Ec funcional para funciones L de Dirichlet

$e(x) = e^{2\pi i x}$

E.F. sólo χ primitivo.

Sea χ mód q. $\qquad \tau(\chi) = \sum_{m=1}^{q} \chi(m) \cdot e\left(\frac{m}{q}\right)$ $\qquad e_q(m)$

$\boxed{(n,q)=1} \Rightarrow \boxed{\chi(n)\,\tau(\bar\chi)} = \sum_{i=1}^{q} \bar\chi(i)\,\chi(n)\, e\left(\frac{i}{q}\right)$

$\chi : \left(\mathbb{Z}/q\right)^{\times} \to S^1 \subseteq \mathbb{C}^{\times}$

$\chi \cdot \bar\chi = 1$

$i \equiv n \cdot h$

$$\boxed{= \sum_{h=1}^{q} \bar\chi(h)\, e\left(\frac{n \cdot h}{q}\right)}.$$

Prop χ primitivo $\Rightarrow \chi(n)\,\tau(\bar\chi) = \sum_{h=1}^{q} \bar\chi(h)\, e\left(\frac{nh}{q}\right)$

$\forall n$.

Dem si $(n,q)=1$ ya lo probamos.

Sea $\frac{n}{q} = \frac{n_1}{q_1}$ con $(n_1, q_1) = 1$. $\qquad \boxed{q_1 < q}$

$\boxed{\chi(n) = 0}$ hay que ver que el lado derecho se anula.

$q = q_1 \cdot q_2 \qquad h = u q_1 + v \qquad 0 \le u < q_2, \quad 1 \le v \le q_1$

$$\sum \bar\chi(h)\, e\left(\frac{nh}{q}\right) = \sum_{v} \underbrace{\left(\sum_{u} \bar\chi(u q_1 + v)\right)}_{S(v) \text{ periódica mód } q_1} \cdot e\left(\frac{n_1 v}{q_1}\right).$$

Sup $(c,q)=1$

$$\bar\chi(c)\, S(v) = \sum_{u \bmod q_2} \bar\chi(c u q_1 + c v) = \sum_{u} \bar\chi(u q_1 + c v) = S(cv).$$

Si $S(\omega) \neq 0 \to \overline{\chi}(c) = \frac{S(c\omega)}{S(\omega)}$ periódico mod q_1 (χ primitivo)

$\Rightarrow S(\omega) = 0 \ \forall \omega$ #.

Prop $\tau(\overline{\chi}) = \overline{\chi(-1)\,\tau(\chi)}$, con $\chi(-1) = \pm 1$ (ejercicio 76)

Prop χ primitivo mod $q \Rightarrow |\tau(\chi)| = q^{1/2}$.

Dem $\tau(\chi)^2 = \cdots = q$ (funciona si q es primo)

$$|\chi(n)|^2 |\tau(\chi)|^2 = \overline{\chi}(n)\tau(\chi) \cdot \chi(-n)\tau(\overline{\chi})$$

con $\tau(\overline{\chi}) = \overline{\chi(-1)\tau(\chi)}$; $|\chi(n)|^2$ es 0 si $(n,q)>1$, 1 si $(n,q)=1$

$$= \left[\sum_{h_1} \chi(h_1)\, e\left(\frac{n \cdot h_1}{q}\right)\right] \cdot \left[\sum_{h_2} \overline{\chi}(h_2)\, e\left(\frac{-n h_2}{q}\right)\right]$$

$$= \sum_{h_1, h_2} \chi(h_1)\,\overline{\chi}(h_2) \cdot \underbrace{e\left(\frac{n(h_1-h_2)}{q}\right)}$$

$\sum_n \left(\cdot\right) = \begin{cases} q & \text{si } h_1 \equiv h_2 \\ 0 & \text{si } h_1 \not\equiv h_2 \end{cases}$

$\sum_n$

$$\varphi(q)\,|\tau(\chi)|^2 = q \sum_n \underbrace{\chi(h) \cdot \overline{\chi}(h)}_{|\chi(h)|} = q \cdot \varphi(q) \quad \#$$

Prop χ mod q inducido por χ_1 mod q_1, $q = q_1 \cdot r$

$\Rightarrow \tau(\chi) = \mu(r)\,\chi_1(r)\,\tau(\chi_1)$.

(ejercicio 77)

(e.g. $(r, q_1) > 1 \Rightarrow \tau(\chi) = 0$)

Corolario: χ primitivo mód q $\Longrightarrow$

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{i=1}^{q} \bar{\chi}(i)\, e\left(\frac{in}{q}\right) \quad \forall n$$

Obs: $\chi(-1)^2 = \chi(1) = 1 \Rightarrow \chi(-1) = \begin{cases} +1 & \chi \text{ es par} \\ -1 & \chi \text{ es impar} \end{cases}$

Caso χ par $(\chi \neq 1)$ $\quad \Gamma(s/2) = \int_0^\infty \cdots$

$\left(t = n^2 \cdot \pi \cdot x/q\right)$ $\quad \left(\frac{q}{\pi}\right)^{s/2} \Gamma(s/2)\, n^{-s} = \int_0^\infty e^{-n^2 \pi x/q}\, x^{s/2} \frac{dx}{x}.$

$$\hat{L}(s,\chi) := q^{s/2} \left(\frac{\Gamma(s/2)}{\pi^{s/2}}\right) L(s,\chi) = \int_0^\infty x^{s/2} \underbrace{\left(\sum_{n=1}^{\infty} \chi(n)\, e^{-n^2\pi x/q}\right)}_{\frac{1}{2}\psi(x,\chi)} \frac{dx}{x}$$

conductor $\quad$ "factor gamma"

$\chi(-n) = \chi(n)$

$\chi(0) = 0$

$$\psi(x,\chi) = \sum_{n=-\infty}^{\infty} \chi(n)\, e^{-n^2 \pi x/q}$$

serie Theta con caracter

$$\hat{L}(s,\chi) = \frac{1}{2} \int_0^\infty x^{s/2}\, \psi(x,\chi) \frac{dx}{x}$$

$$= \frac{1}{2}\int_1^\infty x^{s/2} \psi(x,\chi) \frac{dx}{x} + \frac{1}{2}\int_1^\infty x^{-s/2} \psi(x^{-1},\chi) \frac{dx}{x}$$

Prop $\quad \psi(1/x, \chi) = \frac{\tau(\chi)}{q^{1/2}} \cdot x^{1/2} \cdot \psi(x, \bar{\chi})$

(ejercicio 78)

$$\hat{L}(s,\chi) = \frac{1}{2}\int_1^\infty x^{s/2}\psi(x,\chi)\frac{dx}{x} + \frac{\tau(\chi)}{2q^{1/2}}\int_1^\infty x^{\frac{1-s}{2}}\psi(x,\bar{\chi})\frac{dx}{x}$$

$$\hat{L}(1-s,\bar{\chi}) = \frac{\tau(\bar{\chi})}{2q^{1/2}}\int_1^\infty x^{s/2}\psi(x,\chi)\frac{dx}{x} + \frac{1}{2}\int_1^\infty x^{\frac{1-s}{2}}\psi(x,\bar{\chi})\frac{dx}{x}$$

$$= \frac{\tau(\bar{\chi})}{q^{1/2}}\left[\frac{1}{2}\int x^{s/2}\psi(x,\chi)\frac{dx}{x} + \frac{1}{2}\frac{\tau(\chi)}{q^{1/2}}\int_1^\infty x^{\frac{1-s}{2}}\psi(x,\bar{\chi})\frac{dx}{x}\right]$$

$$\boxed{\tau(\chi)\,\tau(\bar{\chi}) = \chi(-1)\,|\tau(\chi)|^2 = q}$$

$$\boxed{\hat{L}(1-s,\bar{\chi}) = \frac{\tau(\bar{\chi})}{q^{1/2}}\,\hat{L}(s,\chi)}.$$

↳ constante de $|\,|=1$.
"root number"

(ejemplo: $\boxed{\chi \text{ cuadrático}, \quad \chi(-1)=1} \Rightarrow \tau(\chi) = q^{1/2}$

en ese caso $\hat{L}(1-s,\chi) = \hat{L}(s,\chi)$.

↳ χ_d con $d>0$.)

Caso χ impar $\qquad \chi(-n) = -\chi(n)$

$s/2 \rightsquigarrow \frac{s+1}{2}$. $\qquad \Gamma\left(\frac{s+1}{2}\right) = \int_0^\infty \cdots$

$$\left(\frac{q}{\pi}\right)^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) n^{-s} = \int_0^\infty n e^{-n^2\pi x/q} x^{\frac{s+1}{2}} \frac{dx}{x}$$

$$\hat{L}(s,\chi) = q^{\frac{s+1}{2}} \left(\frac{\Gamma\left(\frac{s+1}{2}\right)}{\pi^{\frac{s+1}{2}}}\right) L(s,\chi) = \frac{1}{2}\int_0^\infty x^{\frac{s+1}{2}} \psi_1(x,\chi) \frac{dx}{x}$$

Donde $\psi_1(x,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n) e^{-n^2\pi x/q}$.

Prop $\psi_1(x^{-1},\chi) = \frac{\tau(\chi)}{i q^{1/2}} \cdot x^{3/2} \cdot \psi_1(x,\bar{\chi})$

Dem ejer 79.

$$\hat{L}(s,\chi) = \frac{1}{2}\int_1^\infty \psi_1(x,\chi) x^{\frac{s+1}{2}} \frac{dx}{x}$$
$$+ \frac{1}{2} \cdot \frac{\tau(\chi)}{i q^{1/2}} \int_1^\infty \psi_1(x,\bar{\chi}) x^{1-s/2} \cdot \frac{dx}{x}$$

Ahora $\tau(\chi)\,\tau(\bar{\chi}) = -q$

$$\left(\frac{\tau(\chi)}{i q^{1/2}} \cdot \frac{\tau(\bar{\chi})}{i q^{1/2}} = 1 \right)$$

$$\hat{L}(1-s, \bar{\chi}) = \frac{\tau(\bar{\chi})}{i q^{1/2}} \cdot \hat{L}(s, \chi).$$

(De nuevo: $\left.\begin{array}{l}\chi \text{ cuadrático} \\ \chi(-1) = -1\end{array}\right\} \to \tau(\chi) = i q^{1/2}$).