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$(\pi(x))$

Fórmula explícita para $\psi(x)$.

$$\psi(x) = \sum_{n\le x} \Lambda(n) = \sum_{p^k \le x} \log p$$

$$\psi_0(x) = \psi(x) - \frac{1}{2}\Lambda(x)$$

Teo (von Mangoldt) $x > 1$

caso triviale

$$\psi_0(x) = x - \underbrace{\sum_{\rho} \frac{x^\rho}{\rho}} - \frac{\zeta'(0)}{\zeta(0)} - \overbrace{\frac{1}{2}\log(1-x^{-2})}.$$

$\lim_{T\to\infty} \sum_{|\gamma|<T}$

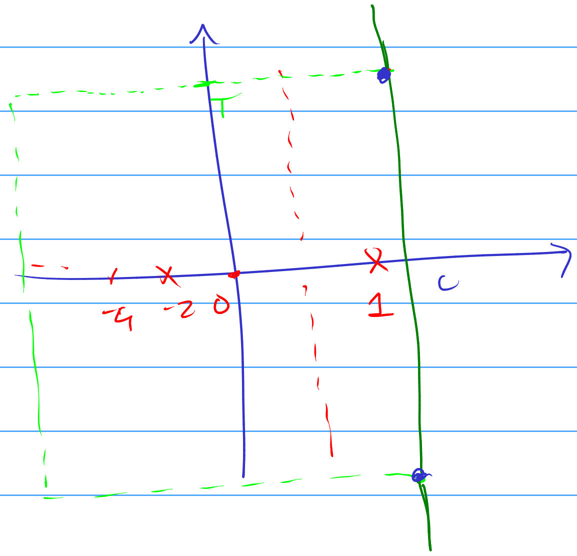
lema $c>0$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & 0<y<1 \\ 1/2 & y=1 \\ 1 & y>1 \end{cases}$$

$\boxed{y = \frac{x}{n}}$

$$\sum \Lambda(n) n^{-s} = -\zeta'(s)/\zeta(s)$$

$$\underset{c>1}{\Longrightarrow} \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(-\zeta'(s)/\zeta(s)\right) \frac{x^s}{s} ds = \psi_0(x)$$



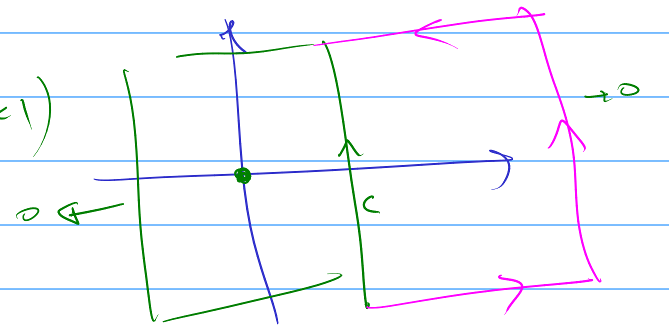
$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s \frac{ds}{s} \quad ; \quad f(y) = \begin{cases} 0 & 0<y<1 \\ 1/2 & y=1 \\ 1 & y>1 \end{cases}$$

Lema $y>0$, $c>0$, $T>0$:

$$\left| I(y,T) - f(y) \right| < \begin{cases} \underbrace{y^c \cdot \min\left(1, \frac{1}{T|\log y|}\right)} & y \neq 1 \\ c/T & y=1 \end{cases}$$

$\boxed{\text{ejercicio}}$

$(0<y<1)$



$y = \frac{x}{n}$

$$J(x,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{-\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s}\, ds.$$

$$\left|\psi_0(x) - J(x,T)\right| < \sum_{\substack{n=1 \\ n\neq x}}^{\infty} \Lambda(n)\left(\frac{x}{n}\right)^c \min\left(1, \frac{1}{T|\log x/n|}\right) + \frac{c}{T}\Lambda(x)$$

Tomar $x\geq 2$, $c = 1 + \frac{1}{\log x} \longleftrightarrow x^c = e\cdot x$

Objetivo: estimar $\sum_n$.

$\boxed{A}$ $n \le \frac{3}{4}x$, $n \ge \frac{5}{4}x$ $\rightarrow$ $\frac{1}{|\log(\frac{x}{n})|} = O(1)$

$$\sum_A \ll \frac{x}{T} \underbrace{\sum_A \Lambda(n) n^{-c}} = \frac{x}{T} \underbrace{\left(-\frac{\zeta'(c)}{\zeta(c)} \right)}_{\overset{\shortparallel}{O(\log x)}} \ll \frac{x \log x}{T}$$

$\boxed{B}$ $\frac{3}{4}x < n < x_1 < x$ (x_1 = mayor pot de primo menor que x)

$v = x_1 - n = 1, 2, 3, \dots < \frac{1}{4}x$, $\log \frac{x}{n} \ge \log \frac{x_1}{n} = -\log\left(1 - \frac{v}{x_1}\right)$

$$\ge \frac{v}{x_1}.$$

$$\sum_B \ll \sum_{v=1}^{x/4} \overbrace{\Lambda(\underline{x_1 - v})}^{\le \log x} \cdot \frac{x_1}{T \cdot v} \ll \frac{x \log^2 x}{T}$$

$\boxed{C}$ $x < x_2 < n < \frac{5}{4}x$ (x_2 = menor P.P. mayor que x)

$v = n - x_2 = 1, 2, \dots < \frac{1}{4}x$ $\left|\log \frac{x}{n}\right| \ge \left|\log \frac{x_2}{n}\right| = -\log\left(1 - \frac{v}{n}\right)$

$$\ge \frac{v}{n}$$

$$\sum_C \ll \sum_{v=1}^{x/4} \overbrace{\Lambda(x_2 + v)}^{\ll \log x} \cdot \frac{\overset{\ll x}{n}}{T v} \ll \frac{x \log^2 x}{T}$$

$\left(\begin{array}{l} 0 < z < 1 \\ \Rightarrow -\log(1-z) \ge z \end{array} \right)$

Falta $[x_1, x_2]$ $\rightsquigarrow$ x_1, x_2, x.

$\boxed{n = x_1}$ $\log \frac{x}{n} = -\log\left(1 - \frac{x - x_1}{x}\right) \ge \frac{x - x_1}{x}$

$$\ll \Lambda(x_1) \cdot \min\left(1, \frac{x}{T(x - x_1)}\right) \ll \log x \cdot \min\left(1, \frac{x}{T\underbrace{(x - x_1)}_{\le \langle x \rangle}}\right)$$

$\boxed{n = x_2}$ ... $\left|\log\frac{x}{n}\right| \geq \frac{x_2 - x}{x_2}$

$$\ll \Lambda(x_2)\cdot\min\left(1, \frac{x_2}{T(x_2-x)}\right) \ll \log x \cdot \min\left(1, \frac{x}{T(x_2-x)}\right)$$

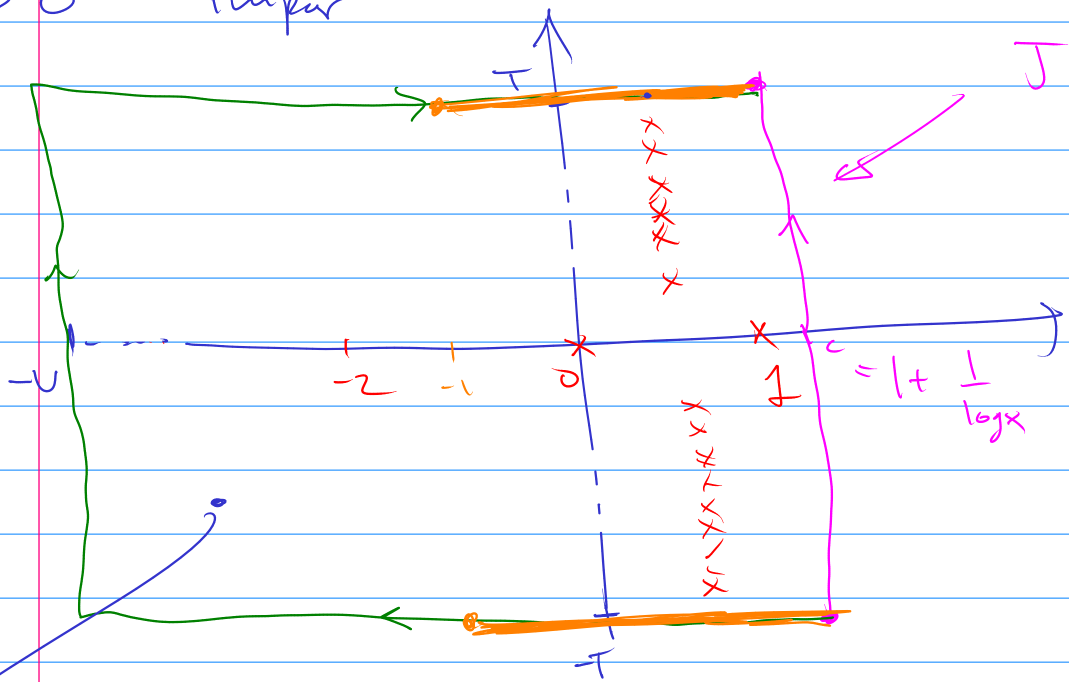
$\langle x \rangle$

$\boxed{n = x}$ $\ll \frac{\log x}{T}$

Notación: $\langle x \rangle = \min(x - x_1, x_2 - x) =$ "distancia a la P.P. más cercana $\neq x$"

Lema $\left|\psi_0(x) - J(x,T)\right| \ll \frac{x\log^2 x}{T} + \log x \cdot \min\left(1, \frac{x}{T\langle x\rangle}\right)$

$U \gg 0$ impar



$J(x,T)$

" $\int \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds$ "

$c = 1 + \frac{1}{\log x}$

$$\sum \operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}$$

$$+ \sum_{0<2m<U}\frac{x^{-2m}}{-2m} \to \frac{1}{2}\log(1-x^{-2})$$

Elección de T:

$T >> 0 \qquad \#\{\text{ceros } |\gamma - T| < 1\} << \log T$

hay un gap de tamaño $>> \dfrac{1}{\log T}$.

"cambiando T por < 1"

Asumir $\quad |\gamma - T| >> \dfrac{1}{\log T} \qquad \forall \rho = \beta + i\gamma$.

Recordar: $\quad \dfrac{\zeta'(s)}{\zeta(s)} = \displaystyle\sum_{|\gamma - T| < 1} \frac{1}{s-\rho} + O(\log T)$

$\boxed{-1 \leq \sigma \leq 2}$

$\boxed{s = \sigma + it}$

$\dfrac{1}{s-\rho} << \log T$

$\# << \log T$

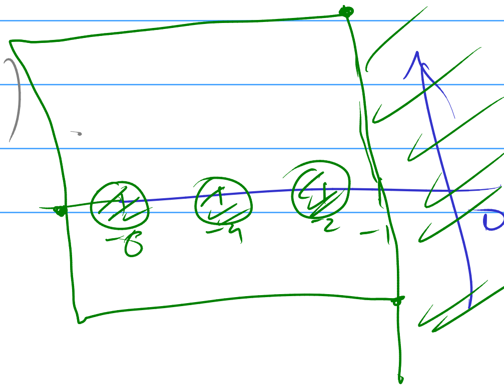
$<< \log^2 T$.

$$\left|\int_{-1 \pm Ti}^{c \pm Ti}\right| << \log^2 T \int_{-1}^{c} \left|\frac{x^s}{s}\right| d\sigma << \frac{\log^2 T}{T} \int_{-1}^{c} x^\sigma \, d\sigma$$

$$<< \frac{x \log^2 T}{T \cdot \log x}$$

Lema $\quad \left.\begin{array}{l} \operatorname{Re} s \leq -1 \\ |s + 2m| > \frac{1}{2} \\ \forall m \end{array}\right\} \Rightarrow \left|\zeta'(s)/\zeta(s)\right| << \log(2|s|)$

ejercicio (usa ec. funcional).



$$\int_{-U\pm iT}^{-1\pm iT} \ll \frac{\log 2T}{T}\int_{-U}^{-1} x^{\sigma}\, d\sigma \ll \frac{\log T}{T\, x \log x}$$

$$\left(\frac{\log 2|s|}{|s|} < \frac{\log 2T}{T}\right)$$

$$\int_{-U-iT}^{-U+iT} \ll \frac{\log 2U}{U}\int_{-T}^{T} x^{-U}\, dt \ll \frac{T \log U}{U x^{U}} \xrightarrow{(U\to\infty)} 0$$

Conclusión:

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}) + R(x,T)$$

$$R(x,T) \ll \frac{x \log^2(xT)}{T} + \log x \cdot \min\left(1, \frac{x}{T\langle x\rangle}\right)$$

$T \to \infty$ $\qquad R(x,T) \longrightarrow 0$

(uniforme en intervalos cerrados que no contengan P.P.)

NOTA T no era cualquiera

Cambiar T por $O(1)$

agrego $O(\log T)$ términos en $\sum_{\rho}$

cada uno de tamaño $O\left(\frac{x}{T}\right)$

diferencia es $O\left(\frac{x \log T}{T}\right) \ll$ cota de $R(T,x)$

obs: $x \in \mathbb{Z} \Rightarrow \langle x \rangle \geq 1 \Rightarrow$

$$R(x,T) \ll \frac{x \log^2(xT)}{T}$$

Nota: $1 < x < 2$ vale con otro $R(x,T)$.