Series de Dirichlet

Davenport 7-16 // Apostol 11-12.

Leer cap 7 [D] como "repaso"

$$c_1 \le \frac{\pi(x)\log x}{x} \le c_2$$

$\log 2$ $\quad 2\cdot\log 2$

$0.92\ldots$ $\quad 1.10\ldots$

Prop $\displaystyle\sum_{m|n} \Lambda(m) = \log n$

derivada logarítmica

$$\frac{d}{ds}\log f(s) = \frac{f'(s)}{f(s)}$$

Dem $\displaystyle\zeta(s) = \prod_p (1-p^{-s})^{-1}$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{1-p^{-s}} = \sum_p\sum_i \log p \cdot p^{-si}$$

$\boxed{\operatorname{Re} s \gg 0}$

$$= \sum_n \Lambda(n)\, n^{-s}.$$

$$\underbrace{\left(\sum \Lambda(n)n^{-s}\right)\zeta(s)}_{"} = -\zeta'(s) = \sum \log n \cdot n^{-s}$$

Ej 63

$$\sum (\Lambda * 1)(n)\, n^{-s} \longrightarrow \Lambda * 1 = \log \quad ✓$$

Prop (Mertens) ① χ car. Dirichlet no principal

$$\Rightarrow \sum_p \frac{\chi(p)}{p} \text{ converge.}$$

$(a,q)=1$ $\Rightarrow$

② $$\sum_{\substack{p\le x\\ p\equiv a \,(\mathrm{mod}\, q)}} \frac{1}{p} = \frac{1}{\varphi(q)}\log\log q + A(q,a) + O\left(\frac{1}{\log x}\right)$$

$$\left(\frac{\sum_{\substack{p\le x\\ p\equiv a}} \frac{1}{p^s}}{\sum_{p\le x}\frac{1}{p^s}} \xrightarrow{x\to\infty} f(s) \xrightarrow{s\to 1} \frac{1}{\varphi(q)} \right)$$

D.B **Riemann** 1859

Demuestra (a) $\zeta(s)$ tiene cont. analítica a $\mathbb{C}$ con un único polo en $s=1$, residuo 1.

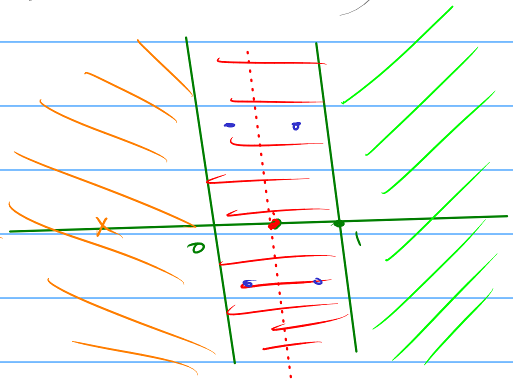
(b) $\hat{\zeta}(s) := \underbrace{\pi^{-s/2}\,\Gamma(s/2)}_{\text{"factor de Euler en } \infty\text{"}}\,\zeta(s) \Rightarrow \hat{\zeta}(1-s) = \hat{\zeta}(s)$

(Γ: leer Ap 12.2 / D io / wikipedia)

Consecuencias: Ceros de $\zeta(s)$ con $\mathrm{Re}\, s<0$

$\Longleftrightarrow$ polos de $\Gamma(s/2)$

$s=-2,-4,-6,\ldots$ ceros triviales



$\{s : 0 \le \mathrm{Re}\, s \le 1\}$ ← banda crítica

$(\zeta(0) = -\tfrac{1}{2})$

Riemann conjeturó ① ζ tiene ∞ ceros en la banda crítica.

Hadamard 1893 (simétricos +)

② $N(T) = \#\{s : \zeta(s)=0,\ 0 \le \mathrm{Re}\, s \le 1,\ 0 \le \mathrm{Im}\, s \le T\}$

VM 1895/1905 $= \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$

③ $\xi(s) = \frac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$ es entera y:

Hadamard 1893 $$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$ (ρ ceros no triviales de $\zeta(s)$)

④ Fórmula explícita para $\pi(x)$ en función de los ceros de $\zeta(s)$ ...

VM 1895

$$\psi(x) = x - \underbrace{\sum_{\rho}}_{\text{sumar } \rho,\bar{\rho} \text{ conjuntamente}} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}) + \frac{1}{2}\Lambda(x)$$

⑤ $\mathrm{Re}\,\rho = \frac{1}{2}$ $\forall$ cero no trivial ρ. ⟸ $\boxed{\text{Hipótesis de Riemann}}$

(a) + (b) $\Gamma(s/2) = \int_0^\infty e^{-t} t^{s/2} \frac{dt}{t}$. **Re s > 0**

$t = n^2 \pi x$

conv. abs

$$\pi^{-s/2}\Gamma(s/2)\, n^{-s} = \int_0^\infty x^{s/2} e^{-n^2\pi x} \frac{dx}{x}$$

$$\hat{\xi}(s) = \pi^{-s/2}\Gamma(s/2)\,\xi(s) = \int_0^\infty x^{s/2}\,\omega(x)\,\frac{dx}{x}$$

Donde $\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$

$\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$ función theta.

$(\Theta = 2\omega + 1)$.

Γ: la $\int$ converge Re s > 0.

- ⊗ partes $\Gamma(z+1) = z\,\Gamma(z)$ cont. analítica a $\mathbb{C}$
- ⊗ $\Gamma(1) = 1$, $\Gamma(n+1) = n!$
- ⊗ polos en $z = 0, -1, -2, \ldots$
- ⊗ no tiene ceros en $\mathbb{C}$

Prop $\Theta(1/x) = x^{1/2}\Theta(x)$ $\forall x > 0$

$$\rightarrow \quad \omega(1/x) = -\frac{1}{2} + \frac{x^{1/2}}{2} + x^{1/2}\omega(x)$$

$$\int_0^\infty x^{s/2}\omega(x)\frac{dx}{x} = \int_1^\infty x^{s/2}\omega(x)\frac{dx}{x} + \int_1^\infty x^{-s/2}\,\omega(1/x)\,\frac{dx}{x}$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \left(x^{s/2} + x^{\frac{1-s}{2}}\right)\omega(x)\frac{dx}{x}$$

$$\boxed{\hat{\xi}(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{s/2} + x^{\frac{1-s}{2}}\right)\omega(x)\frac{dx}{x}}$$

$\int$ converge abs, unif en compactos $\rightsquigarrow$ función entera.

+ ec. funcional $s \leftrightarrow 1-s$. #

$\hat{\xi}(s)$ polos simples residuo 1 en $s = 0$, $s = 1$.

Notas:

① $\pi^{-s/2}\Gamma(s/2)\neq 0$, $\Gamma(0)$ polo $\Big\} \to \xi(s)$ único polo en $s=1$.

② $\Gamma(1/2) = \pi/2 \to \xi(1)$ residuo 1.

③ $\Gamma(0)$ residuo $1 \to \xi(0) = -1/2$

④ $\xi(s) < 0$ para $0<s<1$, $s\in\mathbb{R}$ (ejercicio).

Prop

$$\sum_{n=-\infty}^{\infty} e^{-(n+\alpha)^2\pi/x} = x^{1/2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi x + 2\pi i n\alpha}$$

(precisamos $\alpha=0$)

Dem Poisson:

$x>0$

$$\sum_{n=-N}^{N} e^{-(n+\alpha)^2\pi/x} = \sum_{v=-\infty}^{\infty}\int_{-N}^{N} e^{-(t+\alpha)^2\pi/x}\cdot e^{2\pi i v t}\,dt$$

$(N\to\infty)$

$$\sum_{-\infty}^{\infty} e^{-(n+\alpha)^2\pi/x} = \sum_{v=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t+\alpha)^2\pi/x}\cdot e^{2\pi i v t}\,dt$$

$$u=\frac{t+\alpha}{x}\Big] = x\sum_{v=-\infty}^{\infty} e^{-2\pi i v\alpha}\int_{-\infty}^{\infty} e^{\boxed{-u^2\pi x + 2\pi i v u x}}\,du$$

$$-\pi x(u+iv)^2 - \pi x v^2$$

$$= x\sum_{v} e^{-2\pi i v\alpha - \pi x v^2}\underbrace{\int_{-\infty}^{\infty} e^{-\pi x(u+iv)^2}\,du} = (**)$$

$$\int_{-\infty}^{\infty} e^{-\pi x(u+\beta)^2}\,du = \int_{-\infty}^{+\infty} e^{-\pi x v^2}\,dv = A x^{-1/2}$$

$$\left(A=\int_{-\infty}^{\infty} e^{-\pi v^2}\,dv>0\right)$$

$$\sum_n e^{-(n+\alpha)^2 \pi/x} = A x^{1/2} \sum_\nu e^{-2\pi i \nu \alpha - \pi x \nu^2}$$

- using $\alpha = 0$ dos veces $\rightsquigarrow A^2 = 1 \Rightarrow A = 1$ #