6) Fórmula de Dirichlet para el número de clases

Def ① d es un <u>discriminante</u> si $d \equiv 0, 1\ (4)$

$(d \neq \square)$

② un discriminante d es <u>fundamental</u>

$$\begin{cases} d = N, & N \equiv 1\ (4) \text{ libre de } \square \\ \text{ó} & \\ d = 4N, & N \equiv 2,3\ (4) \text{ libre de } \square . \end{cases}$$

$\to$ $d \equiv 0, 1\ (\text{mod } 4) \rightsquigarrow \chi_d(n) = \left(\frac{d}{n}\right)$ car. de Dirichlet módulo $|d|$

Ej. ① χ_d es primitivo $\iff$ d es fundamental

$(p \to \chi_{p^*},\ \chi_{-4}, \chi_8, \chi_{-8})$

② todo car. primitivo real es χ_d.

{car. prim reales}

$\updownarrow$

Ej (Teo Alg. Números)

$\{K/\mathbb{Q} \text{ grado } 2\} \longleftrightarrow \{\text{disc fund}\}$

$K \longmapsto \text{disc}(K)$

$(K = \mathbb{Q}(\sqrt{N})$ $\rightsquigarrow$ $\text{disc}\, K = \begin{cases} N & N \equiv 1\ (4) \\ 4N & N \equiv 2,3\ (4) \end{cases}$

N libre de $\square$

Formas cuadráticas binarias (FCB)

$f = ax^2 + bxy + cy^2$ $\qquad a,b,c \in \mathbb{Z}$

$\text{disc } f = d = b^2 - 4ac$

Ej:

1. $d \equiv 0, 1 \pmod 4$
2. $d = \square \iff f = L_1 \cdot L_2$
3. d es fundamental $\iff$ $\begin{cases} \text{toda FCB de disc } d \\ \text{cumple } (a,b,c) = 1 \end{cases}$

Equivalencia "unimodular"

$$f(x,y) \sim f(\alpha x + \beta y, \gamma x + \delta y)$$

donde $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ i.e. $\begin{cases} \alpha, \beta, \gamma, \delta \in \mathbb{Z} \\ \alpha\delta - \beta\gamma = 1 \end{cases}$

Lagrange: toda clase contiene al menos una forma con $|b| \leq |a| \leq |c|$

Ej: $\#\{\text{clases FCB de disc } d\}$ es finito.

$d < 0 \rightsquigarrow f$ es definida $\begin{cases} \to \text{def} + & a > 0 \\ \to \text{def} - & a < 0 \end{cases}$

$(a,b,c) \longleftrightarrow (-a,-b,-c)$

K/Q cuadrático imaginario

$h(d) = \#\{\text{clases def positivas disc } d\}$

$d>0 \leadsto f$ es indefinida

$\boxed{h(\text{cuad. real})}$ $h(d) = \#\{\text{clases de disc } d\}$

Obs: $h(d) \geq 1$ (si $d \equiv 0, 1\ (4)$)

$$\begin{cases} d \equiv 0\ (4) & x^2 - \frac{d}{4} y^2 \\ d \equiv 1\ (4) & x^2 + xy - \left(\frac{1-d}{4}\right) y^2 \end{cases}$$

Formula de Dirichlet

$h(d) \longleftrightarrow L(1, \chi_d)$

(d fund)

Autometrías

ej: $f(-x,-y) = f(x,y)$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ orden 2

($d=-4$) $f = x^2+y^2 \leadsto f(-y,x) = f(x,y)$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ orden 4

($d=-3$) $f = x^2+xy+y^2 \leadsto f(-y, x+y) = f(x,y)$

$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ orden 6.

Ej: $d<0 \leadsto$ No voy a dar

$$w = w_d = \begin{cases} 2 & d<-4 \\ 4 & d=-4 \\ 6 & d=-3 \end{cases}$$

$= \#$ autometrías de forma de disc d.

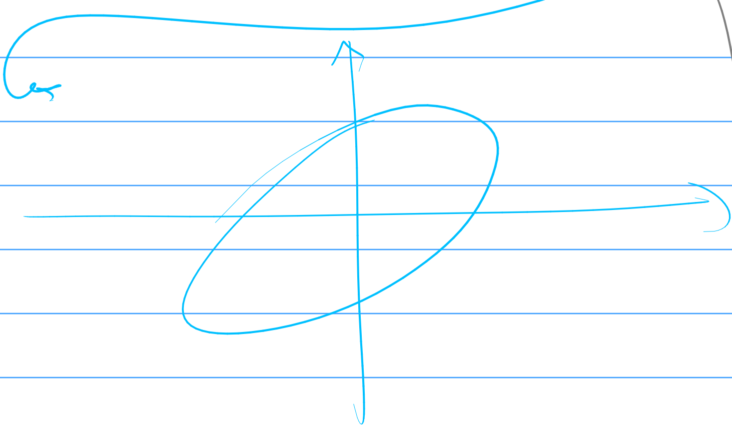
$d>0$: ∞ automorfismos

----- después -----

Finitos
$d<0$
fund

$$R_f(n) = \#\{(x,y)\in\mathbb{Z}^2 : f(x,y)=n\} < \infty$$

Si $f\sim f'$ $\Rightarrow R_f = R_{f'}$



$$R(n) = R_d(n) = \sum_{f\in H(d)} R_f(n).$$

Teo $n>0$ $\quad (n,d)=1 \quad (d<0)$

$$\boxed{R_d(n) = w_d \sum_{m\mid n} \left(\frac{d}{m}\right)}.$$

Rem ... $R(n) \overset{(w)}{\longleftrightarrow}$ sol. de $z^2 \equiv d \ (4n)$

$(n = ax^2+bxy+cy^2 \rightsquigarrow z)$ $\quad\Updownarrow$

$\sum\left(\frac{d}{m}\right)$

Obs: $R(n)$ es multiplicativa $\Big|$ $(m,n)=1 \Rightarrow R(mn)=R(m)R(n)$

$$R(p) = w\left(1+\left(\frac{d}{p}\right)\right) = \begin{cases} 0 \\ w \\ 2w \end{cases} \qquad \left(\frac{d}{p}\right) = -1,\ 0,\ +1$$

$$i = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots$$

$$R(p^i) = \begin{cases} w,\ 0,\ w,\ 0,\ w,\ \ldots & (d/p) = -1 \\ w,\ w,\ w,\ w,\ w,\ \ldots & (d/p) = 0 \\ w,\ 2w,\ 3w,\ 4w,\ 5w,\ \ldots & (d/p) = +1 \end{cases}$$

$$\boxed{x^2+y^2=1}$$

$$w^{-1}\sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n) = \sum_{\substack{m_1 m_2 \le N \\ (m_1 m_2, d)=1}} \left(\frac{d}{m_1}\right)$$

$$= \sum_{m_1 \le \sqrt{N}} \left(\frac{d}{m_1}\right) \sum_{\substack{m_2 \le N/m_1 \\ (m_2, d)=1}} 1 \qquad \frac{N}{m_1}\frac{\varphi(|d|)}{|d|} + O(|d|)$$

$$\boxed{d \neq 1} \qquad \boxed{d \text{ fund}}$$

$$+ \sum_{\substack{m_2 < \sqrt{N} \\ (m_2, d)=1}} \sum_{\sqrt{N} < m_1 \le \frac{N}{m_2}} \left(\frac{d}{m_1}\right) = O(\sqrt{N}) \qquad \text{acertado}$$

$$= N\frac{\varphi(|d|)}{|d|} \cdot \sum_{m_1 \le \sqrt{N}} \frac{1}{m_1}\left(\frac{d}{m_1}\right) + O(\sqrt{N})$$

$$w^{-1}\sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n) = N\frac{\varphi(|d|)}{|d|} \cdot \sum_{m \le \sqrt{N}} \frac{1}{m}\left(\frac{d}{m}\right) + O(\sqrt{N})$$

Ejercicio $\displaystyle\sum_{m > \sqrt{N}} \frac{1}{m}\left(\frac{d}{m}\right) = O(N^{-1/2})$

$\boxed{L(1,\chi_d)}$

Conclusión:

$$\omega^{-1} \sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n) = N \frac{\varphi(|d|)}{|d|} \cdot L(1,\chi_d) + O(\sqrt{N})$$

$$\rightsquigarrow \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} R(n) = \omega \cdot \frac{\varphi(|d|)}{d} \cdot L(1,\chi_d)$$

$\times\, h(d)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} R_f(n) = \text{sólo dep de } d.$$

Jueves