D.13]

$$s = \sigma + it$$

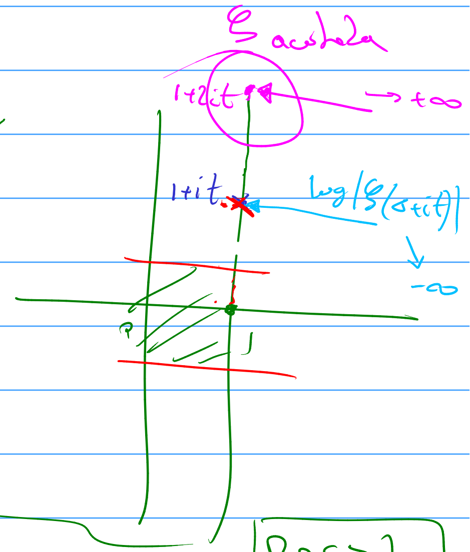
Regiones sin ceros para $\zeta(s)$.

(Hadamard, de la Vallée Poussin)

Prop $\operatorname{Re} s = 1 \Rightarrow \zeta(s) \neq 0$.

$$\log \zeta(s) \doteq \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} e^{-itm\log p}$$

$$\log|\zeta(s)| = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} \underline{\cos(tm.\log p)}$$



$\operatorname{Re} s > 1$

$\log(1-p^{-s})$

$|p^{-s}| < |p^{-1}| < 1$

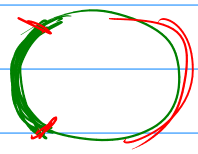
$\sum p^{-s}$ converge

$p^{-sm} = p^{-m\sigma} \cdot e^{-itm\log p}$

Sup: $|\zeta(1+it)| = 0 \rightsquigarrow \log|\zeta(\sigma+it)| \xrightarrow[\sigma\to 1^+]{} -\infty$

$\rightsquigarrow \cos(tm\log p)$ predominantemente negativo...

$\rightsquigarrow \cos(2tm\log p)$ " positivo



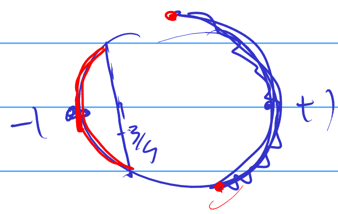
$\rightsquigarrow \log|\zeta(\sigma+2it)| \xrightarrow[\sigma\to 1^+]{} +\infty$ ↯

...

Mertens $\underline{3 + 4\cos\theta + \cos 2\theta \geq 0}$

"

(ej: $2(1+\cos\theta)^2$)



$$\log|\zeta(s)| = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} \cos(tm.\log p)$$

$s = \sigma, \sigma+it, \sigma+2it$

$$\overbrace{\underbrace{3\log|\zeta(\sigma)|}_{\to(+\infty)} + \underbrace{4\log|\zeta(\sigma+it)|}_{\to(-\infty)}}^{\nearrow -\infty} + \underbrace{\log|\zeta(\sigma+2it)|}_{\text{acotado}} \geq 0 \qquad \nearrow +\infty \ ↯$$

$$\exp\Big(\ \ \overbrace{\zeta^3(\sigma)\,|\zeta^4(\sigma+it)|}^{<\,O(\sigma-1)}\ \overbrace{|\zeta(\sigma+2it)|}^{<\,c_2}\ \geq 1$$

$\sigma \to 1^+$

$$\zeta(\sigma) \sim (\sigma-1)^{-1}$$

Si $\boxed{\zeta(1+it)=0} \longrightarrow |\zeta(\sigma+it)| < c_1\,|\sigma-1|$

$$|\zeta(1+2it)| < C_2 \qquad \lightning$$

Cambiamos $\log \zeta(s)$ por $-\zeta'(s)/\zeta(s)$.

$$-\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-\sigma} \cos(t \log n)$$

$$\Rightarrow\ 3\underbrace{\left(\frac{-\zeta'(\sigma)}{\zeta(\sigma)}\right)}_{<\ \frac{1}{\sigma-1}+C_3} + 4\underbrace{\left[-\mathrm{Re}\,\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right]}_{} + \underbrace{\left[-\mathrm{Re}\,\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right]}_{} \geq 0$$

$1<\sigma\leq 2$

$t\geq 2$

influenciados por ceros cerca de $1+it$, $1+2it$.

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - B - \frac{1}{2}\log \pi + \underbrace{\frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)}}_{<\ C_4\cdot\log t} - \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$

$$-\mathrm{Re}\,\frac{\zeta'(s)}{\zeta(s)} < C_5 \cdot \log t - \sum \underbrace{\mathrm{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)}_{\geq 0}$$

$$\left(\rho = \beta + i\gamma \longrightarrow \mathrm{Re}\,\frac{1}{s-\rho} = \frac{\sigma-\beta}{|s-\rho|^2} \geq 0\ ;\quad \mathrm{Re}\,\frac{1}{\rho} = \frac{\beta}{|\rho|^2} \geq 0\right)$$

$$s=\sigma+2it \to -\mathrm{Re}\,\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} < C_5\cdot\log t$$

$$s=\sigma+it \to -\mathrm{Re}\,\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} < C_5\cdot\log t - \frac{1}{\sigma-\beta}$$

$\boxed{\text{Sup } \zeta(\beta+it)=0}$

$$\frac{3}{\sigma-1} + 3C_1 + 5\cdot C_5\cdot\log t - \frac{4}{\sigma-\beta} \geq 0$$

$C_6\cdot\log t$

$$\boxed{\sigma = 1+\frac{\delta}{C_6\cdot\log t}} \leadsto \left(\frac{3+\delta}{\delta}\right) C_6\cdot\log t \geq \frac{4}{\sigma-\beta}$$

$$\leadsto \cdots \leadsto \beta < \sigma - \frac{4\delta}{(3+\delta)\,C_6\log t}$$

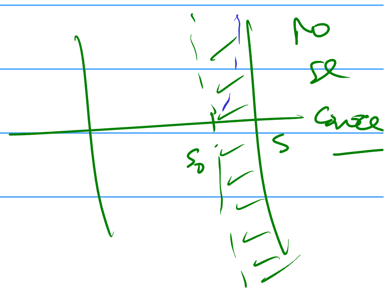
$$\beta < 1 + \frac{\delta\left(\frac{\delta-1}{\delta+3}\right)}{C_6\cdot\log t} \qquad = -\frac{1}{14} \qquad \delta = 1/2$$

Teo $\exists c>0 : \forall \rho = \beta+it, \ \beta\geq 0$

$$\zeta(\rho)=0 \Rightarrow \beta < 1 - \frac{c}{\log(|t|+2)}$$

de la Vallé Poussin (1899)

Littlewood 1922 $\to c\,\frac{\log\log t}{\log t}$



Vinogradov Korobov (1958) $\to \frac{c_\alpha}{(\log t)^\alpha} \quad \forall \alpha > \frac{2}{3}$

$L(s,\chi)$ para χ fijo $\rightsquigarrow$ análogo...

(La dependencia de q explícita es útil.

~~$t/2$~~ $\underline{t \geq 0}$ ($t<0$ corresp. a ceros $L(s,\bar{\chi})$)

① $$-\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-\sigma}\, \chi(n)\, e^{-it\log n}$$

$$\begin{cases} \mathrm{Re}\left(\chi(n)\, e^{-it\log n}\right) = \cos\theta \\ \mathrm{Re}\left(\chi_0(n)\, e^{-i\cdot 0\cdot\log n}\right) = 1 \\ \mathrm{Re}\left(\chi^2(n)\, e^{-i2t\log n}\right) = \cos(2\theta) \end{cases}$$

$$\underbrace{3\left(-\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)}\right)}_{\text{“grande”}} + \underbrace{4\left(-\mathrm{Re}\,\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)}\right)}_{\text{“chico”}} + \underbrace{\left(-\mathrm{Re}\,\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)}\right)}_{\text{acotado.}} \geq 0$$

<u>NOTAR</u>: χ real $\Leftrightarrow$ $\chi^2 = \chi_0$ Cambia el argumento para t pequeño pues $L(s,\chi^2)$ tiene un polo en $s=1$ $\therefore$

χ real