Promedio de $\mu(n)$, $\Lambda(n)$.

Teo $\lim_{x\to+\infty} \frac{1}{x}\sum_{n\le x}\mu(n) = 0$

$\lim_{x\to+\infty} \frac{1}{x}\sum_{n\le x}\Lambda(n) = 1$

Nota: cualquiera de estas dos afirmaciones es equivalente a $\pi(x) \sim \frac{x}{\log x}$!

— x —

$F:(0,\infty)\to\mathbb{C}$
$F(x)=0$ si $x\in(0,1)$

Prop $h = f * g$

$H(x) = \sum_{n\le x} h(n)$, $F(x)=\sum_{n\le x} f(n)$, $G(x)=\sum_{n\le x} g(n)$

$\Rightarrow H(x) = \sum_{(n\le x)} f(n)\, G\left(\frac{x}{n}\right) = \sum_{(n\le x)} g(x)\, F\left(\frac{x}{n}\right)$

$\left(H = f\circ G = g\circ F \right)$.

Dem Sea $U(x) = \begin{cases} 0 & x\in(0,1) \\ 1 & x\ge 1 \end{cases} \to F = f\circ U,\ G = g\circ U$.

$f\circ G = f\circ(g\circ U) = (f*g)\circ U = h\circ U = H$

$g\circ F = - - - - - - - = H$

Corolario $F = \sum_{n\le x} f(n) \Longrightarrow$

$$\sum_{n\le x}\sum_{d\mid n} f(d) = \sum_{n\le x} f(n)\left\lfloor \frac{x}{n}\right\rfloor = \sum_{n\le x} F\left(\frac{x}{n}\right)$$

Dem Aplicar a $f*1$, usando $\sum_{n\le x} 1 = \left\lfloor \frac{x}{n}\right\rfloor$

Prop Si $x \geq 1$ vale

$$\sum_{n\le x}\mu(n)\left\lfloor \frac{x}{n}\right\rfloor = 1$$

$$\sum_{n\le x}\Lambda(n)\left\lfloor \frac{x}{n}\right\rfloor = \log(\lfloor x\rfloor !)$$

Dem $$\sum_{n\le x}\mu(n)\left\lfloor \frac{x}{n}\right\rfloor = \sum_{n\le x}\overbrace{\sum_{d\mid n}\mu(d)}^{\delta(n)} = 1$$

$$\sum_{n\le x}\Lambda(n)\left\lfloor \frac{x}{n}\right\rfloor = \sum_{n\le x}\overbrace{\sum_{d\mid n}\Lambda(d)}^{\log n} = \log(\lfloor x\rfloor !)$$

obs:

$$\sum_{n\le x}\frac{\mu(n)}{n} \longrightarrow 0$$

equivalente al Teo de los números primos (TNP)

Prop $\left|\sum_{n\le x} \frac{\mu(n)}{n}\right| \le 1$ (= sólo si $x<2$)

$(x\ge 2)$

$$1 = \sum_{n\le x} \mu(n)\left\lfloor \frac{x}{n}\right\rfloor = \sum_{n\le x} \mu(n)\left(\frac{x}{n} - \left\{\frac{x}{n}\right\}\right)$$

$$= x\sum_{n\le x}\frac{\mu(n)}{n} - \sum_{n\le x}\mu(n)\underbrace{\left\{\frac{x}{n}\right\}}_{<1}$$

$$x\left|\sum_{n\le x}\frac{\mu(n)}{n}\right| = \left|1 + \sum \mu(n)\left\{\frac{x}{n}\right\}\right| \le 1 + \left|\sum \mu(n)\left\{\frac{x}{n}\right\}\right|$$

$$\le 1 + \sum_{n\le x}\left\{\frac{x}{n}\right\} = 1 + \{x\} + \underbrace{\sum_{n=2}^{x}\overbrace{\left\{\frac{x}{n}\right\}}^{<1}}_{<\lfloor x\rfloor - 1}$$

$$< \{x\} + \lfloor x\rfloor = x.$$ #

Prop (identidad de Legendre) si $x>1$

$$\lfloor x\rfloor ! = \prod_{p\le x} p^{\alpha(p)}$$ donde

$$\alpha(p) = \sum_{m=1}^{\infty}\left\lfloor \frac{x}{p^m}\right\rfloor$$ $\log_p x$

Dem $\log \lfloor x \rfloor ! = \sum_{n \leq x} \Lambda(n) \lfloor \frac{x}{n} \rfloor$

$$= \sum_{p \leq x} \sum_{m=1}^{\infty} \log p \cdot \lfloor \frac{x}{p^m} \rfloor = \sum_{p \leq x} \log p \, \alpha(p)$$

Prop $\log \lfloor x \rfloor ! = x \log x - x + O(\log x)$.

$\left(\lfloor x \rfloor ! \sim C_x \frac{x^x}{e^x} \quad (\text{Stirling } C_x = \sqrt{2\pi x}) \right.$ $\log C_x = o(\log x))$

$$\sum_{n \leq x} \Lambda(n) \lfloor \frac{x}{n} \rfloor = x \log x - x + O(\log x)$$

Dem $f(t) = \log t$ en la Formula de suma de Euler.

$$\sum_{n \leq x} \log n = \int_1^x \log t \, dt + \int_1^x \frac{\{t\}}{t} dt - \{x\} \log x.$$

$$= \underbrace{x \log x - x}_{} \underbrace{+1}_{O(\log x)} + \underbrace{\int_1^x \frac{\{t\}}{t} dt}_{O(\log x)} + O(\log x)$$

Prop $x \geq 2$:

$$\sum_{p \leq x} \lfloor \frac{x}{p} \rfloor \cdot \log p = x \log x + O(x)$$

Dem

$$\sum_{p\le x}\sum_{m=1}^{\log_p x}\left\lfloor \frac{x}{p^m}\right\rfloor \cdot \log p = x\log x + O(x)$$

$$\parallel$$

$$\sum_{p\le x}\left\lfloor\frac{x}{p}\right\rfloor \log p \quad + \quad \sum_{m=2}^{\log_p x}\left\lfloor\frac{x}{p^m}\right\rfloor \log p$$

$$\sum_{p\le x}\log p\sum_{m=2}\left\lfloor\frac{x}{p^m}\right\rfloor \le \sum_{p\le x}\log p\sum_{m=1}^{\infty}\frac{x}{p^m}$$

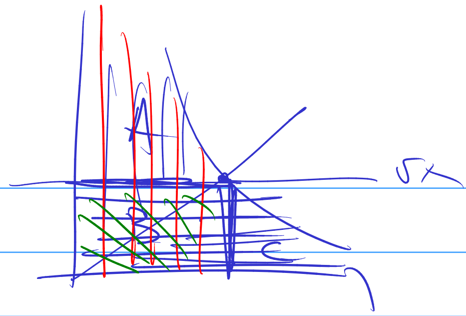
$$= x\sum_{p\le x}\log p\sum_{m=2}^{\infty}\left(\frac{1}{p}\right)^m$$

$$= x\sum_{p\le x}\log p\,\frac{1}{p^2}\left(\frac{1}{1-\frac{1}{p}}\right) = x\sum_{p\le x}\frac{\log p}{p(p-1)}$$

$$\le x\underline{\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)}} = O(x)$$

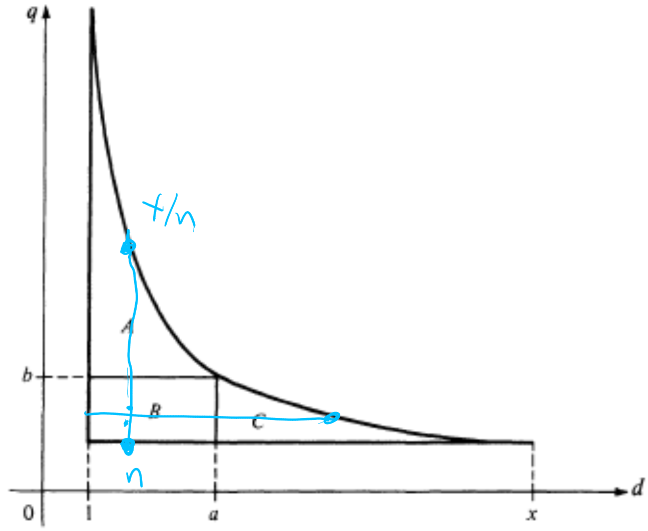
#

$$F(x) = \sum_{n \le x} f(n), \qquad G(x) = \sum_{n \le x} g(n)$$



$$H(x) = \sum_{n \le x} (f * g)(n).$$

$$H = f \circ G = g \circ F$$



Prop Sean $a, b > 0$ tq $ab = x \implies$

$$H(x) = \sum_{\substack{q,d \\ qd \le x}} f(d)\, g(q) = \overbrace{\sum_{n \le a} f(n)\, G\left(\frac{x}{n}\right)}^{A \cup B} + \overbrace{\sum_{n \le b} g(n)\, F\left(\frac{x}{n}\right)}^{C \cup B} - \underbrace{F(a)\, G(b)}_{B}$$

(Obs: $a = 1 \to H = g \circ F$, etc)