On rank jumps on families of elliptic curves

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The new results are from distinct collaborations with
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Plan

- **Motivation**
- **Definitions and examples**
- **The problem and different methods**
- **More on the geometric method**
- **What's next?**

Ranks of elliptic curves

Let k be a number field and E/k an elliptic curve.

$E : y^2 = x^3 + ax + b$, with $a, b \in k$.

Mordell-Weil Theorem: $E(k) \simeq \mathbb{Z}^{r(a,b)} \oplus Tors_{a,b}$.

Consider a family of elliptic curves:

$$(\star) \quad E_t : y^2 = x^3 + a(t)x + b(t), \text{ with } a(t), b(t) \in k[t].$$

For $t \in k$ such that $\Delta(t) \neq 0$, we have $E_t(k) \simeq \mathbb{Z}^{r_t} \oplus Tors_t$.

Natural Question: How does r_t behave as t varies?

TODAY: We'll use surfaces to deal with this question.

Elliptic surface

A smooth projective surface S is called an *elliptic surface* if $\exists \pi : S \to B,\ s.t.$

- $\pi^{-1}(t)$ is a smooth curve of genus 1, for almost all $t \in B$
- $\exists \sigma : B \to S$, a section

We suppose moreover that

- there is at least one singular fiber
- π is relatively minimal

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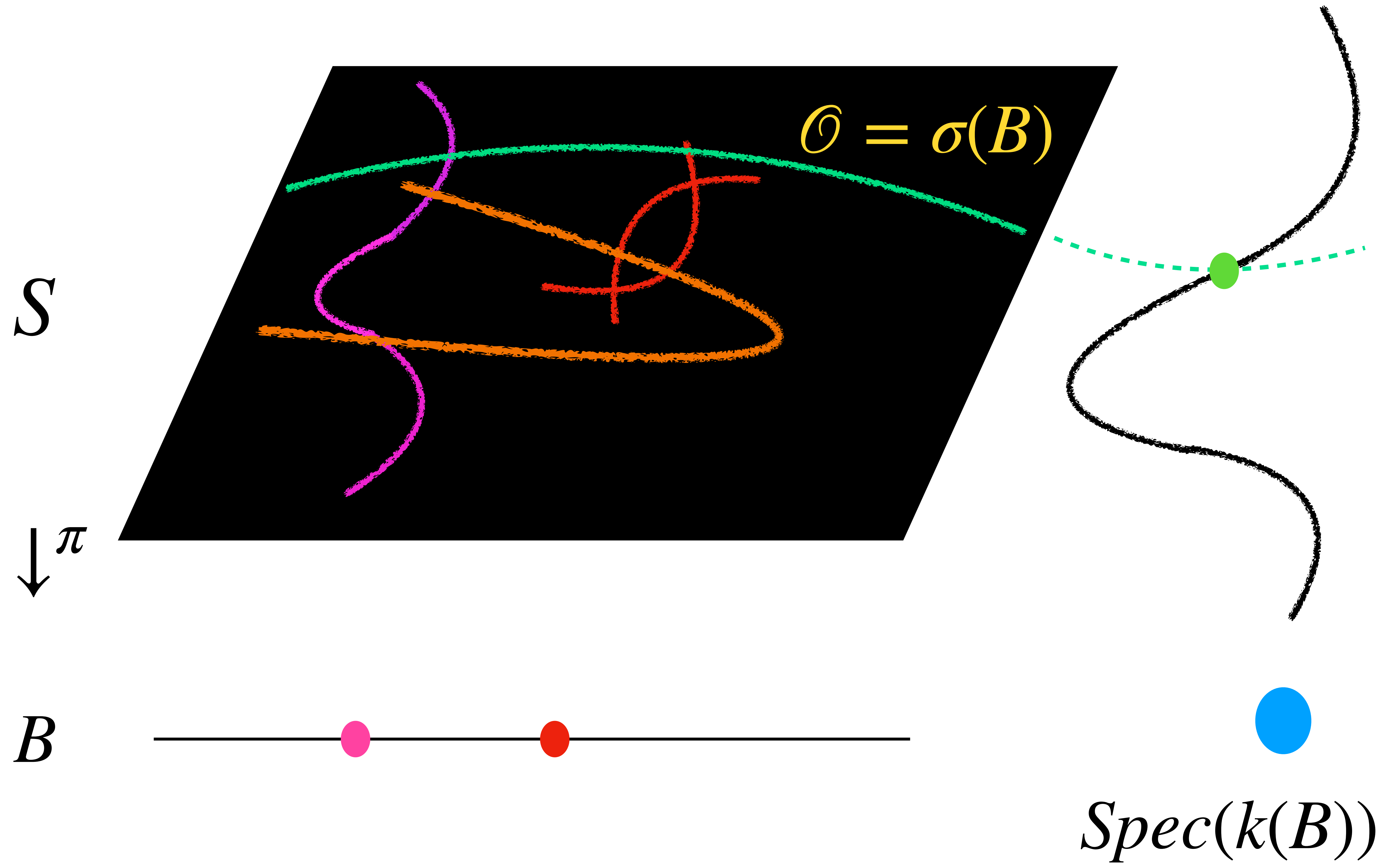


$$Y^2 = X^3 + aX + b\,; \quad a, b \in k(B)$$

In orange: a multisection

Why do we care?

Elliptic surfaces appear in many places

- Shioda-Tate: $\mathrm{NS}(S)/T \simeq \mathrm{MW}(\pi)$
- Zariski density/potential density (Bogomolov-Tschinkel, S.- van Luijk)
- k-unirationality of conic bundles (Kollár-Mella)
- Dense sphere packings (Shioda, Elkies)
- Error correcting codes (S. - Várilly-Alvarado - Voloch)
- High rank elliptic curves over $\mathbb{Q}$ (Elkies - record: 28)

An example

Rational elliptic surfaces

Consider F, G two plane cubics. Then

$F \cap G = 9$ points counted with multiplicities and we have

Blow up $F \cap G$

$$\begin{array}{ccc} & & S \\ & \swarrow & \downarrow \pi \\ \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \end{array}$$

$$(x : y : z) \mapsto (F(x, y, z) : G(x, y, z))$$

Arithmetic of elliptic surfaces

Given a number field k

The Mordell-Weil theorem tells us that:

- For the special fibers $E_t := \pi^{-1}(t)$ with $t \in B(k)$:

$$E_t(k) = \mathbb{Z}^{r_t} \oplus \mathrm{Tors}_t\,.$$

- For the generic fiber:

$$\mathscr{E}_\eta(k(B)) = \mathbb{Z}^r \oplus \mathrm{Tors}.$$

From now on: *rank* = Mordell-Weil rank, and r_t denotes the rank of the special fiber E_t and r denotes the rank of the generic fiber.

Ranks of elliptic curves in families

$(\star) \quad E_t : y^2 = x^3 + a(t)x + b(t)$, with $a(t), b(t) \in k[t]$.

Natural Question: How does r_t behave as t varies?

Given $i \in \mathbb{N}$ and $\mathcal{G}_i := \{t \in \mathbb{P}^1(k); r_t = i\} \subset \mathbb{P}^1(k)$, what can we say about $\#\mathcal{G}_i$?

Néron-Silverman's Specialization Theorem: $r_t \geq r$ for all but finitely many t.

So $i < r \Rightarrow \#\mathcal{G}_i < \infty$.

What about $\mathcal{G}_i$, for $i \geq r$?

We'll look at $\mathcal{F}_{r+i} := \{t \in \mathbb{P}^1(k); r_t \geq r + i\}$.

Ranks of elliptic curves in families

Néron and Silverman Specialization Theorems tell us that:

$$r_t \geq r, \text{ for all but finitely many } t \in B(k).$$

More precisely:

Néron: outside a **THIN** set.

Silverman: outside a set of bounded height.

Can we say more?

When $r_t > r$ we say that the **rank jumps.**

TODAY: Does the **rank jump**? **Where and how large is the jump**?

Methods

- Root Numbers
- Height Theory
- Base change

Root numbers

Given an elliptic curve E/k. The root number of E is the sign of the functional equation:

$$\tilde{L}(E, s) = W(E)\,\tilde{L}(E, 2 - s).$$

Parity conjecture: $W(E) = (-1)^{\mathrm{rank}(E)}$.

In other words: *The parity of the rank of an elliptic curve over a number field is determined by its root number.*

Variation of the root number ⇒ Rank jump

Constant root number with different "parity" from the generic rank ⇒ Rank jump.

Variation of root numbers in families

Isotrivial families (Rohrlich, Gouvêa, Mazur, Várilly-Alvarado, Dokchitser^2, Desjardins)

Ex: $Y^2 = X^3 - (1 + T^4)X$, $W(E_t) = -1, \forall t$ and hence $\#\mathcal{F}_{r+1} = \infty$.

Non-isotrivial families

Expected: Both +1 and -1 occur infinitely often.

Holds under major conjecture and known under hypothesis on the degree of the coefficients.

Height theory approach

Definition: $P \in E_t(\mathbb{Q})$ is a **division point** if $\exists n \in \mathbb{N}$ s.t. $n \cdot P \in \mathrm{Sec}(\pi)(\mathbb{Q})$.

Let $U \subset S$ be a Zariski open. We denote by U_{div} the set of division points in U.

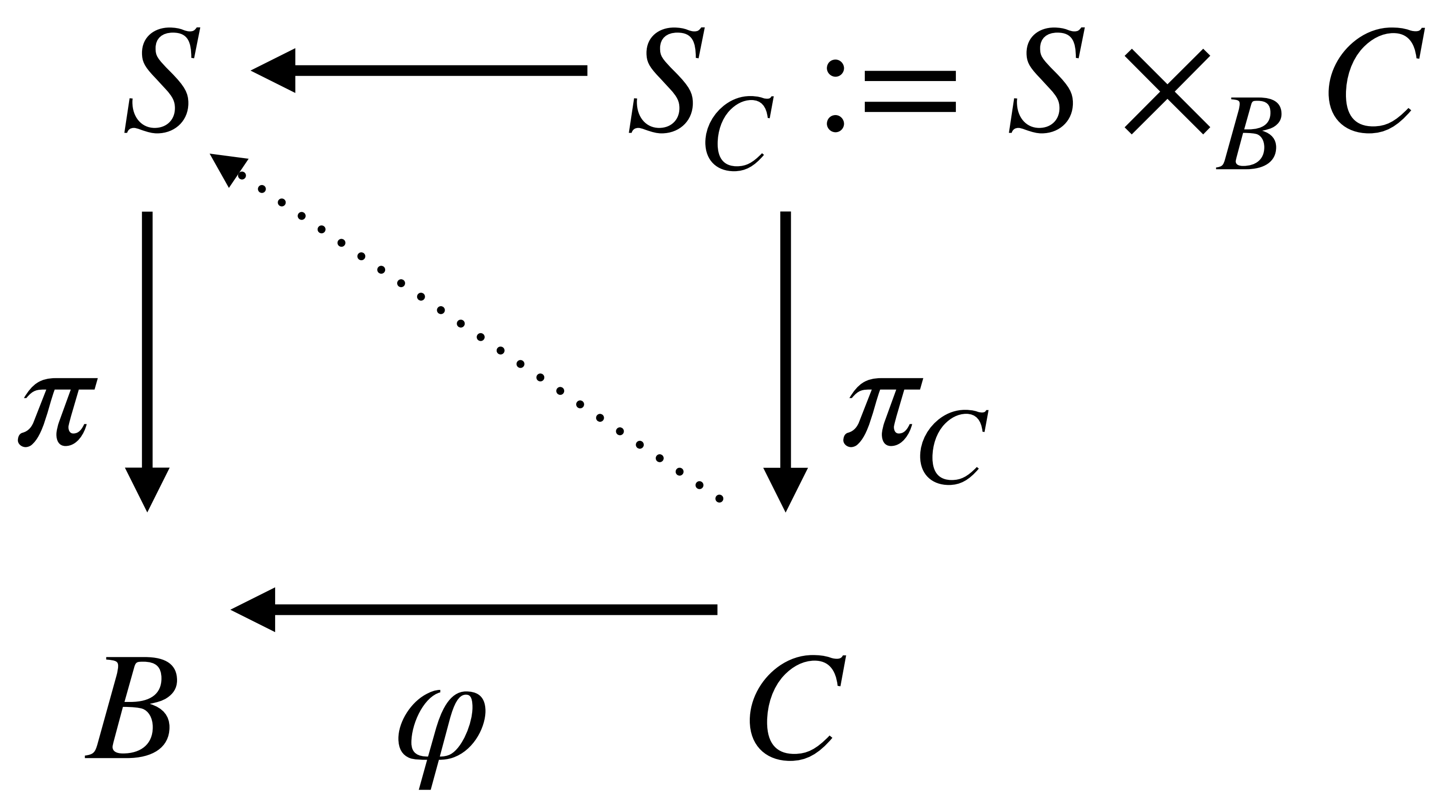
Idea: Count division points of bounded height on fibers and compare with total count (of bounded height).

Billard (2000): Let S be a $\mathbb{Q}-$rational elliptic surface and $D \subset S$ an ample divisor.
There is $\delta > 0$ s.t. $\forall U \subset S$

$N(U(\mathbb{Q}), H_D, B) \gg B^{\delta}$ and $N(U_{div}(\mathbb{Q}), H_D, B) \ll B^{\delta/2}$

Corollary (Billard): $\#\mathscr{F}_{r+1} = \infty$.

Geometric approach: base change



- π_C is an elliptic fibration
- Sections of π induce sections of π_C
- New section $\sigma_C : C \to S \times_B C$
- Hence $\mathrm{rk}(S_C(k(C))) \geq r = \mathrm{rk}(S(k(B)))$
- If σ_C independent of sections of π then:

For $t \in \varphi(C(k)) \subset B(k)$ we have $r_t \geq r + 1$.

Interesting when $\#C(k) = \infty$ because then we get rank jump on an infinite set!

Making sure that the rank jumps

Surfaces with many rational curves

- A key Lemma in (S.2011) shows that if $dim|C| \geq 1$ and C is not a fiber component then all but finitely many curves in $|C|$ make the rank jump after base change by it.

- TO DO: Give hypothesis to assure that S contains linear systems of curves with $\#C(k) = \infty$.

- **Good candidate**: $k-$unirational surfaces!

Rank jumps by base changing

- S. (2011): If S is $k-$unirational then

$$\#\mathscr{F}_{r+1} = \infty \, .$$

- S. (2011) If moreover S has two conic bundle structures then

$$\#\mathscr{F}_{r+2} = \infty \, .$$

What about the quality of these sets?

Quality - Expectation

- Silverman conjectured in the 80’s that

$$r_t = r \text{ or } r + 1,$$

for $100\,\%$ of the fibers when ordered by height

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Thin Sets

Given an algebraic variety V over k. A subset $T \in V(k)$ is said to be:

- Of ***type 1*** if it is contained in a proper Zariski closed subset.
- Of ***type 2*** if is contained in the image of the $k-$points of a dominant morphism of degree at least 2

$$\phi : W \to V, \text{ so } T \subset \phi(W(k)) \subset V(k).$$

T is called ***THIN*** if it is contained in a finite union of subsets of types 1 and 2.

V is said to satisfy the **HILBERT PROPERTY over** k, if $V(k)$ is not thin.

Examples

A. Over number fields, $\mathbb{P}^n$ satisfies the Hilbert Property, for all n.

B. The set of $\square$'s in a number field is THIN. Indeed, they lie in the image of the degree 2 map $t \mapsto t^2$.

Our contribution

Thm. A (Loughran, S. 2019): Let $\pi : S \to \mathbb{P}^1$ be a geometrically rational elliptic surface such that π admits a bisection of arithmetic genus zero then

$$\mathscr{F}_{r+1} = \{t \in \mathbb{P}^1(k); r_t \geq r+1\} \text{ is not thin.}$$

Thm. B (Loughran, S. 2019): If moreover π has at most one non-reduced fiber OR admits a 2-torsion section defined over k, then

$$\mathscr{F}_{r+2} = \{t \in \mathbb{P}^1(k); r_t \geq r+2\} \text{ is not thin.}$$

Our contribution

Thm. C (Dias, S. 2021): Let $\pi : S \to \mathbb{P}^1$ be a geometrically rational elliptic surface such that one of the following holds:

$i)$ It has a non-reduced fibre of type II^*, III^* or I_n^*, for $2 \leq n \leq 4$;

$ii)$ It has a non-reduced fibre of type IV^*, I_1^* or I_0^* and a reducible reduced singular fibre;

then,

$$\mathscr{F}_{r+3} = \{t \in \mathbb{P}^1(k); r_t \geq r + 3\} \text{ is not thin.}$$

An example (due to Rohrlich)

Consider an elliptic curve $E : y^2 = x^3 + ax + b$ with $\mathrm{rk}(E(k)) = 1$. Then

$S : ty^2 = x^3 + ax + b$ is an elliptic surface fibered over the $t-$line. Moreover we have $r = 0$ and $r_t = 1$, for $t = \square$. Thus

$\#\mathcal{F}_{r+1} = \infty$ BUT the set $\{t \in \mathbb{P}^1(k); t = s^2,$ for some $s \in k\}$ is thin!

Using the hypothesis

Thm. A (Loughran, S. 2019): Let $\pi : S \to \mathbb{P}^1$ be a geometrically rational elliptic surface such that π admits a bisection of genus zero then

$$\{t \in \mathbb{P}^1(k); r_t \geq r+1\} \text{ is not thin.}$$

- Since $h^1(S) = 0$ and C is a bisection of genus zero, Riemann-Roch gives that C moves, i.e., $\dim |C| \geq 1$. Moreover $C' \simeq \mathbb{P}^1_k$ for infinitely many curves in $|C|$.
- This together with $K_S^2 = 0$ can be used to show that S is k-unirational.
- At that point we can use a curve in $|C|$ to prove: $\#\mathcal{F}_{r+1} = \infty$.

BUT…

The set is thin!

$\varphi(C(k))$ is a set of type 2 and hence THIN!

STRATEGY: Consider all curves in $|C|$ at once!

Remember that given any finite number of covers of the base, we have to find a curve in $|C|$ that has a $k-$point mapped to a point outside of their image.

How does one show that a subset T of the line is NOT THIN?

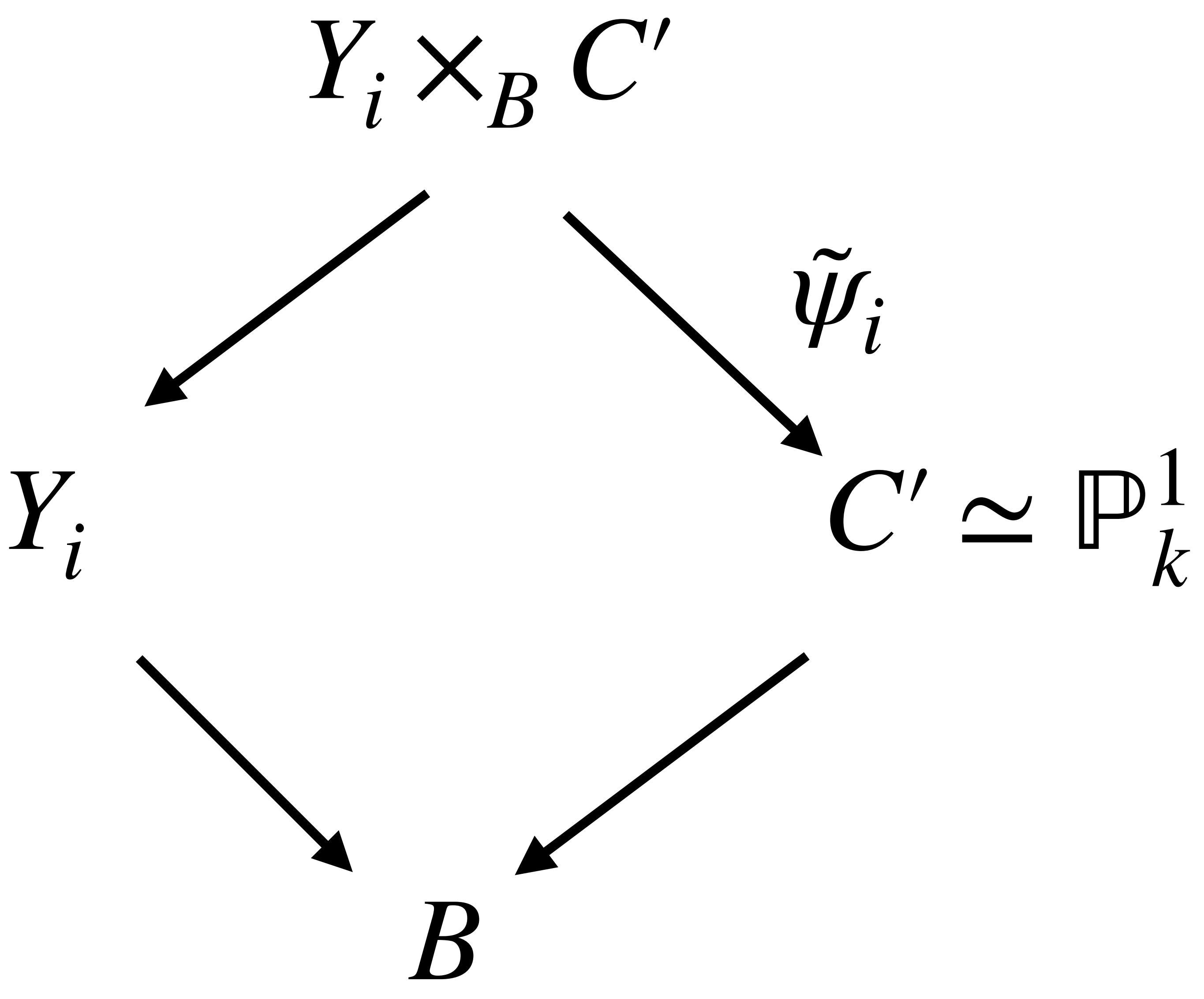
We have to check that given a finite number of arbitrary covers

$$\phi_i : Y_i \to \mathbb{P}^1, i = 1, \cdots, n$$

there exists $t \in (\mathbb{P}^1(k) \cap T) \backslash (\cup_i \phi_i(Y_i(k)))$.

Avoiding the covers

Given a finite number of covers $\psi_i : Y_i \to B$ we have to find $P \in C'(k)$ such that $\varphi(P) \notin \cup\, \psi_i(Y_i(k))$.



If $Y_i \times_B C'$ is an integral curve then

Hurwitz formula tells us that $g(Y_i \times_B C') \geq 1$.

Since $C' \simeq \mathbb{P}^1_k$ satisfies the Hilbert Property,

there is a $P \in C'(k) \backslash \tilde{\psi}_i((Y_i \times_B C')(k))$, i.e.,

$\varphi(P) \in B \backslash (\cup_i \psi_i(Y_i(k)))$.

BUT HOW CAN WE MAKE SURE THAT $Y_i \times_B C'$ is integral?

Integral fibre products

Linearly disjoint function field extensions

- Recall that $k(C')/k(B)$ is a quadratic extension.
- Given $k(Y_i)/k(B)$ a finite number of extensions, then there are only finitely many quadratic sub-extensions. Thus
- If the set $\{k(C')/k(B); C' \in |C|, \#C(k) \neq \emptyset\}$ has infinitely many isomorphism classes of quadratic extensions then there are infinitely many $C' \in |C|$ s.t. $C' \times_B Y_i$ is smooth.
- If the ramification of $C' \to B$ varies this is certainly true! This is the case when there is at most one non-reduced fibre.

More than one non-reduced fibre

Same ramification but non-isomorphic field extensions

- There is only one configuration of reducible fibres with more than one non-reduced fibre, namely $(2I_0^*)$. Its ("almost") Weierstraß equation is of the form:

$$Y^2 = g(t)f(x), \text{ with } f, g \text{ separable of degrees 3 and } \leq 2, \text{ respectively.}$$

- A bisection of genus zero is given by the curve $C_{x_0} : Y^2 = g(t)f(x_0)$, for any choice of x_0 with $f(x_0) \neq 0$.

- The map $C_{x_0} \to B$ ramifies at the zeroes of $g(t)$, for all x_0.

HOW DO WE SHOW THAT $k(C_{x_0})/k(B)$ are linearly disjoint?

HOW DO WE SHOW THAT $k(C_{x_0})/k(B)$ are linearly disjoint?

A secret about the surface that I didn't tell you yet

- $k(C_{x_0}) = k(t)(\sqrt{f(x_0)g(t)})$. Assume that $f(x)$ takes only finitely many values modulo squares. Then we can write
- $S(k) = \bigcup_{d \in F} S_d(k)$, where F is a finite set and $S_d(k) := \{(x, y, t) \in S(k); f(x) = dw^2, \text{ for some } w \in k\}$.
- The sets $S_d(k)$ are THIN! Hence $S(k)$ is also thin.
- BUT this is in contradiction with the fact that S is a Châtelet surface which is known to satisfy the **Hilbert Property** (thanks to Colliot-Thélène and Sansuc).

Directions

- More on the quality of the sets for which the rank jumps?
- Fibrations on abelian varieties.
- Non-geometrically rational elliptic surfaces, e.g. K3! (**Renato Dias**).

Obrigada!
Gracias!