Sumas parciales de μ. $\qquad M(x) = \sum_{n\le x} \mu(n) \overset{o(1)}{\text{"}} = o(x)$

Teo TNP $\Leftrightarrow$ $M(x) = o(x)$ $\left(\frac{M(x)}{x} \to 0 \right)$

$$H(x) = \sum_{n\le x} \mu(n) \log n$$

Prop $\lim \left(\frac{M(x)}{x} - \frac{H(x)}{x\log x} \right) = 0$

Dem Abel con $f(t) = \log t$

$$\underline{H(x)} = \underline{M(x)\log x} - \int_1^x \frac{M(t)}{t}\, dt$$

$$\Rightarrow \left(\frac{M(x)}{x} - \frac{H(x)}{x\log x} \right) = \frac{1}{x\log x} \int_1^x \underbrace{\frac{M(t)}{t}}_{=o(1)}\, dt \longrightarrow 0 .$$

Prop $\psi(x) \sim x \Rightarrow H(x) = o(x\log x)$

Dem Af: $-H(x) = \sum_{n\le x} \mu(n)\, \psi\left(\frac{x}{n}\right)$ ✓

Vimos $\Lambda(n) = -\sum_{d\mid n} \mu(d) \log d = 1 * (-\mu\cdot\log)$ $\left(\Lambda = \mu * \log \right)$

inversión $\Rightarrow$ $\underline{-\mu(n)\log n} = (\mu * \Lambda)(n)$

$$-H(x) = -\sum_{n\le x} \mu(n)\log n = \sum_{n\le x} \mu*\Lambda(n) = \sum_{n\le x} \mu(n)\, \psi\left(\frac{x}{n}\right).$$

$\psi(x) \sim x \rightsquigarrow$ dado $\varepsilon > 0$ $\exists A = A_\varepsilon \,/\, |\psi(x) - x| \leq \varepsilon x$ $\forall x \geq A$.

Si $x \geq A$ $\quad -H(x) = \sum_{n \leq y} \mu(n)\, \psi\big(\underbrace{\tfrac{x}{n}}_{\geq A}\big) + \sum_{y < n \leq x} \cdots$ $\qquad y = \left\lfloor \frac{x}{A} \right\rfloor$

$$\Big|\sum_{n \leq y}\Big| = \Big| \sum_{n \leq y} \mu(n)\frac{x}{n} + \sum_{n \leq y} \overbrace{\mu(n)}^{|\,|\leq 1} \underbrace{\left(\psi\left(\tfrac{x}{n}\right) - \tfrac{x}{n}\right)}_{|\,| \leq \varepsilon \frac{x}{n}} \Big|$$

$$\leq x \Big| \overbrace{\sum_{n \leq y} \frac{\mu(n)}{n}}^{\leq 1} \Big| + \varepsilon x \sum_{n \leq y} \frac{1}{n} \leq x + \varepsilon x (\log x + 1)$$

$$\Big| \sum_{n > y} \Big| = \Big| \sum_{y < n \leq x} \mu(n) \psi(A) \Big| \leq x \cdot \psi(A).$$

$$\Rightarrow |H| \leq x(1 + \varepsilon + \psi(A_\varepsilon)) + \varepsilon x \log x$$

$$\frac{|H(x)|}{x \log x} \leq \frac{2 + \psi(A_\varepsilon)}{\log x} + \varepsilon \longrightarrow \varepsilon$$

i.e. $\limsup \left| \frac{H(x)}{x \log x} \right| \leq \varepsilon \quad \forall \varepsilon > 0.$

$\Rightarrow$ lim existe y es 0.

<u>Prop</u> $M(x) = o(x) \Rightarrow \psi(x) \sim x$

<u>Dem</u> Af: $\psi(x) = x - \sum_{qd \leq x} \mu(d) f(q) + O(1)$

<u>Donde</u> $f(n) = d(n) - \log n - 2\gamma$

$\lfloor x \rfloor = \sum_{n \leq x} 1$ $\qquad \psi(x) = \sum_{n \leq x} \Lambda(x)$ $\qquad 1 = \sum_{n \leq x} \delta(x)$

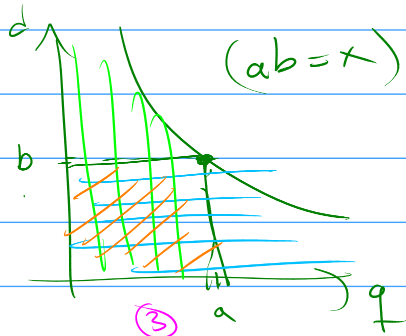
$\rightarrow 1 = \mu * d$ $\qquad \Lambda = \mu * \log$ $\qquad \delta = \mu * 1$

$$\underbrace{\lfloor x\rfloor}_{x+O(1)} - \psi(x) - 2\underbrace{\gamma}_{O(1)} = \sum_{n\le x}\left[1 - \Lambda(n) - 2\gamma\,\delta(n)\right]$$

$$= \sum_{n\le x}\sum_{d\mid n}\mu(d)\left(d\left(\tfrac{n}{d}\right) - \log\left(\tfrac{n}{d}\right) - 2\gamma\right)$$

$n = d\cdot q$

$$= \underbrace{\sum_{d\cdot q\le x}\mu(d)\,f(q)}_{} = \circledast$$



$$\circledast = \overbrace{\sum_{d\le b}\mu(d)F\left(\frac{x}{d}\right)}^{1} + \overbrace{\sum_{q\le a}f(q)M\left(\frac{x}{q}\right)}^{2} - \overbrace{F(a)M(b)}^{3}$$

Donde $F(x) = \sum_{n\le x} f(n)$

$M(x) = \sum_{n\le x}\mu(n)$

$$\sum d(n) = x\log x + (2\gamma-1)x + O(\sqrt{x})$$

$$(-1)\sum \log n = x\log x - x + O(\log x)$$

$$(-1)\sum 2\gamma = 2\gamma x$$

$$F(x) = O(\sqrt{x})$$

$|F(x)| \le B\sqrt{x} \quad \forall x\ge 1$ $\quad (\exists A > B)$

$$1\quad \left|\sum_{d\le b}\underbrace{\mu(d)}_{|\,|\le 1}F\left(\frac{x}{d}\right)\right| \le B\sum_{d\le b}\left(\frac{x}{n}\right)^{1/2} \le A\cdot x^{1/2}\cdot b^{1/2} = \frac{Ax}{a^{1/2}} < \varepsilon x$$

$$\left(\sum_{n\le b} n^{-1/2} = O(b^{1/2})\right)$$

Dado $\varepsilon > 0$, elegimos $\underset{(a_\varepsilon)}{a} \ge 1$ tq $\dfrac{A}{a^{1/2}} < \varepsilon$

$$2\quad M(x) = o(x) \to |M(x)| < \frac{\varepsilon}{K}x \quad \forall x \ge X_K \quad (K > 0 \text{ arbitrario})$$

$$\left|\sum_{q\le a} f(q)\, M\!\left(\frac{x}{q}\right)\right| \le \sum_{q\le a} |f(q)| \cdot \frac{\varepsilon}{k}\frac{x}{q} = \frac{\varepsilon x}{k} \overbrace{\sum_{q\le a} \frac{|f(q)|}{q}}$$

$\forall x \ge a \cdot x_k$

elegimos $k = \sum_{q\le a} \frac{|f(q)|}{q} \rightsquigarrow \le \varepsilon x$

(3) $|F(a) M(b)| \le B \cdot \sqrt{a}\, \underbrace{|M(b)|}_{\frown b} < A\sqrt{a} \cdot b < \varepsilon ab = \varepsilon x.$

$|x - \psi(x) + O(1)| \le |(1)| + |(2)| + |(3)| < 3\varepsilon x$

$\Rightarrow \left|1 - \frac{\psi(x)}{x}\right| < 3\varepsilon \quad \forall x > a \cdot x_k$

Vale $\forall \varepsilon > 0 \Rightarrow \frac{\psi(x)}{x} \to 1$. #

$$A(x) = \sum_{n\le x} \frac{\mu(n)}{n}$$

Prop $A(x) = o(1) \iff$ TNP

En Apostol está $(\Rightarrow)$ (Ejercicio)

$\left(\text{Abel} \Rightarrow \frac{M(x)}{x} = A(x) - \underbrace{\frac{1}{x}\int_1^x A(t)\,dt}_{o(x)}\right)$

El Teorema de los números primos

Teo $\psi(x) \sim x$

$$\log n = \sum_{dq=n} \Lambda(d) \qquad (\Lambda = \mu * \log)$$

$$T(x) = \sum_{n\le x} \log n = \sum_{dq\le x} \Lambda(d) = \sum_{q\le x} \psi\left(\frac{x}{q}\right)$$

(Chebyshev)

inversión

$$\psi(x) = \sum_{n\le x} \mu(n)\, T\left(\frac{x}{n}\right)$$

$$\tilde{G}(x) = \sum_{n\le x} \tilde{F}\left(\frac{x}{n}\right)$$

¿Asintótica?

$$\tilde{F}(x) = \sum_{n\le x} \mu(n)\, \tilde{G}\left(\frac{x}{n}\right)$$

$$\psi(x) - \tilde{F}(x) = \sum_{n\le x} \mu(n) \underbrace{\left[T\left(\frac{x}{n}\right) - \tilde{G}\left(\frac{x}{n}\right)\right]}_{\text{¿pequeño?}}$$

① $\tilde{F}_1(x) = x \rightsquigarrow \tilde{G}_1(x) = \sum \frac{x}{n} = x\log x + \gamma x + O(1)$

$\qquad T(x) = x\log x - x + O(\log x)$

② $\tilde{F}_2(x) = x - C \rightsquigarrow \tilde{G}_2(x) = \tilde{G}_1(x) - C\lfloor x\rfloor = x\log x + (\gamma - C)x + O(1)$

③ $\boxed{F(x) = \psi(x) - x + \gamma + 1} \to G(x) = O(\log x)$

$$F(x) = \sum_{n\le x} \mu(n)\, G\left(\frac{x}{n}\right)$$

Sabemos $G(x) = o(\log x)$

$\rightsquigarrow$ $F(x) = $ ¿? $\quad \cancel{o(x)}$

Aún si fuera $G(x) = o(1) \longrightarrow F(x) = O(x)$

(No sale $o(x)$ sin conocer $M(x) = o(x)$)

$(\alpha < 1)$

Sup $G(x) \le K \cdot x^{1/2}$ (mucho menos de lo que sabemos)

$$\Rightarrow \sum_{n \le x} \mu(n) \, G\left(\frac{x}{n}\right) \le K x^{1/2} \sum_{n \le x} n^{-1/2} = o(x) \; !$$

(en part $\psi(x) = o(x)$)

Idea: "agrandar" G.

$$J(x) = \sum \mu(n) \left[\log \frac{x}{n} \cdot G\left(\frac{x}{n}\right)\right] = o(x)$$

Lema $G(x) = \sum_{n \le x} F(x/n)$ $\quad \left(\Leftrightarrow F(x) = \sum_{n \le x} \mu(n) \, G(x/n)\right)$

$$\Rightarrow J(x) = \sum \mu(n) \log\left(\frac{x}{n}\right) G\left(\frac{x}{n}\right) = F(x) \log x + \sum_{n \le x} F\left(\frac{x}{n}\right) \Lambda(n)$$

Dem Jueves.

Teo (Selberg)

$$\psi(x) \log x + \sum_{n \le x} \Lambda(n) \, \psi(x/n) = 2x \log x + O(x)$$

Dem aplicar el lema a $F(x) = \psi(x) - x + \gamma + 1$

$$\left[\psi(x) - \boxed{x} + O(1)\right]\boxed{\log x} + \sum_{n\le x}\left[\psi\left(\frac{x}{n}\right) - \boxed{\frac{x}{n}} + O(1)\right]\boxed{\Lambda(n)} = O(x)$$

$\sum \Lambda(n) = \psi(x) = O(x)$

$$\sum \frac{x}{n}\Lambda(n) = x\sum \frac{\Lambda(n)}{n} = x\left(\log x + O(1)\right)$$

$$\Rightarrow \underbrace{\psi(x)\log x}_{\sim x\log x} + \underbrace{\sum \psi\left(\frac{x}{n}\right)\Lambda(n)}_{\sim x\log x} = 2x\log x + O(x).$$

#

Esperamos TNP $\Rightarrow$

$$\sum_{n\le x}\Lambda(n)\log n \qquad ? \qquad \sum_{p^\alpha q^\beta\le x}\log p\cdot\log q$$

$$\sum_{p\le x}\log^2 p$$

$$\sum_{n\le x}\Lambda_2(x) = 2x\log x + O(x)$$

$\Lambda_2(n)$ $= \Lambda(n)\log n + \sum_{dq=n}\Lambda(d)\Lambda(q)$, $\Lambda_2 = \mu * \log^2 = \Lambda\cdot\log + \Lambda*\Lambda$

cuenta productos de dos primos