Ap Cap. 2

Def Función aritmética $\quad f : \mathbb{Z}_{>0} \to \mathbb{C}$

Def Producto de Dirichlet

$$(f*g)(n) = \sum_{d|n} f(d)\, g(n/d).$$

$$= \sum_{a \cdot b = n} f(a) \cdot f(b).$$

Ejemplos $\mu, \varphi, \delta, N, 1$

$$\mu * 1 = \delta, \quad \varphi * 1 = N, \quad \mu * N = \varphi$$

Prop $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$

Dem

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}$$

$$= n \left[1 - \sum_{p|n} \frac{1}{p} + \sum_{\substack{p_1|n \\ p_2|n \\ p_1 < p_2}} \frac{1}{p_1 p_2} - \sum \frac{1}{p_1 p_2 p_3} + \cdots \right]$$

$$= n \prod_{p|n} \left(1 - \frac{1}{p}\right) \quad \checkmark$$

Ej $\varphi(p^\alpha) = (p-1)\, p^{\alpha - 1}$, $\quad \varphi(mn) = \varphi(m)\,\varphi(n)$ si $(m,n) = 1$

Prop $*$ es conm., asoc., neutro δ.

Dem $(f*g)(n) = \sum_{a\cdot b=n} f(a)g(b) \Rightarrow$ conm

$(f*g*h)(n) \overset{\downarrow}{=} \sum_{a\cdot b\cdot c=n} f(a)g(b)h(c) \rightsquigarrow$ asoc.

$(f*\delta)(n) = \sum_{a\cdot b=n} f(a)\underbrace{\delta(b)} = f(n)$.

Inversos

Prop f tiene inverso $\Longleftrightarrow f(1)\neq 0$

Dem $(\Rightarrow)$ $\underset{\underset{1=\delta(1)}{\shortparallel}}{(f*g)(1)} = f(1)g(1) \Rightarrow f(1)\neq 0$.

$(\Leftarrow)$ definimos g: $g(1) = \frac{1}{f(1)}$

$n>1$

$\sum_{d|n} \underbrace{f(\tfrac{n}{d})} g(d) = \delta(n) = 0$

$\Longleftrightarrow g(n) := -\frac{1}{f(1)} \sum_{\substack{d|n \\ d<n}} f(\tfrac{n}{d})\cdot g(d)$

es fácil ver que

esta definición recursiva cumple $f*g=\delta$.

Obs: $\{f : f(1)\neq 0\}$ es un grupo abeliano con $*$.

Teo Fórmula de inversión de Möbius.

$$f(n) = \sum_{d|n} g(d) \;\; 1(n/d) \Rightarrow g(n) = \sum_{d|n} f(d)\,\mu\left(\frac{n}{d}\right)$$

Dem $f = g * 1 \Rightarrow f * \mu = g * 1 * \mu = g$.

Eg.

$$\sum_{d|n} \varphi(d) = n \qquad (\varphi * 1 = N)$$

$$\sum_{d|n} d \cdot \mu\left(\frac{n}{d}\right) = \varphi(n) \qquad (N * \mu = \varphi)$$

Función de von Mangoldt

$$\Lambda(n) = \begin{cases} \log p & n = p^k \quad k \geq 1 \\ 0 & \end{cases}$$

Prop

$$\sum_{n|d} \Lambda(d) = \log n$$

$$(\Lambda * 1 = \log)$$

Dem $n = \prod_{i=1}^{t} p_i^{a_i} \to \log n = \sum_{i=1}^{t} a_i \log p_i$.

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{t} \sum_{k=1}^{a_i} \underbrace{\Lambda(p_i^k)}_{\log p_i} = \sum_i a_i \log p_i$$

$d = p_i^k$

eg. $\pi(x) \sim \frac{x}{\log x}$

$$\sum_{n \leq x} 1_p(n) \sim \int_2^x \frac{1}{\log n}\,dn$$

$\Updownarrow$

$$\sum_{n \leq x} \Lambda(n) \sim x$$

Cor $$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d.$$

Dem $\Lambda * 1 = \log \Rightarrow \Lambda = \mu * \log$.

$$\sum_{d} \mu(d) \log\left(\frac{n}{d}\right) = \underbrace{\sum_{d|n} \mu(d) \log n}_{} - \sum_{d|n} \mu(d) \log d$$

$$0 = \begin{cases} \log n \left(\underbrace{\sum_{d|n}'' \mu(d)}_{\delta(n)}\right) \\ \end{cases}$$

\#

Funciones multiplicativas

Def f es multiplicativa si $\begin{cases} f \not\equiv 0 \\ \text{y } f(mn) = f(m) f(n) \\ \text{siempre que } (m,n) = 1 \end{cases}$

f es completamente multiplicativa si $f(mn) = f(m) f(n)$ $\forall m, n$.

MULT. NO COMPLETAMENTE	COMP MULT	NO MULT
φ, μ	1, N, δ	Λ, $\log$

Prop ① f mult $\Rightarrow$ $f(1)=1$ (usar $f\not\equiv 0$)

② f, g mult $\Rightarrow$ $\underbrace{f\cdot g}_{\text{mult "puntual"}}$, f/g, f^{α} mult.

(Ejercicio)

Prop Sea f con $f(1)=1$

(a) f mult $\Leftrightarrow$ $f(p_1^{a_1}\cdots p_t^{a_t}) = f(p_1^{a_1})\cdots f(p_t^{a_t})$

(b) f comp. mult. $\Leftrightarrow$ f mult y $f(p^a) = f(p)^a$

⊛ Ejercicio

Prop f, g mult $\Rightarrow$ $f * g$ es mult.

Dem $h = f * g$, sea $(m,n)=1$

$$h(mn) = \sum_{c \mid mn} f(c)\, g\left(\frac{mn}{c}\right) = \sum_{\substack{a\mid m \\ b \mid n}} f(ab)\, g\left(\left(\frac{m}{a}\right)\left(\frac{n}{b}\right)\right) =$$

$(m,n)=1$

$\hookrightarrow \{c>0 : c \mid m\cdot n\} \xleftrightarrow{\text{biyección}} \{(a,b) : a\mid m,\ b\mid n\}$ $\quad (a,b)=1$

$ab \longleftarrow\!\shortmid (a,b)$ ejercicio

$$= \sum_{\substack{a\mid m \\ b \mid n}} \underbrace{f(a)\, g\left(\frac{m}{a}\right)}\ \underbrace{f(b)\, g\left(\frac{n}{b}\right)} = \left(\sum_{a\mid m} f(a)\, g\left(\frac{m}{a}\right)\right)\left(\sum_{b\mid n} f(b)\, g\left(\frac{n}{b}\right)\right)$$

$$= h(m)\cdot h(n)$$

Pregunta: ¿qué podemos decir sobre $f * g$ si f, g son completamente mult?

Prop ① f mult $\Rightarrow$ f^{-1} mult.

② $g, f*g$ mult $\Rightarrow$ f mult.

① ejercicio

② $f = (f*g) * g^{-1}$ ✓

Corolario: $\{ f : f \text{ mult} \}$ es un subgrupo.

Prop sup. f es mult.

f es completamente mult $\Leftrightarrow$ $f^{-1}(n) = \mu(n) \cdot f(n)$

Dem $(\Rightarrow)$ Sea $g = \mu \cdot f$

$$(g*f)(n) = \sum_{d|n} \mu(d) \overbrace{f(d) f\left(\tfrac{n}{d}\right)}^{\overset{f(n)}{=}} = f(n) \cdot \overbrace{\sum_{d|n} \mu(d)}^{\overset{\delta(n)}{=}}$$

$$= f(n) \cdot \delta(n) = \delta(n)$$

$(\Leftarrow)$ $f^{-1} = \mu \cdot f$

$(\mu \cdot f) * f = \delta \Rightarrow \sum_{d|n} \mu(d) f(d) f\left(\tfrac{n}{d}\right) = \delta(n)$

$\boxed{\alpha \geq 1}$

$\boxed{n = p^\alpha}$ $= \mu(1) \cdot f(1) \cdot f(p^\alpha) + \mu(p) f(p) \cdot f(p^{\alpha-1}) + \underset{=0}{\mu(p^2)} \cdots \overset{0}{}$

$\hookrightarrow$ $\cancel{f(1)} \cdot f(p^\alpha) = f(p) f(p^{\alpha-1}) \overset{IC}{\Longrightarrow} f(p^\alpha) = f(p)^\alpha$

Ejemplo $\varphi = \mu * N$ $\boxed{\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}}$

$$\varphi^{-1} = \mu^{-1} * N^{-1} = 1 * N^{-1} = 1 * \mu . N$$

$N^{-1} = \mu . N$

$$\varphi^{-1}(n) = \sum_{d|n} \mu(d) d$$

Prop f mult $\Rightarrow \sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p))$

Dem $g(n) = \sum_{d|n} \mu(d) f(d)$ es mult

$$g(p^\alpha) = \sum_{d|p^\alpha} \mu(d) f(d) = \underbrace{\mu(1) f(1)}_{1} + \underbrace{\mu(p) f(p)}_{-f(p)}$$

$\leadsto \quad = 1 - f(p)$

$$g(p_1^{\alpha_1} \cdots p_t^{\alpha_t}) = g(p_1^{\alpha_1}) \cdots g(p_t^{\alpha_t}) = \prod_{p|n} (1 - f(p)) \quad \#$$

Cor 1. $\varphi^{-1}(n) = \prod_{p|n} (1 - p)$

2. $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \sum \mu(d) \left(\frac{1}{d}\right) = n \prod \left(1 - \frac{1}{p}\right)$

$\boxed{f(d) = \frac{1}{d}}$