Teo

$$\psi_0(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}) + R(x,T)$$

donde $\displaystyle |R(x,T)| \ll \frac{x\log^2(xT)}{T} + (\log x)\min\left(1, \frac{x}{T\langle x\rangle}\right)$

obs: si $x \in \mathbb{Z}$ no hace falta

18 El teo de los números primos.

Teo $\psi(x) = x + O\left(x \cdot \exp\left(-c(\log x)^{1/2}\right)\right)$

Dem para estimar $\displaystyle\sum_{|\gamma|<T} \frac{x^\rho}{\rho}$ usamos $N(T)$ y regiones libres de ceros. $(\rho = \beta + i\gamma)$

⊛ Si $|\gamma| < T \Rightarrow \beta < 1 - \frac{c_1}{\log T}$.

$$|x^\rho| = x^\beta < x \cdot \exp\left(-c_1 \cdot \log x / \log T\right).$$

⊛ $|\rho| \geq \gamma$ ⟶

$$\sum_{0<\gamma<T} \frac{1}{\gamma} = \int_0^T \frac{1}{t}\, dN(t) \overset{\text{partes}}{=} \underbrace{\frac{N(T)}{T}}_{O(\log T)} + \underbrace{\int_0^T \frac{N(t)}{t^2}\,dt}_{O\left(\int_0^T \frac{\log t}{t}\right) = O(\log^2 T)}$$

$$= O(\log^2 T)$$

Si $x \in \mathbb{Z}$

$$|\psi(x) - x| \ll \frac{x \log^2(xT)}{T} + x \cdot \log^2 T \cdot \exp\left(-c_1 \frac{\log x}{\log T}\right).$$

eligiendo T / $\log^2 T = \log x$

$$T = \exp(\log^{1/2} x) \quad (\ll x^{\varepsilon})$$

$$\dashrightarrow |\psi(x) - x| \ll x \log^2 x \cdot \exp(-\log^{1/2} x) + x \log x \exp(-c_1 \log^{1/2} x)$$

$$\ll x \cdot \exp(-c_2 \log^{1/2} x)$$

#

donde $c_2 < \min(1, c_1)$

(En part $\psi(x) \sim x$)

$$\pi_1(x) = \sum_{n \le x} \frac{\Lambda(n)}{\log n} = \int_2^x \frac{1}{\log u} \, d\psi(u)$$

$$\sum_{p^k \le x} \frac{1}{k} = \frac{\log p}{\log(p^k)}$$

$$= \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t \cdot \log^2 t} dt$$

cambiar $\psi(x)$ por x

$$\frac{x}{\log x} + \int_2^x \frac{t}{t \cdot \log^2 t} dt = \dots = \underbrace{\int_2^x \frac{dt}{\log t}}_{li(x)} + \frac{2}{\log 2}$$

error : $x\cdot\exp(-c_2\log^{1/2}x) + \int_2^x \exp(-c_2\log^{1/2}t)\,dt$

$$\int_2^{x^{1/4}} \ll x^{1/4} \qquad \int_{x^{1/4}}^{x} \ll x\cdot\exp\left(-\frac{c_2}{2}\log^{1/2}x\right).$$

$\log^{1/2}t > \frac{1}{2}\log^{1/2}x$ $\quad \left(c_3 = \frac{c_2}{2}\right)$

luego $\pi_1(x) = \mathrm{li}(x) + O\left(x\exp\left(-c_3\log^{1/2}x\right)\right)$

$$\pi_1(x) = \sum_{p^k\le x} \frac{1}{k} = \pi(x) + \underbrace{\frac{1}{2}\pi(x^{1/2})}_{x^{1/2}} + \underbrace{\frac{1}{3}\pi(x^{1/3})}_{x^{1/3}} + \cdots$$

"pequeño"

Teo $\pi(x) = \mathrm{li}(x) + O\left(x\exp\left(-c_3\cdot\log^{1/2}x\right)\right).$

(si se prueban mejores regiones sin ceros)

$\to$ mejora el error: $\pi(x) = \mathrm{li}\,x + O\left(x\exp\left(-c(\theta)\log^{\theta}x\right)\right)$

para $\theta < 3/5$.

$$\exp(c\cdot\log^{1}x) = x^{c}$$

$$\Downarrow$$

$$\exp(c\cdot\log^{1/2}x) \gg \log^{*}x.$$

Suponiendo la

Hipótesis de Riemann

$$|x^\rho| = x^{1/2} \qquad \sum_{|\gamma|<T} \frac{1}{|\rho|} \ll O(\log^2 T)$$

$$\to \quad |\psi(x) - x| \ll x^{1/2} \cdot \log^2 T + \frac{x \log^2 xT}{T}$$

$$\boxed{T = x^{1/2}}$$

$$\ll x^{1/2} \cdot \log^2 x$$

Teo ① RH $\Rightarrow$ $\psi(x) = x + O(x^{1/2} \cdot \log^2 x)$

② RH $\Rightarrow$ $\pi(x) = \operatorname{li} x + O(x^{1/2} \cdot \log x)$

Obs: si vale $\beta \leq \theta \to \pi(x) = \operatorname{li} x + O(x^\theta \cdot \log x)$

Prop: $\psi(x) = x + O(x^\alpha) \Rightarrow \beta \leq \alpha \quad \forall \rho$.

(En part: $\psi(x) = x + O(x^{1/2+\varepsilon}) \Leftrightarrow$ RH).

Dem $\boxed{\sigma > 1}$ $\dfrac{-\zeta'(s)}{\zeta(s)} = \sum \Lambda(n)\, n^{-s} = \int_1^\infty n^{-s}\, d\psi(n)$

$$\| \quad s \int_1^\infty \psi(x)\, x^{-s-1}\, dx$$

$$\psi(x) = x + R(x)$$

$$\rightsquigarrow \quad -\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} + s\int_1^\infty \underline{R(x)\, x^{-s-1}\, dx}$$

Si $R(x) = O\left(x^{\alpha}\right) \Rightarrow \int$ converge para $\mathrm{Re}\, s > \alpha$. #

Nota curiosa

Si $\psi(x) = x + O\left(x^{\theta+\varepsilon}\right) \not\Rightarrow \psi(x) = x + O\left(x^{\theta} \cdot \log^2 x\right)$