$\chi \neq \chi_0$ primitivo mod $q \geq 3$

$\sigma > 1, \quad t \geq 0$

$$3\left(-\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)}\right) + 4\left(-\mathrm{Re}\,\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)}\right) + \left(-\mathrm{Re}\,\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)}\right) \geq 0$$

$\boxed{A}$ $\chi = \overline{\chi}$ $\quad (1 < \sigma < 2)$

$$-\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} = \sum \chi_0(n)\Lambda(n) n^{-\sigma} \leq -\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma-1} + c_1$$

$$-\mathrm{Re}\,\frac{L'(s,\chi)}{L(s,\chi)} = \underbrace{\frac{1}{2}\log\frac{q}{\pi}} + \overbrace{\frac{1}{2}\mathrm{Re}\,\frac{\Gamma'\left(\frac{s+a}{2}\right)}{\Gamma\left(\frac{s+a}{2}\right)}}^{O(\log(t+2))} - \mathrm{Re}\sum \frac{1}{s-\rho}$$

$$\left(-\cancel{\mathrm{Re}\,B(\chi)} - \cancel{\mathrm{Re}\sum \frac{1}{\rho}}\right)$$

$\log q + \log(t+2)$

$$< O(\overset{\shortparallel}{\mathcal{L}}) - \sum_\rho \frac{\sigma - \beta}{|s-\rho|^2} \quad > 0 \qquad (\rho = \beta + i\gamma)$$

$\chi^2 \neq \chi_0$ podría no ser primitivo

$\rightsquigarrow \chi_1$ induce χ^2

$$\left|\frac{L'(s,\chi^2)}{L(s,\chi^2)} - \frac{L'(s,\chi_1)}{L(s,\chi_1)}\right| \leq \sum_{p|q} \frac{p^{-\sigma}\log p}{1-p^{-\sigma}} \leq \sum_{p|q} \log p \leq \log q \qquad \log q = O(\mathcal{L})$$

Supongamos $\rho = \beta + it$ es un cero de $\mathcal{E}$.

$$-\operatorname{Re} \frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} < c_3 \cdot \mathcal{L} - \frac{1}{\sigma-\beta}$$

$$\rightsquigarrow \quad \frac{4}{\sigma-\beta} < \frac{3}{\sigma-1} + c_4 \mathcal{L}$$

$$\sigma = 1 + \frac{\delta}{c_4 \mathcal{L}} \quad \rightsquigarrow \quad \beta < 1 - \frac{c_5}{\mathcal{L}}.$$

Ej: $$\frac{A}{\sigma-\beta} < \frac{B}{\sigma-1} + K \qquad A > B > 0$$

$$\forall\, 1 < \sigma < 2.$$

$\rightsquigarrow \ \dots \quad \beta < 1 - \dfrac{c}{K}$ $\quad c$ sólo depende de A, B.

(Idea: $\sigma = 1 + \dfrac{\delta}{K} \Longrightarrow \beta < 1 - \dfrac{(\ \ )}{K}$)

Prop $\exists\, c_5 > 0$: $\forall\, \chi$ no 2 q $\bar{\chi} \neq \chi$

$\rho = \beta + it$ cero de $L(s,\chi) \Rightarrow \beta < 1 - \dfrac{c_5}{\mathcal{L}}$.

obs: χ no ppal inducido por χ_1

$$\frac{L(s,\chi)}{L(s,\chi_1)} = \prod_{\text{finito}} (1 - \chi_1(p)\, p^{-s}) \quad \Leftarrow \text{ceros en } \operatorname{Re} s = 0.$$

$\boxed{B}$ $\bar{\chi} = \chi \longrightarrow \chi^2 = \chi_0$.

$$-\mathrm{Re}\,\frac{L'(\sigma+2it,\chi_0)}{L(\sigma+2it,\chi_0)} \quad ?$$

Como antes: $\left|\dfrac{L'(s,\chi_0)}{L(s,\chi_0)} - \dfrac{\zeta'(s)}{\zeta(s)}\right| \leq \log q$.

$$-\mathrm{Re}\,\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}$$ no acotado para t chico

$$< \mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right) + c_6 \cdot \log(t+2)$$

$$\rightsquigarrow \frac{4}{\sigma-\beta} < \frac{3}{\sigma-1} + \mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right) + c_8 \cdot \mathcal{L}$$

$\sigma = 1 + \dfrac{\varepsilon}{c_8\mathcal{L}} \rightsquigarrow t \geq \dfrac{\varepsilon}{c_8\mathcal{L}} \geq \sigma-1 \longrightarrow \leq \dfrac{1}{5(\sigma-1)}$

$\left(\mathrm{Re}\left(\dfrac{1}{1+2i}\right) = \dfrac{1}{1^2+2^2}\right)$

$\boxed{4 > 3 + \frac{1}{5}}$

$$\beta < 1 - \left(\frac{4-5\varepsilon}{16+5\varepsilon}\right)\frac{\varepsilon}{c_8\mathcal{L}} < 1 - \frac{1}{5}\,\frac{\varepsilon}{c_8\mathcal{L}}$$

$\forall \quad \varepsilon < \varepsilon_0$

$\left(c_9 = \dfrac{\varepsilon_0}{c_8}\right)$

Prop: $\exists\, c_q > 0 : \ \forall\, 0 < \delta < c_q$

$\forall\, \chi \bmod q$, $\bar{\chi} = \chi$, $\chi \neq \chi_0$

$\forall\, \rho = \beta + it$ cero de $L(s,\chi)$ con $|t| \geq \dfrac{\delta}{\log q}$

$$\beta < 1 - \frac{\delta}{5\mathcal{L}} \qquad \left(\geq \frac{\delta}{\mathcal{L}} \right)$$

¿Qué pasa con $|t| < \dfrac{\delta}{\log q}$.

existe a lo sumo UN cero con $\beta > 1 - \dfrac{\delta'}{\log q}$

(debe ser real.)

$$-\frac{L'(\sigma,\chi)}{L(\sigma,\chi)} < c_{10} \log q - \overbrace{\underbrace{\sum_\rho \frac{1}{\sigma-\rho}}}$$

real por pares conjugados

$$\parallel$$

$$\sum \chi(n)\Lambda(n) n^{-\sigma} \geq -\sum \Lambda(n) n^{-\sigma} = \frac{\zeta'(\sigma)}{\zeta(\sigma)} > \underline{\frac{-1}{\sigma-1} - c_{11}}$$

$$\Rightarrow \frac{-1}{\sigma-1} < c_{12} \log q - \sum_\rho \frac{1}{\sigma-\rho} \qquad ** $$

Sup $\rho^{\pm} = \beta \pm i\gamma$ son dos ceros con $\gamma \neq 0$

$$|\gamma| < \frac{\delta}{\log q}.$$

$$\frac{1}{\sigma-\rho^+} + \frac{1}{\sigma-\rho^-} = \frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2}$$

$$\frac{-1}{\sigma-1} < C_{12}\cdot \log q - \frac{2(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2}. \quad < \frac{2}{\left(1+\frac{1}{4}\right)(\sigma-\beta)}$$

$$\sigma = 1 + \frac{2\delta}{\log q} \rightsquigarrow |\gamma| < \frac{\delta}{\log q} = \frac{1}{2}(\sigma-1) < \frac{1}{2}(\sigma-\beta)$$

$$\frac{-1}{\sigma-1} < C_{12}\log q - \frac{8/5}{(\sigma-\beta)} \qquad 8/5 \geq 1.$$

Eligiendo δ' pequeño: $\beta < 1 - \frac{\delta'}{\log q}$,

Dos $\beta_1 \leq \beta_2$ dos ceros reales

$$\rightsquigarrow \frac{-1}{\sigma-1} < C_{12}\log q - \frac{1}{\sigma-\beta_1} - \frac{1}{\sigma-\beta_2}$$

$$< C_{12}\log q - \frac{2}{\sigma-\beta_1}$$

$$\rightsquigarrow \beta_1 < 1 - \frac{\delta'}{\log q}.$$

Dado $c, q > 0$, sea

$$\mathcal{R}_{c,q} = \left\{ \sigma + it \; : \; \sigma \geq 1 - \frac{c}{\log q + \log(|t|+2)} \right\}$$

Teo $\exists c > 0$ tq $\forall q$ $\forall \chi$ mod q

① $\chi \neq \bar{\chi} \Rightarrow L(s,\chi)$ no tiene ceros en $\mathcal{R}_{c,q}$

② $\chi = \bar{\chi} \Rightarrow L(s,\chi)$ tiene a lo sumo

un cero real simple en $\mathcal{R}_{c,q}$.

Gronwall 1913 / Titchmarsh (1930/1933).

Teo (Landau 1918) $\chi_1 \neq \chi_2$ reales (prim) ($\chi_1\chi_2 \neq \chi_0$)

mod q_1, q_2. $L(s,\chi_i)$ cero real β_i

$$\min(\beta_1, \beta_2) < 1 - \frac{c_{15}}{\log(q_1 q_2)}$$

Dem Considerar $\chi_1\chi_2 \neq \chi_0$ mod $q_1 q_2$

$$\frac{-L'(\sigma,\chi_1\chi_2)}{L(\sigma,\chi_1\chi_2)} < c_{16} \cdot \log q_1 q_2$$

$$\frac{-L'(\sigma,\chi_j)}{L(\sigma,\chi_j)} < c_{17} \cdot \log q_j - \frac{1}{\sigma - \beta_j}$$

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{L'(s,\chi_1)}{L(s,\chi_1)} - \frac{L'(s,\chi_2)}{L(s,\chi_2)} - \frac{L'(s,\chi_1\chi_2)}{L(s,\chi_1\chi_2)}$$

$$= \sum \Lambda(n) \underbrace{\left(1+\chi_1(n)\right)}_{\geq 0} \underbrace{\left(1+\chi_2(n)\right)}_{\geq 0} n^{-s} \geq 0$$

$$\Rightarrow \underbrace{\frac{1}{s-\beta_1} + \frac{1}{s-\beta_2}}_{\geq \frac{2}{s-\beta_1}} < \frac{1}{s-1} + c_{18} \log q_1 q_2$$

$(\beta_1 \leq \beta_2)$

$$\Rightarrow \beta_1 < 1 - \frac{c_{15}}{\log q_1 q_2}$$