Más ejemplos

Función de Liouville completamente mult. $\lambda(p)=-1$

$$\left(\lambda(p_1^{a_1}\cdots p_t^{a_t}) = (-1)^{a_1+\dots+a_t} \right)$$

Prop ① $\overbrace{\sum_{d|n} \lambda(d)}^{g(n)} = \begin{cases} 1 & n \text{ cuadrado} \\ 0 & \text{\_\_\_\_} \end{cases}$

② $\lambda^{-1}(n) = |\mu(n)| \quad \forall n.$

Dem ① $g(n) = 1 * \lambda$ es mult $\qquad g(p^\alpha) = 1-1+1-1+\dots$
$= \begin{cases} 1 & \alpha \text{ par} \\ 0 & \alpha \text{ impar} \end{cases}$

② $\lambda^{-1} = \mu \cdot \lambda = \mu \cdot \mu = |\mu|.$

Funciones sumas de divisores

$\alpha \in \mathbb{C} \qquad \sigma_\alpha(n) = \sum_{d|n} d^\alpha = 1 * N^\alpha$ mult.

$(\sigma_0 = d, \quad \sigma_1 = \sigma)$

$$\sigma_\alpha(p^a) = 1^\alpha + p^\alpha + p^{2\alpha} + \dots + p^{a\alpha} = \begin{cases} \dfrac{p^{\alpha(a+1)}-1}{p^\alpha - 1} & \alpha \neq 0 \\ a+1 & \alpha = 0 \end{cases}$$

Prop $\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d) \cdot \mu\left(\frac{n}{d}\right)$

Dem $\sigma_\alpha = 1 * \underbrace{N^\alpha}_{\text{comp. mult}} \to \sigma_\alpha^{-1} = \mu * (\mu N^\alpha)$

Convolución generalizada

$$\begin{cases} F : (0,\infty) \to \mathbb{C} \\ \alpha : \mathbb{Z}_{a>0} \to \mathbb{C} \end{cases} \qquad \boxed{F|_{(0,1)} \equiv 0}$$

Def $(\alpha \circ F)(x) := \sum_{(n \leq x)} \alpha(n) F\left(\frac{x}{n}\right)$

Prop ① $\alpha \circ (\beta \circ F) = (\alpha * \beta) \circ F$

② $\delta \circ F = F$

Dem ejercicio ① $\sum_{mn \leq x} \alpha(m) \beta(n) F\left(\frac{x}{mn}\right)$

Corolario: ① $\alpha(1) \neq 0$ $\quad G = \alpha \circ F \iff F = \alpha^{-1} \circ G$

② α es comp. mult $\quad G = \alpha \circ F \iff F = (\mu\alpha) \circ G$.

Series de potencias formales (Ver Ap. §2.15)

$\mathbb{C}[[x]] = \{ a : \mathbb{Z}_{\geq 0} \to \mathbb{C} \}$ $\qquad a \xleftrightarrow{\text{representa}} A(x) = \sum_{i=0}^{\infty} a_i x^i$

$\cup|$

$\mathbb{C}[x]$ $\qquad b \longleftrightarrow B(x) = \sum_{i=0}^{\infty} b_i x^i$

$c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i} \longleftarrow A \cdot B(x) = \sum_{i=0}^{\infty} c_i x^i$

Producto de Cauchy

$\mathbb{C}[[x]]$ anillo commutativo $\qquad A$ es invertible $\iff a_0 \neq 0$

e.g. $(1 - ax)^{-1} = 1 + \sum_{i=1}^{\infty} a^i x^i$.

Series de Bell

f aritmética, p primo

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n) x^n .$$

Obs: f, g multiplicativas: $f = g \iff f_p = g_p$

Ejemplos

① $$\mu_p(x) = 1 - x$$

② $$\varphi_p(x) = 1 + \sum_{n=1}^{\infty} (p^n - p^{n-1}) x^n =$$

$$= \sum_{n=0}^{\infty} p^n x^n - x \sum_{n=0}^{\infty} p^n x^n = \frac{1-x}{1-px}$$

③ f completamente mult

$$f_p(x) = \sum f(p)^n x^n = \frac{1}{1-f(p)x}$$

$$\begin{cases} \delta_p(x) = 1 \\ \mathbb{1}_p(x) = \dfrac{1}{1-x} \\ N_p^{\alpha}(x) = \dfrac{1}{1-p^{\alpha}x} \\ \lambda_p(x) = \dfrac{1}{1+x} \end{cases}$$

Prop $h = f * g \Rightarrow h_p = f_p \cdot g_p$

Dem $$h(p^n) = \sum_{i=0}^{n} f(p^i) \cdot g(p^{n-i})$$ ✓

(Cor $(f^{-1})_p = \dfrac{1}{f_p}$)

$|\mu| = \mu^2 = \lambda^{-1}$ $\quad = \mu^2_p = \dfrac{1}{\lambda_p} = 1 + X$

$\sigma_\alpha = N^\alpha * 1$

$\hookrightarrow \sigma_{\alpha,p}(x) = \dfrac{1}{1-p^\alpha x} \cdot \dfrac{1}{1-x}$

$$= \frac{1}{1 - \sigma_a(p) X + p^\alpha x^2}$$

Sea $f(n) = 2^{\omega(n)}$ $\quad \omega(n) = \#\{p \text{ primo} : p | n\}$

es multiplicativa $\quad \begin{cases} \omega(mn) = \omega(m) + \omega(n) \\ \text{si } (m,n) = 1 \end{cases}$

$$f_p(x) = 1 + \sum_{i=1}^{\infty} 2x^i = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$$

$f_p = 1_p / \lambda_p \rightsquigarrow f = 1 * \lambda^{-1} = 1 * \mu^2$

$$\boxed{2^{\omega(n)} = \sum_{d|n} \mu^2(d)}$$

<u>Derivadas de funciones aritméticas</u>

<u>Def</u> $\quad f'(n) := f(n) \cdot \log(n)$

<u>Ej:</u> $\quad \delta(n) \cdot \log(n) = 0 \longrightarrow \delta' = 0$

$1' = \log \rightsquigarrow \boxed{\Lambda * 1 = 1'}$

Prop (a) $(f+g)' = f' + g'$

(b) $(f*g)' = f'*g + f*g'$

(c) $(f^{-1})' = -f' * (f*f)^{-1}$

Dem (a) ✓ $\qquad \log n = \log d + \log\left(\frac{n}{d}\right)$

(b) $$(f*g)'(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) \log n$$

$$= \sum_{d|n} \overbrace{f(d) \log d}^{f'(d)} \cdot g\left(\frac{n}{d}\right) + \sum_{d|n} f(d)\, \underbrace{g\left(\frac{n}{d}\right) \log \frac{n}{d}}_{g'\left(\frac{n}{d}\right)}$$

$$= f' * g + f * g' \quad ✓$$

(c) $$(f * f^{-1})' = \delta' = 0$$
$$\| $$
$$f' * f^{-1} + f * (f^{-1})'$$

$$\Rightarrow (f^{-1})' = f' * f^{-1} * f^{-1} = f' * (f*f)^{-1} \quad \#$$

Identidad de Selberg

Prop $$\Lambda(n) \log n + \sum_{d|n} \Lambda(d)\, \Lambda\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \cdot \log^2\left(\frac{n}{d}\right)$$

Dem $\quad \Lambda * 1 = 1'$

derivar

$$\Lambda' * 1 + \Lambda * 1' = 1''$$

$$\Lambda' * 1 + \Lambda * \Lambda * 1 = \log^2 \longrightarrow \boxed{\Lambda' + \Lambda * \Lambda = \mu * \log^2}$$