D cap 12

Productos infinitos para $\xi(s)$, $\xi(s,\chi) = \hat{L}(s,\chi)$

$\hat{\zeta}(s) \cdot \frac{s(s-1)}{2}$

$$\xi(s) := \underbrace{\tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}}\ \underbrace{\Gamma(s/2)}\ \underbrace{\zeta(s)}.$$

$\Gamma(\frac{s}{2}+1)$

Prop $|\xi(s)| < \exp(c\,|s| \log|s|) = o(e^{|s|^{1+\varepsilon}})$

Dem podemos suponer $\operatorname{Re} s \geq 1/2$ $\quad (\xi(1-s) = \xi(s))$.

(A) $\left|\frac{1}{2} s(s-1)\pi^{-s/2}\right| < \exp(C_1 |s|)$ ✓

(B) $|\Gamma(s/2)| < \exp(C_2 |s| \log |s|)$

Stirling $\left(\log \Gamma(z) = z\log z - z + O(\log z)\right)$ $\operatorname{Re} z > 0$

(C) $\zeta(s) = \frac{s}{s-1} - s \underbrace{\int_1^\infty \{x\}\, x^{-s-1}\, dx}_{1.1 \leq \dots \leq \frac{1}{\operatorname{Re} s} < 2.} = O(|s|)$

(clase 5)

Obs: s real $\to +\infty$ $\quad \zeta(s) \to 1$, $\quad \log \Gamma(s/2) \sim s \log s$

$\leadsto |\xi(s)| \neq o(e^{c|s|})$

$= o(e^{|s|^{1+\varepsilon}})$

Ej: $\sum |\rho_n|^{-1}$ converge $\Rightarrow \not\exists\ |f(z)| < e^{C|z|}$.

Cor $\xi(s)$ tiene ∞ ceros $\rho_1, \rho_2, \dots$ tq

$\sum |\rho_n|^{-1-\varepsilon}$ converge $\forall \varepsilon > 0$, $\quad \sum |\rho_n|^{-1}$ no converge!

$$\xi(s) = e^{A+Bs} \prod_\rho (1 - s/\rho)\, e^{s/\rho}.$$

$\hookrightarrow$ ceros de $\zeta(s)$ no triviales

Ej: f cero ó polo

$\updownarrow$

f'/f polo simple $\operatorname{Res} = \operatorname{ord}(a)$

Der log

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \qquad s\Gamma(s)=\Gamma(s+1)$$

$$\| \quad \underbrace{\frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)}}_{\frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)}} + \frac{\zeta'(s)}{\zeta(s)}$$

$$\rightsquigarrow \frac{\zeta'(s)}{\zeta(s)} = B + \frac{1}{2}\log\pi \underbrace{- \frac{1}{s-1}}_{\text{polo } s=1} + \underbrace{\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)}_{\text{ceros no triviales}} \underbrace{- \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)}}_{\text{ceros triviales}}$$

$$-\frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} = \frac{1}{2}\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right)$$

$$\frac{1}{s\Gamma(s)} = e^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}$$

fórmula de Weierstrass

A, B

$$\xi(0) = \xi(1) = \frac{1}{2}\pi^{-1/2}\overbrace{\Gamma(1/2)}^{\pi^{1/2}}\underbrace{\lim_{s\to1}(s-1)\zeta(s)}_{1} = \frac{1}{2}.$$

$$\to A = \log\frac{1}{2} \approx -0{,}69\ldots$$

$$B = \frac{\xi'(0)}{\xi(0)} = -\frac{\xi'(1)}{\xi(1)} = \frac{1}{2}\log\pi - \overbrace{\frac{1}{2}\frac{\Gamma'(3/2)}{\Gamma(3/2)}}^{1-\log 2-\gamma/2} - \lim_{s\to1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right)$$

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty}\{x\}x^{-s-1}\,dx$$

$$\hookrightarrow \lim(\ ) = 1 - \int_1^{\infty}\{x\}x^{-2}\,dx = \cdots = \gamma$$

$$B = -\frac{\gamma}{2} - 1 - \frac{1}{2}\log 4\pi \approx -0{,}023\ldots$$

$\sum |\rho|^{-1}$ diverge. Sin embargo: $\sum \rho^{-1}$ "converge"

"Truco": Agrupar $\rho, \bar{\rho}$.

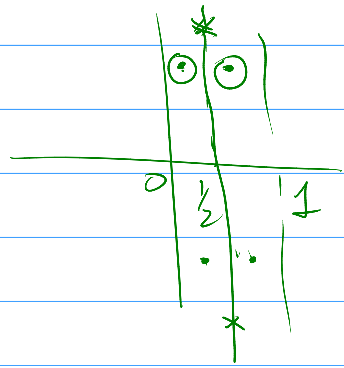
$\rho = \beta + i\gamma$ $(0 \le \beta \le 1)$ $\leadsto$ $\dfrac{1}{\rho} + \dfrac{1}{\bar{\rho}} = \dfrac{2\beta}{\beta^2+\gamma^2} \le \dfrac{2}{|\rho|^2}$

$$\sum_{\substack{\rho \\ (\gamma>0)}} \frac{1}{\rho} + \frac{1}{\bar{\rho}} = \sum_{\rho} |\rho|^{-2} \text{ converge!}$$

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \cancel{\frac{1}{s-\rho}} + \frac{1}{\rho} = -\frac{\xi'(1-s)}{\xi(1-s)} = -B - \sum_{\rho} \cancel{\frac{1}{1-s-\rho}} + \frac{1}{\rho}$$

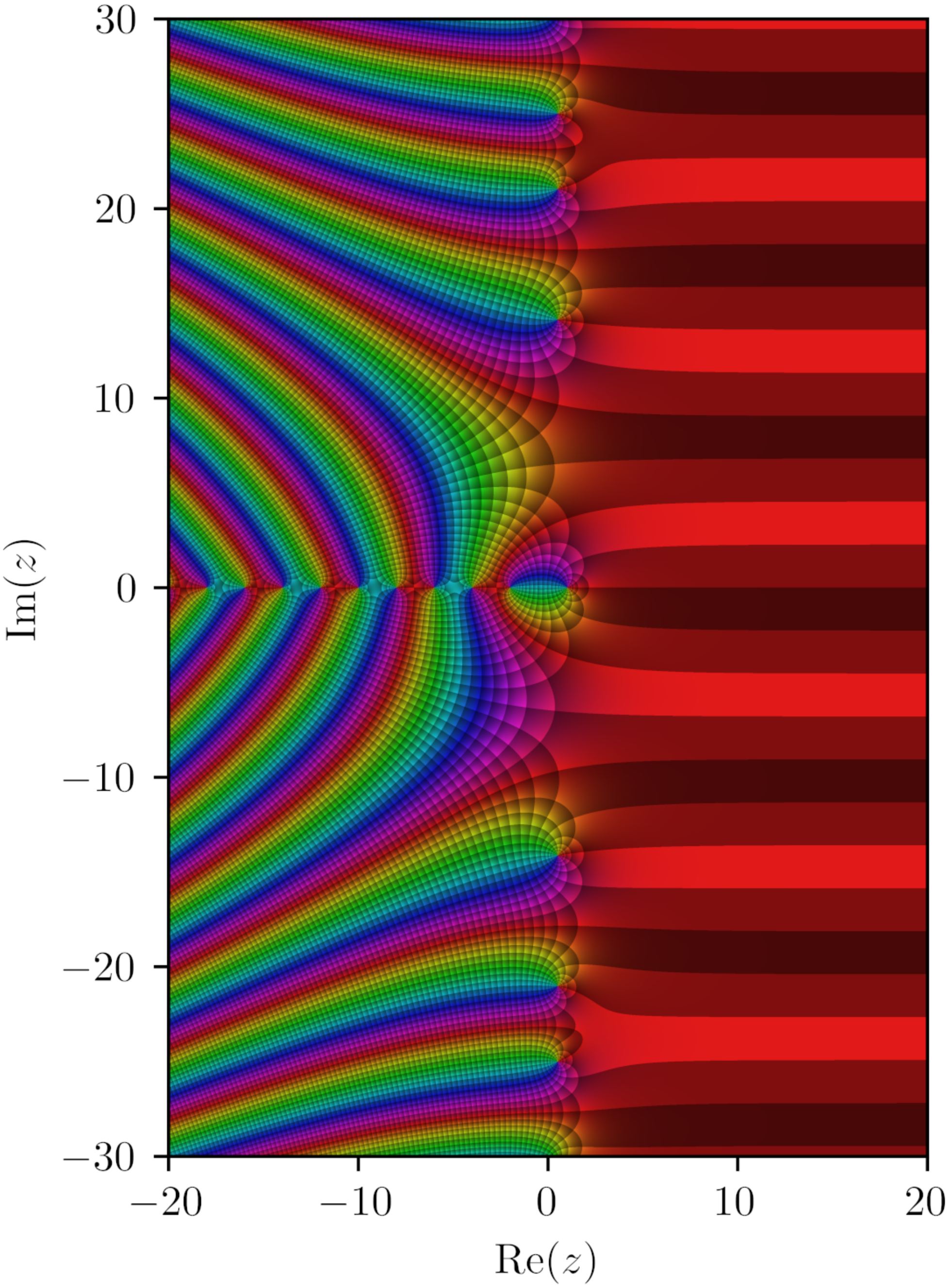
$$\sum_{\rho} \frac{1}{s-\rho} = \sum_{\rho} \frac{1}{s-(1-\rho)}$$

$$-B = \sum_{\rho} \frac{1}{\rho} = \sum_{\gamma>0} \boxed{\frac{\beta}{\beta^2+\gamma^2}} = 0.023$$



Ej $|\mathrm{Im}\,\rho| > 6.5$ $\forall \rho$.

De hecho $|\mathrm{Im}\,\rho_1| = 14.13\ldots$



https://en.wikipedia.org/wiki/Riemann_zeta_function

$L(s,\chi)$ χ primitivo mod $q > 1$ $a = \begin{cases} 0 & \chi(-1) = 1 \\ 1 & \chi(-1) = -1 \end{cases}$

$$\hat{L}(s,\chi) = \xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi)$$

vimos $\xi(s,\chi)$ entera, $\xi(1-s,\bar{\chi}) = \frac{\tau(\bar{\chi})}{i^a q^{1/2}} \cdot \xi(s,\chi)$ $\to |1 \cdot 1| = 1$

① $|L(s,\chi)|$

clases $L(s,\chi) = s \int_1^\infty S(x)\, x^{-s-1}\, dx$ $\left(S(x) = \sum_{n \le x} \chi(n) \right.$, $\left. |S(x)| \le q \right)$

$\mathrm{Re}\, s \ge \frac{1}{2}$:

$$|L(s,\chi)| \le 2q|s|.$$

$$\leadsto |\xi(s,\chi)| \le 2q^{\frac{\mathrm{Re}\, s + 3}{2}} \cdot |s| \cdot \left|\Gamma\left(\frac{s+a}{2}\right)\right|$$

$$= q^{\frac{\mathrm{Re}\, s + 3}{2}} \cdot \exp(C|s| \log|s|).$$

No se puede mejorar: $L(s,\chi) \to 1$ $s \to +\infty$

$L(s,\chi)$ infinitos ceros en $0 \le \mathrm{Re}\, s \le 1$

① $\sum |\rho_n|^{-1-\varepsilon}$ converge

② $\sum |\rho_n|^{-1}$ diverge

③ $\xi(s,\chi) = e^{A+Bs} \prod_\rho (1 - s/\rho) e^{s/\rho}.$

$A = A(\chi)$, $B = B(\chi)$

Ej $\frac{L'(s,\chi)}{L(s,\chi)} = B(\chi) - \frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'\left(\frac{s+a}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$

$$\overline{B(\chi)}$$

$$B(\chi) = \frac{\xi'(0,\chi)}{\xi(0,\chi)} = -\frac{\xi'(1,\bar\chi)}{\xi(1,\bar\chi)} = -B(\bar\chi) - \sum_\rho \left(\frac{1}{1-\bar\rho} + \frac{1}{\bar\rho}\right)$$

$$\frac{1}{\rho}$$

$\left(\rho \text{ cero } L(s,\chi) \overset{(\text{def})}{\Longleftrightarrow} \bar\rho \text{ cero de } L(s,\bar\chi)\right.$ $\Updownarrow$ $\left.\overset{\text{ec. func.}}{\Longleftrightarrow} 1-\rho \text{ cero de } L(s,\bar\chi)\right)$

$$\operatorname{Re} B(\chi) = -\frac{1}{2}\sum_\rho \left(\frac{1}{\rho} + \frac{1}{\bar\rho}\right) = -\sum_\rho \operatorname{Re}\left(\frac{1}{\rho}\right).$$

Dificultad: estimar $B(\chi)$ (como función de q).

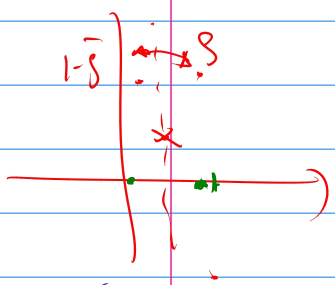
→ excluir que $L(s,\chi)$ tenga ceros cerca de 0.

(ζ no tiene este problema por el polo en $s=1$

$L(0,\chi)$ muy pequeño.)

Notas: χ real $\chi = \bar\chi$

$$B(\chi) = -\sum_\rho \frac{1}{\rho} < 0$$



ceros: simétricos resp $\operatorname{Re} s = \frac{1}{2}$, resp $\mathbb{R}$

χ complejo: $L(s,\chi) \longleftrightarrow L(1-s,\bar\chi) = \overline{L(1-\bar s,\chi)}$

ceros tienen simetría $\rho \leftrightarrow 1-\bar\rho$ ($\operatorname{Re} s = \frac{1}{2}$)

pero NO resp a $\mathbb{R}$.