Lema $G(x) = \sum_{n\le x} F(x/n)$ $\quad (G = 1\circ F \leftrightarrow F = \mu\circ G)$

$$\Rightarrow \quad J(x) := \sum \mu(n)\log\left(\tfrac{x}{n}\right) G\left(\tfrac{x}{n}\right) = F(x)\log x + \sum_{n\le x} F\left(\tfrac{x}{n}\right)\Lambda(n)$$

Dem $J = \mu\circ \grave{G} = \mu\circ \grave{(1\circ F)} = \mu\circ(1\circ \grave{F} + \grave{1}\circ F)$

$$= \underbrace{(\mu * 1)}_{\delta}\circ \grave{F} + \overbrace{(\mu * \grave{1})}^{\Lambda}\circ F = \grave{F} + \Lambda\circ F$$

$F(x) = \psi(x) - x + \gamma + 1 \rightsquigarrow G = O(\log x)$, $\grave{G} = O(\log^2 x)$

$\Rightarrow \quad J = O(x) \rightsquigarrow \grave{F} + \Lambda\circ F = O(x)$

$$\begin{cases} \grave{F} = \grave{\psi} - x\log x + O(\log x) \\ \Lambda\circ F = \Lambda\circ\psi - x\log x + O(x) \end{cases}$$

$\sum \frac{\Lambda(n)}{n} = \log x + O(1)$

$\sum \Lambda(n) = O(x)$

Teo (Selberg) $\grave{\psi} + \Lambda\circ\psi = 2x\log x + O(x)$

Obs: $\sum \Lambda(n)\log n = \overbrace{\psi(x)\log x}^{\grave{\psi}} - \underbrace{\int_1^x \frac{\psi(t)}{t}\,dt}_{O(x)}$

$\Lambda\circ\psi = \Lambda\circ(\Lambda\circ 1) = (\Lambda * \Lambda)\circ 1$

$\Lambda_2 = \grave{\Lambda} + \Lambda * \Lambda = \mu * \log^2$ $\longrightarrow$ $\begin{cases} \Lambda_2(p) = \log^2 p \\ \Lambda_2(pq) = \log p \cdot \log q \end{cases}$

$$\boxed{\psi_2(x) = \sum_{n\le x} \Lambda_2(x) = 2x\log x + O(x)}$$

otra: $\sum_{p\le x} \log^2 p + \sum_{pq\le x} \log p\cdot\log q = 2x\log x + O(x)$

$\boxed{\text{ej } 5+B}$

En general $\Lambda_k = \mu * \log^k$ $\Big\}$ "productos de k primos"

$$\psi_k(x) = \sum_{n\le x} \Lambda_k(n)$$

$$\psi_k(x) \sim k\cdot x\cdot \log^{k-1} x$$

($k=1$ es TNP
$k=2$ fórmula de Selberg

Nota: ej 63: $1*\mu=\delta$ $\quad \zeta(s)\cdot\sum \mu(n)n^{-s} = 1$

$\log = 1'$ $\to$ $\sum \log(n)n^{-s} = -\zeta'(s)$

$\Lambda = \mu * \log$

$$\to \boxed{\sum \Lambda(n)\, n^{-s} = -\zeta'(s)/\zeta(s).}$$

(von Mangoldt) (1895) $\to$ $$\psi(x) = \sum_{n\le x}\Lambda(n) = \underbrace{x}_{\text{polo de } \zeta} - \sum \frac{x^\rho}{\underbrace{\rho}_{\rho \text{ cero de } \zeta}} + \cdots \qquad x^\rho \to O(x^{\operatorname{Re}\rho})$$

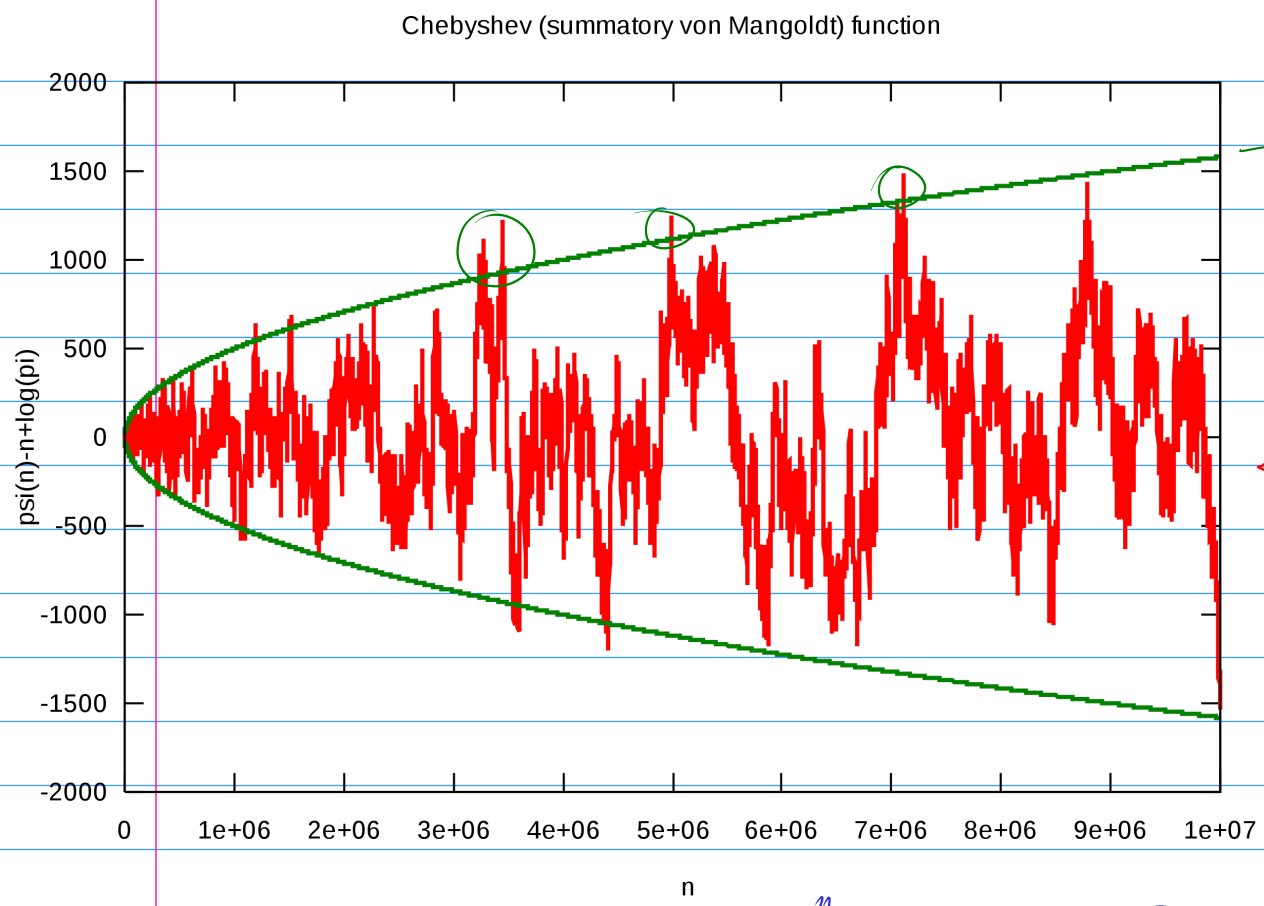
(Para $o(x)$ precisamos que ζ no tenga ceros con $\operatorname{Re} s = 1$.

$$\sum \Lambda_2(n)\, n^{-s} = \zeta''(s)/\zeta(s)$$

$$\psi_2(x) = \sum \Lambda_2(n) = 2x\log x + \overbrace{\sum \frac{\zeta''(\rho)}{\zeta(\rho)}\frac{x^\rho}{\rho}}^{o(x)} + \cdots$$

Basta que ζ no tiene ceros $\operatorname{Re} s > 1$.

$$R(x) = \psi(x) - x \qquad x \geq 2 \qquad (0 \quad x<2)$$



$\to \frac{1}{2}\sqrt{x}$

$\leftarrow R(x)$

TNP: $\boxed{R(x) = o(x)}$ Queremos. // H.R: $R(x) = O(x^{\frac{1}{2}+\varepsilon})$

No se conoce $O(x^{\alpha})$ $\alpha<1$

Selberg $\to$ $R' + \Lambda \circ R = O(x)$

Levinson 1969: $R(x)\log x + \sum \Lambda(n)\, R(\frac{x}{n}) = O(x)$

Lema 1 $\quad R'' = (\Lambda_2 - 2\Lambda') \circ R + O(x\log x)$

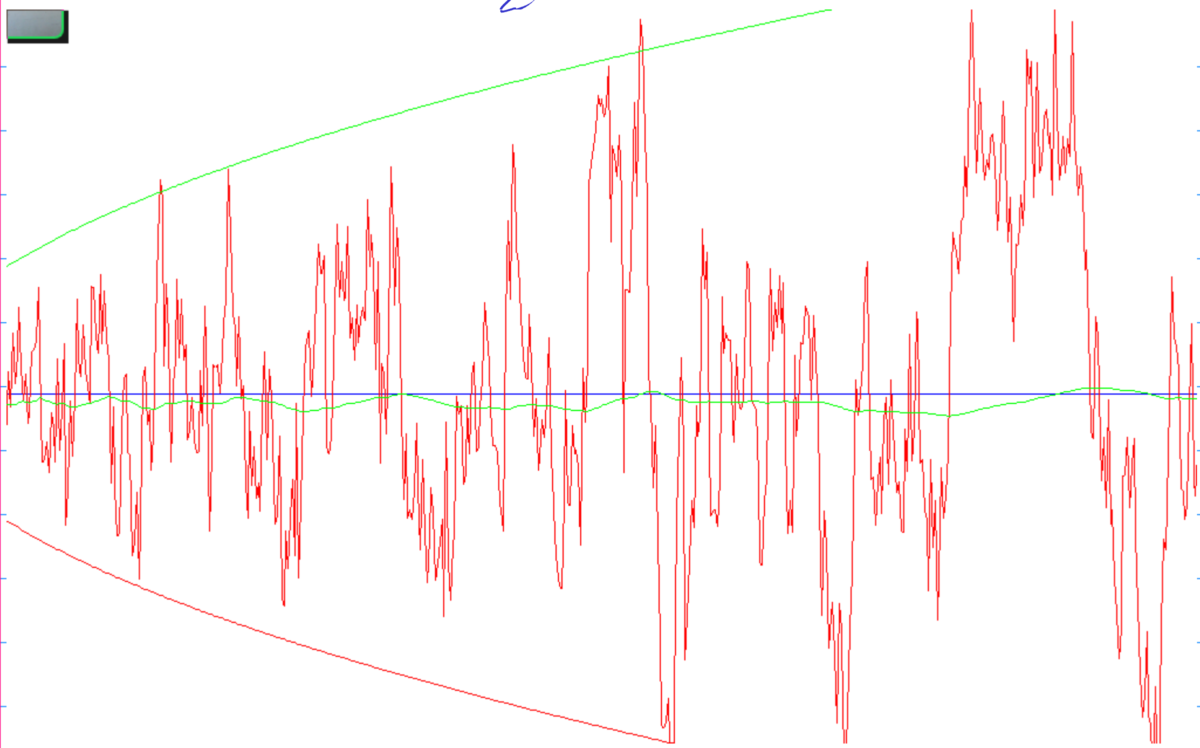
Dem $\quad R' = -\Lambda\circ R + O(x) \to R'' = -\underline{\Lambda\circ R'} - \Lambda'\circ R + O(x\log x)$

$$-\Lambda\circ R' = \Lambda\circ\Lambda\circ R + \underbrace{\Lambda\circ O(x)}_{O(x\log x)} = (\Lambda*\Lambda)\circ R + O(x\log x)$$

$\Lambda_2 = \Lambda*\Lambda + \Lambda'$

$$R'' = \dots = (\Lambda*\Lambda - 2\Lambda')\circ R + O(x\log x)$$

Sumatoria $S(y) = \int_2^y \frac{R(x)}{x}\,dx \qquad (y>2, 0)$



Veremos que $S(y) = o(y) \rightsquigarrow R(x) = o(x)$.

Lema 2 S es Lipschitz $\left(\exists c : |S(y_1)-S(y_2)| \le c|y_1-y_2|\right)$

Dem $\psi(x) = O(x) \rightsquigarrow |R(x)| \le cx$

$S' = R(x)/x \Rightarrow S$ Lipschitz "a trozos" y continua #

Lema 3 $S'' = (\Lambda_2 - 2\hat{\Lambda}) \circ S + O(x \log x)$

Dem $R'' = (\Lambda_2 - 2\hat{\Lambda}) \circ R + O(x \log x)$

$$\left(\underbrace{\int_2^y \frac{R''(x)}{x}\,dx}_{=} + \sum (\Lambda_2 - 2\hat{\Lambda})(n) \underbrace{\int_2^y \frac{R(x/n)}{x/n} \cdot \frac{dx}{n}}_{S(y/n)}\right.$$

$$\left(S(y)\log^2 y - 2\int \frac{\overbrace{S(x)}^{=o(x)} \log x}{x}\,dx\right.$$

Lema 4 $\quad |S(y)|\log^2 y \le \sum \Lambda_2(m)\left|S\left(y/m\right)\right| + O(y\log y)$

Dem $\quad 2\hat{\Lambda}\geq 0$ + desigualdad Δ.

Recuerdo $\quad \sum \Lambda_2 = 2x\log x + O(x) \to \overbrace{\sum_{1\le n\le x}(\Lambda_2(n)-2\log n)}^{Q(x)} = O(x)$

Lema 5 (A) $|S(y)|\log^2 y \le 2\sum \log m \cdot \left|S(y/m)\right| + O(y\log y)$

(B) $|S(y)|\log^2 y \le 2\int_2^y \log u \left|S(y/u)\right| du + O(y\log y)$

Dem. (A) $\sum(\Lambda_2(m) - 2\log m)\left|S(y/m)\right|$

$$= \sum (Q(m)-Q(m-1))\,|S(y/m)|$$

$$= \sum \underbrace{Q(m)}_{O(m)} \underbrace{\left(\left|S\left(\tfrac{y}{m}\right)\right| - \left|S\left(\tfrac{y}{m+1}\right)\right|\right)}_{\text{Lipschitz} \le c\left|\left(\frac{y}{m}\right)-\left(\frac{y}{m+1}\right)\right|}$$

$$= O\left(\sum m\left(\frac{y}{m} - \frac{y}{m+1}\right)\right) = O\left(y\sum\frac{1}{m+1}\right)$$

$$= O(y\log y)$$

(B) .. (1) log creciente
(2) S Lipschitz . . . .

$W(x) := \dfrac{S(e^x)}{e^x}$ $\quad (x=\log y)$ $\quad$ (S Lipschitz) $\quad |W(x)|\le c$

$\int_v^x du$

$$|W(x)| \le \frac{2}{x^2}\underbrace{\int_0^x (x-v)\,|W(v)|\,dv}_{\downarrow} + O\left(\frac{1}{x}\right)$$

$$\int_0^x \left(\frac{1}{u}\int_0^u |w(v)|\,dv\right) u\,du$$

$$\tilde{w}(x) = \frac{1}{x}\int_0^x |w(v)|\,dv$$

$$\boxed{|w(x)| \leq \frac{\int_0^x \tilde{w}(u)\cdot u\,du}{\int_0^x u\,du}} \quad (**)$$

$$\alpha = \limsup |w(x)| \quad , \quad \gamma = \limsup \tilde{w}(x)$$

Prop ① $\alpha < \infty$ ($\varphi(x) = o(t)$) ; $\alpha \leq \gamma$ ($**$)

② w Lipschitz

③ $\exists M : \int_{v_1}^{v_2} |w(v)|\,dv \leq M$

siempre que $w(v) \neq 0$ en (v_1, v_2).

Lema ① + ② + ③

$\Rightarrow \alpha = 0$

$$\boxed{\text{LEVINSON} \quad 1969}$$