m_r \nu_r}{\varphi(p_r^{\alpha_r})}\right)$$

$$\chi \longleftrightarrow (m_0, m_0', m_1, \dots, m_r) \quad \Big\} \cong C(2)\times C(2^{\alpha-2}) \times C(\varphi(p_1^{\alpha_1}))\times\cdots\times C(\varphi(p_r^{\alpha_r}))$$

$$n \bmod q \longleftrightarrow (\nu_0, \nu_0', \nu_1, \dots, \nu_r)$$

Prop ① $$\sum_{n\in U(q)} \chi(n) = \sum_{n=0}^{q-1} \chi(n) = \begin{cases} \varphi(q) & \chi = \chi_0 \\ 0 & \text{—} \end{cases}$$

② $$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \varphi(q) & n \equiv 1\ (q) \\ 0 & \text{—} \end{cases}$$

Ejercicio ① directo ② estructura

③ $X : \mathbb{Z} \to \mathbb{C}$ mult, periodo q, $X(n) = 0$ $(n,q)>1$, $X \not\equiv 0$

$\Rightarrow$ es uno de los χ construidos

Res $(c,q)=1 \to \underbrace{\sum_n X(n)\,\overline{\chi}(n)} = \sum_n X(cn)\,\overline{\chi}(cn)$

$$= X(c)\,\overline{\chi}(c) \sum_n X(n)\,\overline{\chi}(n)$$

La suma $=0$ excepto si $X(c) = \chi(c)$ $\forall c \in U(q)$

$$\sum_\chi \chi(m) \underbrace{\sum_n X(n)\,\overline{\chi}(n)} = \sum_n X(n) \sum_\chi \underbrace{\chi(m)\,\overline{\chi}(n)}_{\chi(m/n)}$$

$$= X(m)\cdot \varphi(q) \not\equiv 0$$

$\Rightarrow$ Algún χ cumple $\sum_n X(n)\,\overline{\chi}(n) \neq 0 \Rightarrow X = \chi$

4 Si $(a,q)=1 \Rightarrow$

$$\frac{1}{\varphi(q)}\sum_{\chi}\bar{\chi}(a)\,\chi(n)=\begin{cases}1 & n\equiv a\,(q)\\ 0 & -\end{cases}$$

$\underline{\text{Def}}$ $L(s,\chi)=\sum \chi(n)\,n^{-s}$ $\quad s>1$

Euler $\leadsto$ $L(s,\chi)=\prod_{p\nmid q}\left(1-\chi(p)\,p^{-s}\right)^{-1}$ $\quad s>1$

$(\neq 0 \quad s>1)$

$$\log L(s,\chi)=\sum_{p}\sum_{m} m^{-1}\chi(p^m)\,p^{-ms}$$

$$⊛=\frac{1}{\varphi(q)}\sum_{\chi}\bar{\chi}(a)\log L(s,\chi)=\sum_{p}\sum_{\substack{m\\ p^m\equiv a(q)}} m^{-1}p^{-ms}$$

$s\to 1^+$

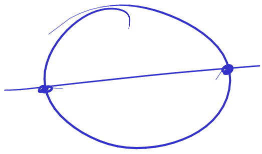
$$=\sum_{p\equiv a}p^{-s}+O(1)$$

Debemos probar que ⊛ $\to +\infty$ cuando $s\to 1^+$

$\boxed{\chi_0}$ $L(s,\chi_0)=\zeta(s)\cdot\underbrace{\prod_{p\mid q}\left(1-p^{-s}\right)}_{\neq 0}\to +\infty$

$\boxed{\chi\neq\chi_0}$ $L(s,\chi)$ continua $s>0$

$\log L(s,\chi)$ acotada $\Longleftrightarrow$ $\underline{L(1,\chi)\neq 0}$



Caso $\chi \neq \overline{\chi}$

mismo argumento: $\prod_{\chi} |L(s,\chi)| \geq 1 \quad s>1$

$\Rightarrow$ A lo sumo un factor podría ser 0. $L(1,\chi)$

Caso real $\chi = \overline{\chi} \neq \chi_0$

Ejercicio: ¿cuántos $\chi = \overline{\chi}$ hay?

$L(1,\chi) \neq 0$

$L(s,\chi)$ variable compleja $s = \sigma + it$

↳ abs convergente $\sigma > 1$, unif en $\sigma \geq \delta > 1$

⇝ $L(s,\chi)$ es holomorfa en $\sigma > 1$

Prop $L(s,\chi)$ tiene continuación analítica a $\sigma > 0$ excepto por un polo simple en $s=1$ para $\chi = \chi_0$

Dem ① $\chi = \chi_0$ lo mismo que para $\zeta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n\left(n^{-s} - (n+1)^{-s}\right) = s\sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s-1}\,dx$$

$\lfloor x \rfloor = x - \{x\}$

$$= s\int_1^{\infty} \lfloor x \rfloor x^{-s-1}\,dx = \frac{s}{s-1} - s\int_1^{\infty} \{x\} x^{-s-1}\,dx$$

$$\int_1^\infty \underbrace{\{x\}}_{\text{acotada}} x^{-s-1}\,dx \quad \begin{cases} \text{abs conv } \sigma>0 \\ \text{unif } \sigma \geq \delta > 0 \end{cases}$$

$\rightarrow$ holomorfa $\sigma > 0$

$\zeta(s)$ polo simple en $s=1$, residuo 1.

$$\boxed{L(s,\chi_0)} = \zeta(s) \cdot \underbrace{\prod_{p|q}(1-p^{-s})}_{>0\ (s=1)}$$

$L(s,\chi_0)$ tiene polo simple en $s=1$, residuo

$$\prod_{p|q}(1-p^{-1}) = \frac{\varphi(q)}{q}.$$

② $\underline{\chi \neq \chi_0}$ $\qquad S(x) = \sum_{n\leq x}\chi(n)$

$$L(s,\chi) = \sum \chi(n) n^{-s} = \sum S(n)\left[n^{-s} - (n+1)^{-s}\right]$$

$$= s\int_1^\infty \underbrace{S(x)}_{\text{acotada}} x^{-s-1}\,dx \qquad \text{es holomorfa en } \sigma > 0.$$

Teo $\chi = \overline{\chi} \neq \chi_0 \Rightarrow L(1,\chi) \neq 0$.

Dem Sup. $L(1,\chi) = 0$

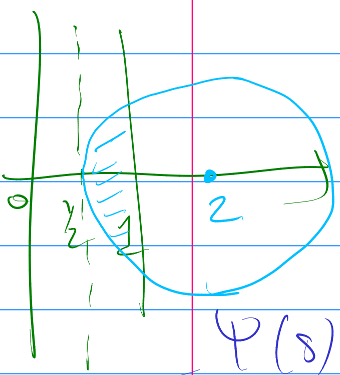
$L(s,\chi)\, L(s,\chi_0)$ holomorfa $\sigma > 0$

$L(2s,\chi_0)$ holomorfa $\sigma > \frac{1}{2}$ (Ejercicio $\neq 0$, $\sigma > 1/2$)

$$\Psi(s) = \frac{L(s,\chi)\, L(s,\chi_0)}{L(2s,\chi_0)} \quad \text{holomorfa } \sigma > 1/2$$

Además $s \to 1/2^+ \Rightarrow \Psi(s) \to 0$

$$\Psi(s) = \prod_{p \nmid q} \frac{(1-\chi(p)p^{-s})^{-1}\,\cancel{(1-p^{-s})^{-1}}}{\underbrace{(1-p^{-2s})^{-1}}_{(1+p^{-s})^{-1}\,\cancel{(1-p^{-s})^{-1}}}}$$



$$\Psi(s) = \prod_{\chi(p)=+1} \frac{(1+p^{-s})}{(1-p^{-s})} = \prod_{\chi(p)=+1} (1+p^{-s})(1+p^{-s}+p^{-2s}+\cdots)$$

$\boxed{\sigma > 1}$ $\rightarrow$ $= \sum a_n n^{-s}$ con $a_n \geq 0$, $a_1 = 1$

Taylor en $s=2$

$$\Psi(s) = \sum_{m=0}^{\infty} \frac{1}{m!} \Psi^{(m)}(2)\,(s-2)^m \quad \Leftarrow \quad |s-2| < 3/2$$

$$\varphi^{(m)}(2) = (-1)^m \sum_n a_n (\log n)^m n^{-2} = (-1)^m b_m$$

$$\boxed{b_m \geq 0}$$

$$\varphi(s) = \sum \frac{1}{m!} b_m (2-s)^m$$

Si $\frac{1}{2} < s < 2$ $\quad \varphi(s) \geq \varphi(e) \geq 1$

$$\left(\varphi(s) \to 0 \text{ cuando } s \to 1/2^+ \right)$$

<u>Teorema</u> $q > 1$, $(a, q) = 1$

Existen ∞ primos $p \equiv a \ (q)$ y

la serie $\sum_{p \equiv a} p^{-1}$ diverge.