Teo Fórmula de sumación de Euler

$f:[y,x] \to \mathbb{C}$ con derivada continua $(0<y<x)$

$$\sum_{y<n\le x} f(n) = \underbrace{\int_y^x f(t)\,dt} + \int_y^x \{t\} f'(t)\,dt - \{x\} f(x) + \{y\} f(y).$$

Dem Sea $m=\lfloor y \rfloor$, $k=\lfloor x \rfloor$

$$\int_{n-1}^{n} \lfloor t \rfloor f'(t)\,dt = (n-1)\int_{n-1}^{n} f'(t)\,dt = (n-1)\big(f(n)-f(n-1)\big)$$

$$= \underbrace{\big(n f(n) - (n-1) f(n-1)\big)} - f(n)$$

$$\int_{m+1}^{k} = k f(k) - (m+1) f(m+1) - \sum_{n=m+2}^{k} f(n)$$

$$\sum_{y<n\le x} f(n) = f(m+1) + k f(k) - (m+1) f(m+1) - \underbrace{\int_{m+1}^{k} \lfloor t \rfloor f'(t)\,dt}$$

$$= -\int_y^x \lfloor t \rfloor f'(t)\,dt + \underbrace{\int_y^{m+1} - m f(m+1)}_{-\lfloor y \rfloor f(y)} \quad \underbrace{\int_k^x + k f(k)}_{\lfloor x \rfloor f(x)}$$

PARTES $$\int_y^x f(t)\,dt = x f(x) - y f(y) - \int_y^x t f'(t)\,dt$$

$du = dt$
$v = f(t)$

$\leadsto$ resultado #

Corolario

① $$\sum_{n\le x}\frac{1}{n}=\log x+\gamma+O(1/x)$$

$$\left(\Rightarrow \gamma=\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n}-\log x\right)\right)$$

② $$\sum_{n\le x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+\zeta(s)+O(x^{-s}) \qquad \begin{cases} s>0\\ s\neq 1\end{cases}$$

$$\left(\Rightarrow \zeta(s)=\lim_{x\to\infty}\left(\frac{x^{1-s}}{s-1}+\sum_{n\le x}\frac{1}{n^s}\right) \qquad \begin{matrix} s>0\\ s\neq 1\end{matrix}\right)$$

③ $$\sum_{n>x}\frac{1}{n^s}=O(x^{1-s}) \qquad s>1$$

④ $$\sum_{n\le x}n^{\alpha}=\frac{x^{\alpha+1}}{\alpha+1}+O(x^{\alpha}) \qquad \alpha\geq 0$$

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① vimos el jueves ②

$$\zeta(s)=\frac{s}{s-1}-s\underbrace{\int_1^{\infty}\{x\}x^{-s-1}\,dx}_{(\text{Clase 5})} \qquad s>0$$

$$\sum_{n\le x}\frac{1}{n^s}=1+\underbrace{\int_1^{x}\frac{dt}{t^s}}-s\underbrace{\int_1^{x}\frac{\{t\}}{t^{s+1}}dt}-\frac{\{x\}}{x^s}.$$

$$1+\overbrace{\frac{x^{1-s}}{1-s}}-\underbrace{\frac{1}{1-s}}\underbrace{-s\int_1^{\infty}\frac{\{t\}}{t^{s+1}}dt}_{\left(\zeta(s)-\frac{s}{s-1}\right)}+\underbrace{\int_x^{\infty}\frac{\{t\}}{t^{s+1}}dt}_{O(x^{-s})}-\underbrace{\frac{\{x\}}{x^s}}_{O(x^{-s})}$$

$$\underbrace{\sum_{n\le x}\frac{1}{n^s}} = \underbrace{\frac{x^{1-s}}{1-s}} + \underbrace{\zeta(s)} + \underbrace{O\left(x^{-s}\right)}.$$

③ $$\overline{\sum_{n>x}}\frac{1}{n^s} \overset{(s>1)}{=} \frac{x^{1-s}}{s-1} + O\left(x^{-s}\right)$$

④ $$\sum_{n\le x} n^{\alpha} = 1 + \underbrace{\int_1^x t^{\alpha}\,dt}_{\frac{x^{\alpha+1}-1}{\alpha+1}} + \overbrace{\alpha\int_1^x \{t\}\, t^{\alpha-1}\,dt}^{\overset{O(x^{\alpha})}{\shortparallel}} \underbrace{-\{x\}\,x^{\alpha}}_{O(x^{\alpha})}$$

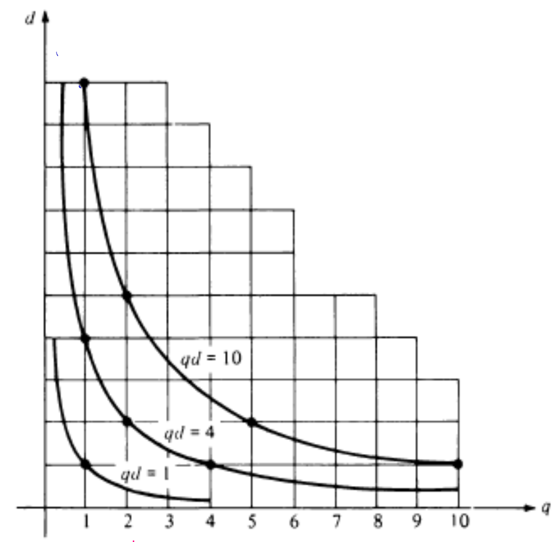
$$= \frac{x^{\alpha+1}}{\alpha+1} + O\left(x^{\alpha}\right).$$

Promedio de $d(n)$

Huxley 2003 $\frac{131}{416} \simeq 0.315$

Prop $$\sum_{n\le x} d(n) = x\log x + (2\gamma-1)x + O(x^{1/2})$$

Dem $$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d\mid n} 1 = \sum_{\substack{q,d\\ qd\le x}} 1$$



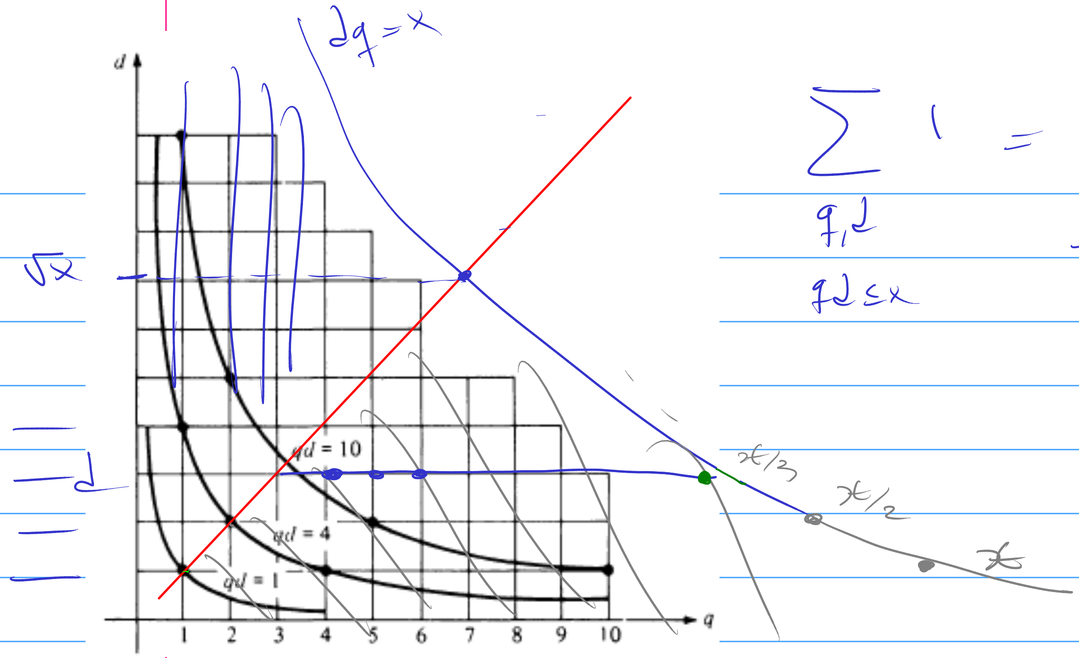
$$= \sum_{d\le x}\underbrace{\sum_{q\le \frac{x}{d}} 1}_{\frac{x}{d}+O(1)}$$

$$= \sum_{d\le x}\left(\frac{x}{d}+O(1)\right)$$

$$= x\sum_{d\le x}\frac{1}{d} + O(x)$$

$$= x\left(\log x + \gamma + O\left(\tfrac{1}{x}\right)\right) + O(x) = x\log x + O(x)$$

$$\Longrightarrow \sum_{n\le x} d(n) \sim x\log x \qquad \left(\overline{d}(n) \sim \log n\right)$$



$$\sum_{\substack{q,d\\ qd\le x}} 1 = \#\{q=d\} + 2\#\{\underbrace{d<q}_{\Downarrow\ d\le\sqrt{x}}\}$$

$$= \underbrace{\sum_{d\le\sqrt{x}} 1}_{\lfloor\sqrt{x}\rfloor = \sqrt{x} + O(1)} + 2\sum_{d\le\sqrt{x}}\left(\left\lfloor\frac{x}{d}\right\rfloor - d\right)$$

$$= \sqrt{x} + O(1) + 2\sum_{d\le\sqrt{x}}\left(\frac{x}{d} - d + O(1)\right)$$

$$= \sqrt{x} + 2x \underbrace{\sum_{d\le\sqrt{x}}\frac{1}{d}}_{=\left(\log\sqrt{x}+\gamma+O\left(\frac{1}{\sqrt{x}}\right)\right)} - 2\underbrace{\sum_{d\le\sqrt{x}} d}_{=\frac{(\sqrt{x})^2}{2}+O(\sqrt{x})} + O(\sqrt{x})$$

$$= x\log x + (2\gamma-1)x + O(\sqrt{x})$$

Promedio de $\sigma_\alpha(n)$ $\quad$ ($\alpha=0 \to d(n)$ ✓)

$\boxed{\alpha=1}$ **Prop**
$$\sum_{n\le x}\sigma(n) = \frac{1}{2}\zeta(2)x^2 + O(x\log x)$$

Dem
$$\sum_{n\le x}\sigma(n) = \sum_{d\le x}\underbrace{\sum_{q\le x/d} q}$$
$$= \sum_{d\le x}\left(\frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)\right) = \frac{x^2}{2}\underbrace{\sum_{d\le x}\frac{1}{d^2}} + O\left(x\sum_{d\le x}\frac{1}{d}\right)$$
$$= \frac{x^2}{2}\left(-\frac{1}{x} + \zeta(2) + O\left(\frac{1}{x^2}\right)\right) + O(x\log x)$$
$$= \frac{x^2}{2}\zeta(2) + O(x\log x).$$

Prop $\alpha>0$, $\alpha\neq 1$ $\quad$ $\beta=\max(1,\alpha)$
$$\sum_{n\le x}\sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1} + O(\beta)$$
$$\text{-------} = \square + \underbrace{O(x) + O(x^\alpha)}_{\shortparallel}$$

$$O(x^\beta) = \begin{cases} O(x^\alpha) & \alpha\ge 1 \\ O(x) & \alpha\le 1\end{cases}$$

$\delta = \max(0,1-\beta)$

Prop $\beta>0$
$$\sum_{n\le x}\sigma_{-\beta}(n) = \begin{cases}\zeta(\beta+1)x + O(x^\delta) & \beta\neq 1\\ \zeta(2)x + O(\log x) & \beta=1\end{cases}$$

Promedio de $\varphi(n)$

Lema $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$.

Dem $\to$ después

$$O\Big(\sum_{n>x} \frac{1}{n^2}\Big) = O(1/x)$$

$$\rightsquigarrow \sum_{n\le x} \frac{\mu(n)}{n^2} = \sum^{\infty} - \sum_{n>x} \frac{\mu(n)}{n^2}$$

$$= \frac{6}{\pi^2} - O(1/x).$$

Prop $\displaystyle\sum_{n\le x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$

Dem $\displaystyle\sum_{n\le x} \varphi(n) = \sum_{d\le x} \mu(d) \sum_{q\le x/d} q$

$$\varphi(n) = \sum_{dq=n} \mu(d)\, q$$

$$= \sum_{d\le x} \mu(d) \left(\frac{1}{2}\Big(\frac{x}{d}\Big)^2 + O\Big(\frac{x}{d}\Big)\right)$$

$$= \frac{x^2}{2} \sum_{d\le x} \frac{\mu(d)}{d^2} + O\Big(x \sum_{d\le x} \frac{1}{d}\Big)$$

$$= \frac{x^2}{2}\left(\frac{6}{\pi^2} + O\Big(\frac{1}{x}\Big)\right) + O(x \log x)$$

$$= \frac{3}{\pi^2} x^2 + O(x \log x).$$

Puntos visibles desde el origen

Def decimos que $P, Q \in \mathbb{Z}^2$ "se ven" si $\overline{PQ} \cap \mathbb{Z}^2 = \{P, Q\}$

Obs: (a,b), (m,n) son visibles sii $mcd(a-m, b-n) = 1$

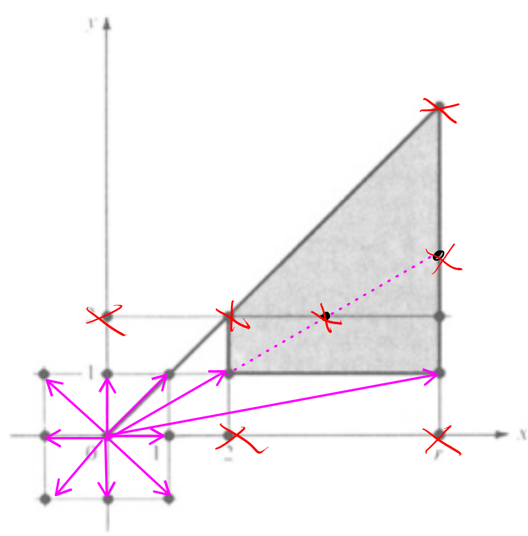
Dem ej, suponer $m = n = 0$.

$(\rightarrow)$ $a = a_0 d$, $b = b_0 d \rightsquigarrow (a_0, b_0) \in \overline{PQ}$

$(\leftarrow)$ — — . \

Def $N(r) = \#\{(x,y) : |x| \le r, |y| \le r\}$

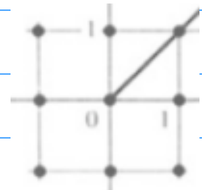
$\grave{N}(r) = \{(x,y) : |x| \le r, |y| \le r, (x,y) = 1\}$.



Prop $\dfrac{\grave{N}(r)}{N(r)} \longrightarrow \dfrac{6}{\pi^2}$.

Dem $N(r) = (2\lfloor r \rfloor + 1)^2 = 4r^2 + O(r)$

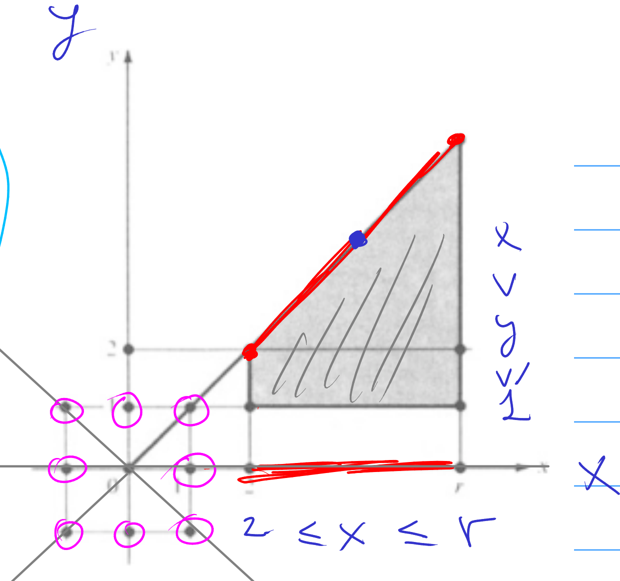
$\grave{N}(r) = 8 + 8\#$



$\{(x,y)$ visible, $2 \le x \le r$, $1 \le y \le x\}$

$$\dot{N}(r) = 8\cdot\varphi(1) + 8 \sum_{2\le n\le r} \sum_{\substack{1\le m\le n \\ (m,n)=1}} 1$$

$$\varphi(n)$$



$$\boxed{\dot{N}(r) = 8 \sum_{n\le r} \varphi(n).}$$

$$= \frac{24}{\pi^2} r^2 + O(r \log r)$$

$$\frac{\dot{N}(r)}{N(r)} \longrightarrow \frac{6}{\pi^2} \quad \checkmark$$

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