$$\psi(x) = \sum_{n \le x} \Lambda(x) = \sum_{p^n \le x} \log p$$

$$\vartheta(x) = \sum_{p \le x} \log p .$$

Obs: $$\psi(x) = \sum_{m=1}^{\log_2 x} \vartheta(x^{1/m})$$

Prop $$0 \le \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \le \frac{(\log x)^2}{2 \log 2 \sqrt{x}} \to 0$$

Dem: $$0 \le \psi(x) - \vartheta(x) = \underbrace{\sum_{m=2}^{\log_2 x} \vartheta(x^{1/m})}.$$

$$\vartheta(x) \le \sum_{p \le x} \log x \le x \log x$$

$$0 \le \psi(x) - \vartheta(x) \le \sum_{m=2}^{\log_2 x} x^{1/m} \log(x^{1/m}) \le \underbrace{\log_2 x \cdot x^{1/2} \cdot \frac{1}{2} \log x}_{\frac{(\log x)^2 \sqrt{x}}{2 \log 2}} \#$$

Teo Abel $$A(x) = \sum_{n \le x} a(n), \quad f \in C^1$$

$$\sum_{y < n \le x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) \, dt$$

Dem: $$\sum_{y < n \le x} a(n) f(n) =: \int_y^x f(t) \, dA(t) = f(t) A(t) \Big|_y^x - \int A(t) \, df(t)$$

$$= f(x) A(x) - f(y) A(y) - \int_y^x A(t) f'(t) \, dt . \#$$

Riemann-Stieltjes sirve para probar Euler:

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,d\lfloor t\rfloor = f(x)\lfloor x\rfloor - f(y)\lfloor y\rfloor - \int_y^x \lfloor t\rfloor f'(t)\,dt$$

$$\int_y^x f(t)\,dt = f(x)\,x - f(y)\,y - \int_y^x t\,f'(t)\,dt$$

restar

... sumación de Euler

Prop $(x\geq 2)$

① $\vartheta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,dt$

② $\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(x)}{t\log^2 t}\,dt$

Dem ①

$\vartheta(x) = \sum_{n\le x} a(n)\log n$

$a(n) = \begin{cases} 1 & n \text{ primo} \\ 0 & \text{—} \end{cases}$

$\omega * \mu$.

$\pi(x) = \sum_{n\le x} a(n)$

$\left(= \int_1^x \log t \cdot d\,\pi(t) \right)$

$= \pi(x)\log(x) - \int_1^x \frac{\pi(t)}{t}\,dt$

$\vartheta(x) = \sum a(n)\log n$

②

$\pi(x) = \sum (a(n)\log n)\cdot\frac{1}{\log n}$

$= \int_1^x \frac{1}{\log t}\cdot d\,\vartheta(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(x)}{t\log^2 t}\,dt$

Teo Son equivalentes:

(i) $\pi(x) \sim \dfrac{x}{\log x}$ (TNP)

(ii) $\theta(x) \sim x$

(iii) $\psi(x) \sim x$

$\Updownarrow$ ✓

Dem

(i) → (ii)

$$\frac{\theta(x)}{x} = \underbrace{\boxed{\frac{\pi(x)\log x}{x}}}_{\to 1} - \underbrace{\overbrace{\frac{1}{x}\int_2^x \frac{\pi(t)}{t}\,dt}^{\to 0}}$$

(i) $\to$ $\dfrac{\pi(t)}{t} \sim \dfrac{1}{\log t}$ $\to$ $\dfrac{1}{x}\displaystyle\int_2^x \frac{\pi(t)}{t}\,dt = O\left(\frac{1}{x}\int_2^x \frac{1}{\log t}\,dt\right) \to 0$

$Li(x) = \displaystyle\int_2^x \frac{1}{\log t}\,dt$

$$\int_2^x \frac{1}{\log t}\,dt = \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_{\sqrt{x}}^x \frac{dt}{\log t} \le \frac{\sqrt{x}}{\log 2} + \frac{x}{\frac{1}{2}\log x}$$

$$= O\left(\frac{x}{\log x}\right)$$

(ii) → i

$$\frac{\log x\,\pi(x)}{x} = \overbrace{\frac{\theta(x)}{x}}^{\to 1} + \underbrace{\frac{\log x}{x}\int_2^x \frac{\theta(t)}{t\log^2 t}\,dt}_{\to 0}$$

(ii) $\to \theta(x) = O(x)$

$$= \int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x$$

$$\int_2^x \frac{\theta(t)}{t\log^2 t}\,dt = O\left(\int_2^x \frac{1}{\log^2 t}\,dt\right) = O\left(\frac{x}{\log^2 x}\right)$$

Prop Sea p_n el n-ésimo primo.

(TNP) (1) $\pi(x) \cdot \log x \sim x$

(2) $\pi(x) \cdot \log \pi(x) \sim x$

(3) $p_n \sim n \log n$.

Dem (1) $\to$ (2) $\log \pi(x) + \log\log x - \log x \to 0$

$$\underbrace{\log x}_{\to \infty} \left(\underbrace{\frac{\log \pi(x)}{\log x}}_{\to 1} + \underbrace{\frac{\log\log x}{\log x}}_{\to 0} - 1 \right) \to 0$$

$\Rightarrow \log \pi(x) \sim \log x$, (1) $\Rightarrow$ (2)

(2) $\to$ (1) misma idea (ejercicio)

(2) $\to$ (3) Sea $x = p_n$, $\pi(x) = n \Rightarrow \pi(x) \log \pi(x) = n \log n$

(2) $\rightsquigarrow n \log n \sim p_n$ ✓

(3) $\to$ (2) Dado x, sea $n = \pi(x) \rightsquigarrow p_n \leq x < p_{n+1}$

$$1 \leftarrow \frac{p_n}{n \log n} \leq \frac{x}{n \log n} \leq \frac{p_{n+1}}{n \log n} = \underbrace{\frac{p_{n+1}}{(n+1)\log(n+1)}}_{\to 1} \cdot \underbrace{\frac{(n+1)\log(n+1)}{n \log n}}_{\to 1}$$

$\frac{x}{n \log n} \to 1$ i.e. $\frac{x}{\pi(x) \log \pi(x)} \to 1$ #

Prop $\forall n \geq 2$

$$\frac{1}{6}\frac{n}{\log n} \leq \pi(n) \leq 6\frac{n}{\log n}$$

Dem $\boxed{\text{Af 1}}$ $2^n \leq \binom{2n}{n} < 4^n$

IC

$$4^n = (1+1)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k} > \binom{2n}{n}$$

$$n\log 2 \leq \overbrace{\log(2n)! - 2\cdot\log n!} < n\cdot\log 4.$$

Vimos $\log n! = \sum_{p\leq n}\alpha(p)\log p$

donde $\alpha(p) = \sum_{m=1}^{\log_p n}\left\lfloor\frac{n}{p^m}\right\rfloor$

$$\log(2n)! - 2\log n! = \sum_{p\leq 2n}\left(\sum_{m=1}^{\log_p 2n}\left\lfloor\frac{2n}{p^m}\right\rfloor - 2\left\lfloor\frac{n}{p^m}\right\rfloor\right)\log p.$$

Obs: $\lfloor 2x\rfloor - 2\lfloor x\rfloor = 0, 1 \quad \leq 1$

$$n\log 2 \leq \sum_{p\leq 2n}\Big(\underbrace{\sum_{m=1}^{\log_p 2n} 1}_{\log_p 2n}\Big)\log p \leq \underbrace{\sum_{p\leq 2n}\log 2n}_{\pi(2n)\cdot\log 2n}.$$

$$\pi(2n) \geq \frac{n\log 2}{\log 2n} = \frac{2n}{\log 2n}\cdot\frac{\log 2}{2} > \frac{1}{4}\frac{2n}{\log 2n}$$

$$\pi(2n+1) \geq \pi(2n) > \frac{1}{4}\frac{2n}{\log 2n} > \frac{1}{4}\left(\frac{2n}{2n+1}\right)\frac{2n+1}{\log 2n+1} \geq \frac{1}{6}\frac{2n+1}{\log(2n+1)}$$

$$\log(2n!) - 2\log n! \geq \sum_{p \leq 2n}\left(\left\lfloor\frac{2n}{p}\right\rfloor - 2\left\lfloor\frac{n}{p}\right\rfloor\right)\log p$$

$$n < p \leq 2n \longrightarrow \left\lfloor\frac{2n}{p}\right\rfloor - 2\left\lfloor\frac{n}{p}\right\rfloor = 1$$

$$\geq \sum_{n<p\leq 2n} \log p = \vartheta(2n) - \vartheta(n)$$

$$\vartheta(2n) - \vartheta(n) < n\log 4$$

$$\hookrightarrow \vartheta(2^{r+1}) - \vartheta(2^r) < 2^r \log 4$$

$r = 0, \ldots, k$

$$\vartheta(2^{k+1}) < 2^{k+1}\log 4$$

$2^k \leq n < 2^{k+1}$

$$\vartheta(n) \leq \vartheta(2^{k+1}) < 2^{k+1}\log 4 \leq 2n \cdot \log 4.$$

$$\boxed{0<\alpha<1}$$

$$\left(\pi(n)-\pi(n^\alpha)\right)\log(n^\alpha) < \sum_{n^\alpha < p \le n} \log p = \vartheta(n)-\vartheta(n^\alpha)$$

$$< \vartheta(n)$$

$$< 2n\log 4$$

$$\pi(n) < \frac{2n\log 4}{\alpha \log n} + \overbrace{\pi(n^\alpha)}^{<n^\alpha}$$

ejercicio

$$\frac{1}{(1-\alpha)e} \geq$$

$$< \frac{2n\log 4}{\alpha\log n} + n^\alpha = \frac{n}{\log n}\underbrace{\left(\frac{2\log 4}{\alpha} + \frac{\log n}{n^{1-\alpha}}\right)}$$

$$\rightsquigarrow \quad \alpha = \frac{2}{3} \Rightarrow \quad 3\cdot\log 4 + \frac{3}{e} < 6$$

$$\Rightarrow \quad \pi(n) < 6\,\frac{n}{\log n} \qquad \#$$

Coro:

$$\frac{1}{6}\, n\log n < p_n < 12\left(n\log n + n\log\frac{12}{e}\right)$$

ejercicio