Funciones enteras de orden 1

$f: \mathbb{C} \to \mathbb{C}$ holomorfa.

Si f es "meromorfa en ∞" $\to$ f es un polinomio.

$f(1/z) = \sum_{n=-n_0}^{\infty} a_n z^n$ | en particular tiene $< \infty$ ceros

$\iff |f(z)| = O(|z|^{\alpha})$ para z "grande"

Def ① f es de orden finito si $|f(z)| = O\left(e^{|z|^{\alpha}}\right)$ para $\alpha \in \mathbb{R}$.

Obs si f no es constante $\alpha > 0$

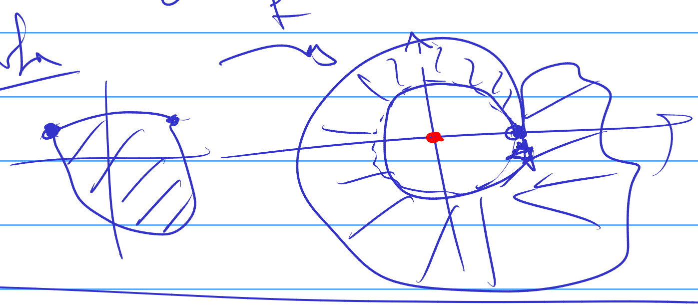
② el orden de f es el ínfimo de esos α.

Prop f entera de orden $\rho < \infty$, sin ceros en $\mathbb{C}$

$\Rightarrow f(z) = e^{g(z)}$ con g polinomio de grado ρ.

Dem Dado f, podemos definir $g(z) = \log f(z)$.

$g: \mathbb{C} \to \mathbb{C}$ $\quad g(z) = \int_{z_0}^{z} \frac{f'}{f}\, dz$ holomorfa



($\mathbb{C}$ es simplemente conexo)

$|e^{g(z)}| \leq C \cdot e^{|z|^{\alpha}} \Rightarrow \mathrm{Re}(g(z)) \leq \log C + |z|^{\alpha}$.

cte.

si $|z| = R$ (R grande)

$$\left| \operatorname{Re}(g(z)) \right| \leq 2R^{\alpha}$$

$$g(z) = \sum_{n=0}^{\infty} (a_n + i b_n) z^n \qquad a_n, b_n \in \mathbb{R}.$$

$z = Re^{i\theta}$

$z^n = R^n \cos n\theta + i R^n \operatorname{sen} n\theta$.

$$\hookrightarrow \operatorname{Re} g(Re^{i\theta}) = \underbrace{\sum a_n R^n \cos n\theta - \sum b_n R^n \operatorname{sen} n\theta}.$$

$\boxed{\operatorname{Sup}\ g(0) = 0}$

$$\int_0^{2\pi} \operatorname{Re} g(Re^{i\theta}) \begin{pmatrix} \cos n\theta \\ \operatorname{sen} n\theta \end{pmatrix} \leq \int_0^{2\pi} \left| \operatorname{Re} g(Re^{i\theta}) \right| d\theta$$

$$= \int_0^{2\pi} \left[\left| \operatorname{Re} g(Re^{i\theta}) \right| + \overbrace{\operatorname{Re} g(Re^{i\theta})} \right] d\theta$$

$|x| + x \leq 2x$ $\qquad \leq 8\pi R^{\alpha}$

$$\left. \begin{array}{l} \pi |a_n| R^n \\ \pi |b_n| R^n \end{array} \right\} \leq$$

Ahora $R \to \infty \implies a_n = b_n = 0 \quad \forall n > \alpha$

Conclusión: g es un polinomio de grado $\leq \alpha$.

Prop Fórmula de Jensen

$f(z)$ holomorfa en $|z| \leq R$, con ceros (con mult). $z_1, \dots, z_n \neq 0$, $|z_i| < R$.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)|$$

$$= \log\left(\frac{R^n}{|z_1| \cdots |z_n|}\right)$$

$$= \int_0^R r^{-1} \cdot n(r)\, dr$$

$n(r) = \#\{z_i : |z_i| \leq r\}$

Dem

$$f(z) = (z-z_1)(z-z_2)\cdots(z-z_n)F(z)$$

$F(z)$ sin ceros.

Probar para cada factor. (ejercicio)

ejercicio. $\int_0^R = 0\int_0^{r_1} + 1\int_{r_1}^{r_2} + \dots + n\int_{r_n}^R$

$(r_i = |z_i|)$ $\qquad \log(r_2/r_1) + \dots$

Sea f de orden ρ. $\alpha > \rho$

$$\log|f(Re^{i\theta})| < \tfrac{3}{2}R^{\alpha} \qquad (R \text{ grande})$$

$$\int_0^R r^{-1} n(r)\,dr < \tfrac{3}{2}R^{\alpha} - \overset{\text{cte}}{\log|f(0)|} < 2R^{\alpha}$$

$$2(2R)^{\alpha} > \int_R^{2R} r^{-1} n(r)\,dr \geq n(R)\int_R^{2R} r^{-1}\,dr = n(R)\cdot\log 2$$

$$\Rightarrow \boxed{n(R) = O(R^{\alpha})} \quad \forall \alpha > \rho.$$

En particular $\sum r_n^{-\beta} < \infty$ si $\beta > \rho$. (ejercicio)

Para simplificar $\boxed{\rho = 1}$.

$$\Rightarrow \sum |z_n|^{-2} \text{ converge.}$$

$$P(z) = \prod_{n=1}^{\infty}\left(1 - z/z_n\right) e^{z/z_n}$$

converge abs, unif en compactos que no contengan los $\{z_i\}$ (ejercicio)

"$P(z_i) = 0$"

¿? $P(z)$ es entera con ceros z_i.

$\Rightarrow$ $f(z) = P(z) \cdot F(z)$

con $F(z)$ entera <u>sin ceros</u>.

<u>Af</u> $F(z)$ tiene orden finito

<u>Idea</u>: acotar $|f(z)|$ "por abajo"

$\rightsquigarrow$ $|F(z)|$ por arriba.

<u>Fijemos</u> R (tal que $R \neq r_i$)

(suf grande)

<u>más aún</u>: $|R - r_n| > r_n^{-2}$ (*)

<u>Ejercicio</u> Si $\sum r_n^{-2}$ converge $\Rightarrow$

existen R <u>arbitrariamente grandes</u> que cumplen (*)

$$P_1(z) = \prod_{|z_n| < \frac{R}{2}} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

$$P_2(z) = \prod_{\frac{R}{2} \leq |z_n| \leq 2R} \qquad P_3(z) = \prod_{2R < |z_n|}$$

ip. $P(z) = P_1(z) \cdot P_2(z) \cdot P_3(z)$

Fijemos $\varepsilon > 0$

$|z| = R$

① $|z_n| < R/2$.

$$\left|\left(1-\frac{z}{z_n}\right) e^{z/z_n}\right| \geq \underbrace{\left(|z/z_n| - 1\right)}_{\geq 1} e^{-|z|/|z_n|} \geq e^{-R/r_n}$$

$$\sum_{r_n \leq \frac{R}{2}} r_n^{-1} \leq \left(\frac{R}{2}\right)^{\varepsilon} \sum_{r_n \leq \frac{R}{2}} r_n^{-1-\varepsilon} \leq \left(\frac{R}{2}\right)^{\varepsilon} \underbrace{\sum r_n^{-1-\varepsilon}}_{\leq R^{\varepsilon} \text{ (si } R \text{ suf grande)}}$$

$$|P_1(z)| > \exp\left(-R^{1+2\varepsilon}\right)$$

② $R/2 \leq |z_n| \leq 2R$.

$$\left|\left(\frac{z_n - z}{z_n}\right) e^{z/z_n}\right| \geq e^{-2} \frac{\overbrace{|z - z_n|}^{\geq r_n^{-2}}}{2R} > \underline{C_1 \cdot R^{-3}}.$$

$\#$ Factores en $P_2 \leq n(2R) = O(R^{1+\varepsilon})$

$$\Rightarrow |P_2(z)| > \left(C_1 \cdot R^{-3}\right)^{R^{1+\varepsilon}} > \exp\left(-R^{1+2\varepsilon}\right).$$

(3) $|z_n| > 2R$.

$$\Big|\underbrace{\left(1 - z/z_n\right)}_{> \frac{1}{2}} e^{z/z_n}\Big| > e^{-c_2 (R/r_n)^2}$$

$\left|\frac{z}{z_n}\right| < \frac{1}{2}$

$$\sum_{r_n > 2R} r_n^{-2} < (2R)^{-1+\varepsilon} \sum_{n=1}^{\infty} r_n^{-1-\varepsilon}$$

$$\Rightarrow |P_3(z)| > \exp\left(-R^{1+2\varepsilon}\right).$$

$$\Rightarrow |P(z)| > \exp\left(-R^{1+3\varepsilon}\right)$$

$$|f(z)| < \exp\left(R^{1+\varepsilon}\right)$$

$$F(z) < \exp\left(R^{1+4\varepsilon}\right)$$

$\Rightarrow F(z)$ es entera de orden 1, sin ceros

i.e. $F(z) = e^{A+Bz}$ $\quad A, B \in \mathbb{C}$.

Prop $f(z)$ es entera de orden 1, con ceros $\{z_n\}$

$\exists A, B \in \mathbb{C}$:
$$\boxed{f(z) = e^{A+Bz} \prod_{n=1}^{\infty} \left(1 - z/z_n\right) e^{z/z_n}}.$$

Además: $\sum r_n^{-1-\varepsilon}$ converge para $\varepsilon > 0$

Si $\sum r_n^{-1}$ converge $\Rightarrow |f(z)| < e^{c|z|}$ ($\forall z$)

(Rem: usar $|(1-x)e^{x}| \leq e^{2|x|}$)