20 TNP para progresiones aritméticas

$$\psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a\,(q)}} \Lambda(n)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a)\, \psi(x,\chi)$$

todos los car. mód q.

① $$|\psi(x,\chi_0) - \psi(x)| \le \sum_{\substack{n\le x \\ (n,q)>1}} \Lambda(n) \ll \log x \cdot \log q .$$

(ej)

② $$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} - \sum_{|\gamma|<T}' \frac{x^{\rho}}{\rho} + R_3(x,T) .$$

$\boxed{2\le T\le x}$

$$|R_3(x,T)| \ll \frac{x}{T}\log^2 xq + x^{1/4}\cdot \log x$$

a lo sumo 1 para q fijo

③ $$\sum_{|\gamma|<T}' \frac{x^{\rho}}{\rho} \ll \underbrace{x\exp\left(-c_2\frac{\log x}{\log qT}\right)}_{x^{\beta}=|x^{\rho}|} \cdot \underbrace{\log^2(qx)}_{\sum \frac{1}{\rho}}$$

$$\rightarrow \psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + R_4(x,T)$$

$$|R_4(x,T)| \ll x\log^2 qx \cdot \exp\left(-c_2\frac{\log x}{\log qT}\right) + \frac{x}{T}\log^2 xq + x^{1/4}\log x$$

Acotar q ! $q \leq \exp\left(c \cdot \log^{1/2} x\right)$

eligiendo $T = \exp\left(c \cdot \log^{1/2} x\right)$

$\rightsquigarrow R_4(x,T) \ll x \cdot \exp\left(-\tilde{c} \log^{1/2} x\right)$

Teo dado $c > 0 \Rightarrow \exists \tilde{c} > 0$ tq

$$\psi(x; q, a) = \frac{x}{\varphi(q)} - \frac{\overline{\chi}_1(a)}{\varphi(q)} \frac{x^{\beta_1}}{\beta_1} + O\left(x \cdot \exp\left(-\tilde{c} \log^{1/2} x\right)\right)$$

$\forall q \leq \exp\left(c \cdot \log^{1/2} x\right)$?

Vimos $\beta_1 < 1 - \dfrac{c_3}{q^{1/2} \log^2 q}$

$$\ll \frac{x}{\varphi(q)} \exp\left(-c_3 \frac{\log x}{q^{1/2} \log^2 q}\right)$$

Caso $q \leq \log^{1-\delta} x$ ($\delta > 0$ fijo)

$$\Rightarrow \psi(x; q, a) = \frac{x}{\varphi(q)} + O\left(x \exp\left(-c_4 \cdot \log^{1/2} x\right)\right)$$

GRH: $q \leq x \Rightarrow \psi(x; q, a) = \dfrac{x}{\varphi(q)} + O\left(x^{1/2} \cdot \log^2 x\right)$

(No es también si $q \gg x^{1/2}$)

21 Teo. Siegel (1935)

① $\forall \varepsilon > 0 \quad \exists\, C_1(\varepsilon) > 0$: no efectiva $\varepsilon < 1/2$

$\forall\, \chi$ real, primitivo mod $q \Rightarrow L(1,\chi) > C_1(\varepsilon) \cdot q^{-\varepsilon}$

$\left(> C_2 \cdot q^{-1/2} \right)$ efectiva

$$\Rightarrow \begin{cases} h(d) > C_2(\varepsilon)\, |d|^{1/2-\varepsilon} & (d<0) \\ h(d) \log \varepsilon_d > C_2(\varepsilon)\, |d|^{1/2-\varepsilon} & (d>0) \end{cases}$$

$d<0$ Gauss conoce q con $h(d)=1$

Siegel $\Rightarrow$ hay a lo sumo 10. (Heegner, Stark, Baker son 9) 60

② $\forall \varepsilon > 0 \quad \exists\, C_3(\varepsilon) > 0$

χ real mod q, $\beta > 1 - C_3(\varepsilon)\, q^{-\varepsilon} \Rightarrow L(\beta,\chi) \neq 0$

Prop ① $\Rightarrow$ ② usando $L'(s,\chi) = O(\log^2 q)$

en $1 - \frac{1}{\log q} \leq \beta \leq 1$.

Usar 2 caracteres χ_1, χ_2 reales primitivos

mód $q_1 \neq q_2$

$\chi_1 \chi_2$ mód $q_1 q_2$ (no nec. primitivo)

$$\boxed{F(s) = \zeta(s) L(s,\chi_1) L(s,\chi_2) L(s,\chi_1\chi_2)}$$

holomorfa excepto un polo simple en $s=1$

con residuo $\lambda = L(1,\chi_1) L(1,\chi_2) L(1,\chi_1\chi_2)$

Lema $F(s) > \frac{1}{2} - \frac{c_4 \lambda}{1-s} (q_1 q_2)^{8(1-s)}$

$\forall \left(\frac{7}{8} < s < 1\right)$

Dem del Teo de Siegel

dado $\varepsilon > 0$.

$\boxed{A}$ $\exists \chi_1$: $L(s,\chi_1)$ tiene cero real $\beta_1 > 1 - \frac{\varepsilon}{16}$.

$\Rightarrow F(\beta_1) = 0 \quad \forall \chi_2$

$\boxed{B}$ Si no elegimos χ_1 cualquiera, $\beta_1 > 1 - \frac{\varepsilon}{16}$

$\Rightarrow F(\beta_1) < 0 \quad \forall \chi_2$

~~En cualquier caso: $F(\beta_1) \leq 0$~~

Lema
$\Rightarrow$ $$c_4 \lambda > \frac{1}{2}(1-\beta_1)(q_1 q_2)^{-8(1-\beta_1)}$$

Ahora χ_1, β_1 están fijos

Sea χ_2 real primitivo mod $q_2 > q_1$

Sabemos $L(1,\chi_1) < c_{11} \cdot \log q_1$

$L(1,\chi_1\chi_2) < c_{11} \cdot \log(q_1 q_2)$

$$\Rightarrow \lambda < c_{11}^2 \cdot \log q_1 \cdot \log(q_1 q_2) \cdot L(1,\chi_2)$$

$$\Rightarrow L(1,\chi_2) > C \cdot q_2^{-\overbrace{8(1-\beta_1)}^{<\frac{\varepsilon}{2}}} (\log q_2)^{-1}$$

$\beta_1 > 1 - \frac{\varepsilon}{16}$

Obs: C depende solo de χ_1

i.e. depende solo de ε.

Dado $C>0$

$$q \leq \exp\left(C \log^{1/2} x\right)$$

$$\psi(x,\chi) = \frac{-x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-\grave{c} \log^{1/2} x\right)\right)$$

($\grave{c}$ depende de C)

Siegel) dado $\varepsilon>0$

$$\beta_1 < 1 - C_1(\varepsilon)\, q^{-\varepsilon}.$$

$$\hookrightarrow \quad x^{\beta_1} < x \exp\left(-C_1(\varepsilon) \frac{\log x}{q^{\varepsilon}}\right)$$

Tomar $q \leq (\log x)^N$ (Algún $N>0$)

Sea $\varepsilon = \frac{1}{2N} \longrightarrow q^{\varepsilon} \leq \log^{1/2} x$

$$\Rightarrow x^{\beta_1} < x \exp\left(-C_2(N) \cdot \log^{1/2} x\right)$$

$$\rightsquigarrow \psi(x,\chi) \ll x \exp\left(-C_3(N) \log^{1/2} x\right)$$

<u>Teo</u> Sea $N \geq 0$. $\exists\, C_3(N) > 0$ tq

$$q \leq (\log x)^N \Rightarrow \psi(x; q, a) = \frac{x}{\varphi(q)} + O\left(x \exp\left(-C_3(N) \log^{1/2} x\right)\right)$$

(unif en q)

$\rightarrow$ resultados similares para

$$\pi(x; q, a) = \frac{\operatorname{li} x}{\varphi(q)} + O(\ \cdots\ )$$