Prop $n \geq 2$ $\quad \frac{1}{6}\frac{n}{\log n} < \pi(n) < 6\frac{n}{\log n}$

Coro $\quad \frac{1}{6} n \log n < p_n < 12\left(n \log n + n \log \frac{12}{e}\right)$

Dem $k = p_n \qquad n = \pi(k) < 6\frac{k}{\log k} = \frac{6\, p_n}{\log p_n}$

$$\Rightarrow p_n > \frac{1}{6} n \log p_n > \frac{1}{6} n \log n \qquad (p_n > n)$$

$$p_n < 6 n \log p_n \leq \frac{12}{e} n\, p_n^{1/2} \qquad \Big|\ (x \geq 1)\ \ \log x \leq \frac{2}{e} x^{1/2}$$

$$p_n^{1/2} < \frac{12}{e} n$$

$$\frac{1}{2}\log p_n < \log n + \log\frac{12}{e}$$

$$p_n < 6 n \log p_n < 6n\left(2\log n + 2\log\frac{12}{e}\right)$$

$$\text{''} \quad 12n\left(\log n + \log\left(\frac{12}{e}\right)\right)$$

TNP $\Leftrightarrow \psi(x) = \sum_{x \leq n} \Lambda(n) \sim x$

Vimos $\quad \sum_{n \leq x} \Lambda(n)\left\lfloor \frac{x}{n} \right\rfloor = x \log x - x + O(\log x)$

¿Podemos deducir algo sobre $\sum_{x \leq n} \Lambda(n)$?

Teo (Shapiro) $a(n) \geq 0$

Sup. $T(x) = \sum_{n \leq x} a(n) \left\lfloor \frac{x}{n} \right\rfloor = x \log x + O(x)$ $\forall x \geq 1$

$\Rightarrow$ (a) $\sum_{n \leq x} \frac{a(n)}{n} = \log x + O(1)$ $\boxed{\forall x \geq 1}$

(b) $\exists B > 0$ tq

$$S(x) = \sum_{n \leq x} a(n) \leq Bx \qquad \forall x \geq 0$$

(c) $\exists A > 0$ tq

$$\sum_{n \leq x} a(n) \geq Ax \qquad \forall x \geq x_0$$

Dem $S(x) = \sum_{n \leq x} a(n)$, $T(x) = \sum_{n \leq x} a(n) \left\lfloor \frac{x}{n} \right\rfloor$

(b)

Af: $S(x) - S(x/2) \leq T(x) - 2T(x/2)$

$$T(x) - 2T(x/2) = \sum_{n \leq x/2} a(n) \underbrace{\left(\left\lfloor \frac{x}{n} \right\rfloor - 2 \left\lfloor \frac{x/2}{n} \right\rfloor \right)}_{0,1} + \sum_{\frac{x}{2} < n \leq x} a(n) \overbrace{\left\lfloor \frac{x}{n} \right\rfloor}^{\overset{1}{\shortparallel}}$$

$$\geq S(x) - S(x/2)$$

$$S(x) - S(x/2) < T(x) - 2T(x/2) = \overbrace{x \log x} + O(x) - \overbrace{\cancel{2} \frac{x}{\cancel{2}} \log \frac{x}{2}} + O(x)$$

$$= O(x)$$

$S(x) - S(x/2) \leq kx \qquad \forall x$

$S(x/2) - S(x/4) \leq \frac{k}{2} x$

$S(x/4) - S(x/8) \leq \frac{k}{4} x$

$\vdots$

$S(x) < 2kx \longrightarrow$ (b) con $B = 2k$.

(a) $\lfloor x/n \rfloor = x/n + O(1)$

$$T(x) = \sum_{n \leq x} a(n) \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n \leq x} \frac{a(n) \cdot x}{n} + \overbrace{\sum_{n \leq x} a(n) \cdot O(1)}^{= O(x)}$$

$$= x \sum_{n \leq x} \frac{a(n)}{n} + O(x) \checkmark$$

(c) $A(x) = \sum_{n \leq x} \frac{a(n)}{n} = \log x + \overbrace{R(x)}^{= O(1)}$

(donde $|R(x)| \leq M$)

Sea $0 < \alpha < 1$

$$A(x) - A(\alpha x) = \log x - \log(\alpha x) + \overbrace{R(x) - R(\alpha x)}^{\geq -2M}$$

$$\geq -\log \alpha - 2M = 1$$

elegir α tq

$x \geq 1$, $\alpha x \geq 1$ $\Big\}$ $x \geq \frac{1}{\alpha}$

Tenemos con ese α vale

$$A(x) - A(\alpha x) \geq 1 \qquad \forall x \geq 1/\alpha.$$

$$1 \le \sum_{\alpha x < n \le x} \frac{a(n)}{n} \le \frac{1}{\alpha x} \sum_{n \le x} a(n) = \frac{S(x)}{\alpha x}$$

$$\Rightarrow \quad S(x) \ge \alpha x \quad \forall x \ge \frac{1}{\alpha}. \quad \checkmark$$

Corolario ① $\displaystyle\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1)$

$$\exists\, c_1, c_2 > 0 : \begin{cases} \psi(x) \le c_1 x & \forall x \ge 1 \\ \psi(x) \ge c_2 x & \forall x \ge x_0 \end{cases}$$

② $\displaystyle\sum_{p \le x} \frac{\log p}{p} = \log x + O(1)$

$$\exists\, c_3, c_4 > 0 : \begin{cases} \theta(x) \le c_3 x & \forall x \ge 1 \\ \theta(x) \ge c_4 x & \forall x \ge x_1 \end{cases}$$

$$\left(\underbrace{\sum_{p \le x} \left\lfloor \frac{x}{p} \right\rfloor \log p}_{\displaystyle \| \atop \displaystyle \sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor \Lambda_1(n)} = x \log x + O(x) \right)$$

Recordar $F(x) = \sum_{n \le x} f(n) \rightsquigarrow \sum_{n \le x} f(n) \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n \le x} F(x/n)$

$$\sum_{n \le x} \psi(x/n) = x \log x - x + O(\log x)$$

$$\sum_{n \le x} \theta(x/n) = x \log x + O(x)$$

Asintótica para $\sum 1/p$.

Prop $$\sum_{p\le x}\frac{1}{p} = \log\log x + \underbrace{A}_{} + O\left(\frac{1}{\log x}\right)$$

constante de Mertens.
0,261497....

Dem $$A(x) = \sum_{p\le x}\frac{\log p}{p} \qquad a(n) = \begin{cases} 1 & n \text{ primo} \\ 0 & \end{cases}$$

$$\sum_{p\le x}\frac{1}{p} = \sum\frac{a(n)}{n} \qquad A(x) = \sum\frac{a(n)}{n}\cdot\log n$$

Abel con $f(t) = \frac{1}{\log t}$

$$\sum_{p\le x}\frac{1}{p} = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t\log^2 t}\,dt =$$

$$\boxed{A(x) = \log x + \underset{}{\overset{\text{,,}O(1)}{R(x)}}}$$

$$= \frac{\log x + R(x)}{\log x} + \int_2^x \frac{\log t + R(t)}{t\log^2 t}\,dt$$

$$= 1 + O\left(\frac{1}{\log x}\right) + \underbrace{\int_2^x\frac{dt}{t\log t}}_{\log\log x - \log\log 2} + \underbrace{\int_2^x\frac{R(t)}{t\log^2 t}\,dt}_{}$$

$$\int_2^x\frac{R(t)}{t\log^2 t}\,dt = \overbrace{\int_2^\infty}^{} - \int_x^\infty = O\left(\int_x^\infty\frac{dt}{t\cdot\log^2 t}\,dt\right) = O\left(\frac{1}{\log x}\right)$$

$$\rightsquigarrow \sum_{p\le x}\frac{1}{p} = \log\log x + \underbrace{\left(1-\log\log 2+\int_2^\infty \frac{R(t)}{t\log^2 t}\,dt\right)}_{\text{"}A\text{"}} + O\left(\frac{1}{\log x}\right)$$

$$\gamma + \sum_{m=2}^{\infty}\frac{\mu(m)}{m}\log\zeta(m)$$

Sumas parciales de μ

$$M(x) = \sum_{n\le x}\mu(x)$$

Conj (FALSA)

$$M(x) < \sqrt{x} \quad \forall_{x>1}$$

LATeN: Lola Thompson

¿cómo calcular $M(x)$?

Vamos a probar:

$$\text{TNP} \Longleftrightarrow \frac{M(x)}{x}\to 0$$

$$\left(M(x) = o(x)\right)$$

$$H(x) = \sum_{n\le x}\mu(n)\log n$$

① Prop $\lim\left(\frac{M(x)}{x} - \frac{H(x)}{x\log x}\right) = 0$

Esta prop implica $M(x)=o(x) \Leftrightarrow H(x) = o(x\log x)$

② $\psi(x)\sim x \Rightarrow H(x) = o(x\log x)$

③ $M(x)=o(x) \Rightarrow \psi(x)\sim x$.

$$A(x) = \sum_{n \le x} \frac{\mu(n)}{n}$$

Prop $A(x) \to 0 \iff$ TNP

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

$\mu * 1 = \delta$

ejercicio

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad \text{si } \operatorname{Re}(s) > 1$$

(Nota: si $s=1 \rightsquigarrow A(x) \to 0$!?)