De ABC a L: Conjectura do ABC uniforme e a não-existência de zeros de Siegel

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Sumário

Sumário

"N.e. zeros de Siegel" para $D < 0$

$$\Longleftrightarrow$$

Minorantes para o "número de classes $\times$ ~~regulador~~" de $\mathbb{Q}(\sqrt{D})$

N.e. = não existem

Notação

- $D < 0$ discriminante fundamental (i.e., $\sqrt{D} \notin \mathbb{R}$, $D = \operatorname{disc} \mathbb{Q}(\sqrt{D})$);
- $C\ell(D)$ = grupo de classes de $\mathbb{Q}(\sqrt{D})$;
- $h(D) = |C\ell(D)|$ número de classes de $\mathbb{Q}(\sqrt{D})$;
- j = função j-invariante:

$$j(\tau) := \frac{\left(1 + 240 \sum_{n\geq 1} \left(\sum_{d|n} d^3\right) q^n\right)^3}{q \prod_{n\geq 1}(1-q^n)^{24}} \qquad \left(\Im(\tau) > 0,\ q = e^{2\pi i \tau}\right)$$

$$\boxed{\begin{aligned} j(\tau+1) &= j(\tau), \\ j(-1/\tau) &= j(\tau); \end{aligned}} \qquad \boxed{\begin{gathered} j(i) = 1728, \\ j(e^{2\pi i/3}) = 0, \\ \text{“}j(i\infty) = 1\infty\text{”}; \end{gathered}}$$

<u>q-expansão de j</u>:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$$

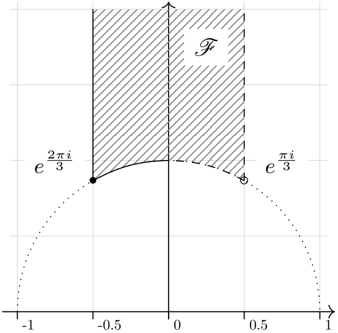
Módulos singulares (1/2)

Seja $\tau \in \mathfrak{h}$ $(\Im(\tau) > 0)$

- Ponto CM: $\tau \mid A\tau^2 + B\tau + C = 0$ $\left(\begin{matrix} A, B, C \in \mathbb{Z},\ A > 0, \\ \gcd(A, B, C) = 1,\ \text{única} \end{matrix}\right)$
- Módulo singular: $j(\tau)$ $\quad (j = j\text{-invariante}, \tau \text{ ponto CM})$

Pontos CM		Formas quadráticas
$\tau \in \mathfrak{h}$	$\longleftrightarrow$	$Ax^2 + Bxy + Cy^2$

- $\mathrm{disc}(\tau) := B^2 - 4AC$
- $\tau \sim \tau' \iff (A, B, C) \sim (A', B', C')$
- $\tau \in \mathscr{F} \iff (A, B, C)$ *reduzida*



Módulos singulares (2/2)

$$\underline{\text{Pontos de Heegner } \Lambda_D} \qquad \underline{\text{F.Q.B.s prim.s reduzidas de disc. } D}$$

$$\left\{\begin{matrix}\text{Pontos CM } \tau \in \mathscr{F}, \\ \operatorname{disc}(\tau) = D\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}(a,b,c) := ax^2 + bxy + cy^2 \\ \text{t.q. } b^2 - 4ac = D, \text{ e} \\ -a < b \leq a < c \text{ ou } 0 \leq b \leq a = c\end{matrix}\right\}$$

$$\uplus \qquad\qquad\qquad \uplus$$

$$\boxed{\tau_D := \underbrace{\frac{\sqrt{D}}{2}}_{D\equiv 0(4)} \text{ ou } \underbrace{\frac{-1+\sqrt{D}}{2}}_{D\equiv 1(4)}} \leftrightarrow \underbrace{\text{Forma principal}}_{(1,0,-\frac{D}{4}) \text{ ou } (1,1,\frac{1-D}{4})} \qquad \mathbb{Z}[\tau_D] = \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$$

Escreva $\mathrm{H}_D :=$ corpo de classes de Hilbert de $\mathbb{Q}(\sqrt{D})$.

- $\mathrm{H}_D = \mathbb{Q}(\sqrt{D}, j(\tau_D)) \quad \Big([\mathrm{H}_D : \mathbb{Q}(\sqrt{D})] = [\mathbb{Q}(j(\tau_D)) : \mathbb{Q}] = h(D)\Big)$
- $\{j(\tau) \mid \tau \in \Lambda_D\} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$-conjugados de $j(\tau_D)$
- $j(\tau_D)$ é um **inteiro (!)** algébrico

A equação $x^3 - Dy^2 = 1728$

Considere a equação diofantina $x^3 - Dy^2 = 1728 \quad (= j(i))$.

- Fatoração de diferenças de módulos singulares
- Soluções do tipo $(j(\tau_D), j(\tau_D) - 1728) = (x^3, Dy^2)$
- Ideia: $ABC \implies$ poucas sol.s assim $\implies$ **minorantes pra** $h(D)$

ABC : $(a + b = c)$

$$\boxed{\begin{array}{c} a, b, c \in \mathbb{Z} \text{ coprimos. } \forall \varepsilon > 0,\ \exists C_\varepsilon > 0 \text{ t.q.} \\ \max\{|a|, |b|, |c|\} < C_\varepsilon \Big(\prod_{p \mid abc} p \Big)^{1+\varepsilon} \end{array}}$$

Exemplo (Problema do número de classes 1)

Gross–Zagier: $h(D) = 1 \implies x,\ y\sqrt{|D|} \in \mathbb{Z}$. Mas então

$$\begin{aligned} |j(\tau_D)| \le \max\{|x|^3, |y\sqrt{D}|^2, 1728\} &\overset{\text{abc}}{\le} C_\varepsilon \cdot |x \cdot y\sqrt{D}|^{1+\varepsilon} \\ &\le C_\varepsilon \cdot |j(\tau_D)|^{\frac{5}{6}(1+\varepsilon)}, \end{aligned}$$

então há apenas um número finito de $D < 0$ com $h(D) = 1$. △

Como cresce $h(D)$?

O quê sabemos?

- $h(D) \to \infty$ quando $D \to -\infty$ (Heilbronn, 1934)
  - Listar $h(D)$ (valores pequenos de $|D|$)
  - Estimar $h(D)$ (valores grandes de $|D|$)

- $-D \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$ são *todos* os D t.q. $h(D) = 1$
  - Resolvido por Heegner ($\sim$1952) [baseado em Weber]
  - Refeito e clarificado por Stark, Birch (1967$\sim$9)
  - Solução de Baker ($\sim$1970) [baseado no Teorema de Baker]

- $h(D)$ cresce *essencialmente* como $|D|^{1/2}$ (HRG $\Rightarrow$ err $\approx (\log\log|D|)^{\pm 1}$)
  - Hecke (1917): Se $L(s, \chi_D) \neq 0$ **próx.** 1, então $\dfrac{h(D)}{|D|^{1/2}} \gg (\log|D|)^{-1}$
  - Landau (1918): $\max\left\{\dfrac{h(D_1)}{|D_1|^{1/2}}, \dfrac{h(D_2)}{|D_2|^{1/2}}\right\} \gg (\log|D_1 D_2|)^{-1}$
  - Siegel (1935): $\dfrac{h(D)}{|D|^{1/2}} \gg_\varepsilon |D|^{-\varepsilon}$ (não-condicional/não-efetivo)

Sumário

O teorema de Granville–Stark

Teorema (Granville–Stark, 2000)

Conj. abc uniforme $\Longrightarrow$ *"N.e. zeros de Siegel" pra $L(s,\chi_D)$, $D<0$*

- Estudar $x^3 - y^2 = 1728$
- ABC uniforme em corpos de números (abc-U)
- abc-U $\Longrightarrow$ minorantes para $h(D)$
- Minorantes $h(D)$ $\Longleftrightarrow$ "n.e. zeros de Siegel"

A. Granville

H. Stark

A equação $x^3 - y^2 = 1728$

- Soluções do tipo $(j(\tau_D), j(\tau_D) - 1728) = (x^3, y^2)$

$$\underline{\textbf{ABC}} \text{ (em } \mathbb{Q}) : \quad \log\max\{|a|,|b|,|c|\} < (1+\varepsilon)\underbrace{\boxed{\sum_{p|abc} \log p}}_{\text{log-condutor}} + O_\varepsilon(1)$$

- Aplicando a mesma lógica para $x^3 - y^2 - 1728 = 0$, obteríamos:

$$\begin{aligned}\log\max\{|x|^3,|y|^2\} &\leq \log\max\{|x|^3,|y|^2,1728\} \\ &\overset{\text{abc}}{\leq} (1+\varepsilon)\log\text{-cond}(12|xy|) + \textbf{erro} \\ &\leq \frac{5}{6}(1+\varepsilon)\log\max\{|x|^3,|y|^2\} + \textbf{erro}\end{aligned}$$

Como tornar isto preciso? $\longrightarrow$ $\boxed{\log\max\{|x|^3,|y|^2\} < (6+\varepsilon')\cdot\textbf{erro}}$

ABC em corpos de números (1/2)

Seja $K/\mathbb{Q}$ um corpo, $\mathbb{V}(K)$ lugares, $\mathbb{V}(K)^{\text{non}} \subseteq \mathbb{V}(K)$ lugares não-arq..

Para um ponto $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n_K$, definimos:

- altura (ingênua, abs, log) $\mathrm{ht}(P)$

$$\boxed{\frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathbb{V}(K)} \log\left(\max_i \{\|x_i\|_v\}\right)}$$

- (log) condutor $\mathcal{N}_K(P)$

$$\boxed{\frac{1}{[K:\mathbb{Q}]} \sum_{\substack{v \in \mathbb{V}(K)^{\text{non}} \\ \exists i,j \leq n \text{ t.q.} \\ v(x_i) \neq v(x_j)}} f_v \log(p_v)}$$

$$\updownarrow$$

$$\begin{cases} v \sim \mathfrak{p} = \mathfrak{p}_v \\ p_v \sim \mathfrak{p}_v \cap \mathbb{Q} \\ f_v := [K_v : \mathbb{Q}_{p_v}] \end{cases}$$

Para $a, b, c \in \mathbb{Z}$ coprimos,

- $\mathrm{ht}([a:b:c]) = \log \max\{|a|, |b|, |c|\}$
- $\mathcal{N}_{\mathbb{Q}}([a:b:c]) = \log\left(\prod_{p|abc} p\right)$

Pra $\alpha \in \overline{\mathbb{Q}}$, $\mathrm{ht}(\alpha) := \mathrm{ht}([\alpha : 1])$.

$$\boxed{\alpha \text{ integral} \Rightarrow \mathrm{ht}(\alpha) = \frac{1}{|\mathcal{A}|} \sum_{\alpha^* \in \mathcal{A}} \log^+ |\alpha^*|}$$

$\mathcal{A} = \{\text{conjugados of } \alpha\}$

ABC em corpos de números (2/2)

abc em corpos de números

Fixe $K/\mathbb{Q}$ um corpo de números. Então, para todo $\varepsilon > 0$ existe $\mathcal{C}(K,\varepsilon) \in \mathbb{R}_+$ tal que, $\forall a,b,c \in K \mid a+b+c=0$, temos

$$\mathrm{ht}([a:b:c]) < (1+\varepsilon)\left(\mathcal{N}_K([a:b:c]) + \log(\mathrm{rd}_K)\right) + \mathcal{C}(K,\varepsilon),$$

onde $\mathrm{rd}_K := |\Delta_K|^{1/[K:\mathbb{Q}]}$ *é o* discriminante-raiz *de K.*

- Ideia [G–S]: Resolver $x^3 - y^2 = j(i)$ em diferenças de módulos singulares:

$$(j(\tau_D),\, j(\tau_D) - 1728) = (x^3, y^2) \qquad \boxed{x^3 - y^2 = 1728}$$

$$j(\tau_D) \in \mathrm{H}_D \qquad \widetilde{\mathrm{H}}_D := \mathrm{H}_D(x,y) \quad ([\widetilde{\mathrm{H}}_D : \mathrm{H}_D] \leq 6)$$

Lema

$$\mathrm{rd}_{\widetilde{\mathrm{H}}_D} \leq 6\sqrt{|D|}$$

Proposição (para $0 < \varepsilon < \frac{1}{5.01}$)

$$\mathrm{ht}(j(\tau_D)) < \frac{3}{1-5\varepsilon}\left((1+\varepsilon)\log|D| + 2\mathcal{C}(\widetilde{\mathrm{H}}_D,\varepsilon)\right) + O(1)$$

O quê obtemos?

Assim, para todo $\varepsilon > 0$ pequeno, temos:

$$\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{3\log|D|} \leq \frac{1+\varepsilon}{1-5\varepsilon} + \limsup_{D\to-\infty} \frac{2\mathcal{C}(\widetilde{\mathrm{H}}_D, \varepsilon)}{(1-5\varepsilon)\log|D|}.$$

Como progredir?

$$\left(\text{Note que: } \begin{array}{c} [\widetilde{\mathrm{H}}_D : \mathbb{Q}] \asymp h(D), \\ \log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D}) = \log|D| + O(1) \end{array}\right)$$

- Uniformidade: $\mathcal{C}(K, \varepsilon) = \mathcal{C}(\varepsilon)$ $\quad (\implies \limsup \leq 1)$
- **Uniformidade fraca**: $\mathcal{C}(K, \varepsilon) = o_\varepsilon(\log \mathrm{rd}_K)$ $\quad (\implies \limsup \leq 1)$
- Uniformidade O-fraca: $\mathcal{C}(K, \varepsilon) = O_\varepsilon(\log \mathrm{rd}_K)$ $\quad (\implies \limsup < +\infty)$
- S. Mochizuki (2020):† $\mathcal{C}(K, \varepsilon) = O_\varepsilon([K : \mathbb{Q}]^{4+\varepsilon})$ $\quad$ (insuficiente!)

† Ignorando certas tecnicalidades.

Minorantes para $h(D)$

Lema [G–S]

$$ABC \text{ uniforme (fraca)} \implies \mathrm{ht}(j(\tau_D)) \leq (3 + o(1)) \log |D|$$

Mas ao mesmo tempo,

$$\mathrm{ht}(j(\tau_D)) = \frac{1}{h(D)} \sum_{(a,b,c)} \log^+ |j(\tau_{(a,b,c)})|$$

$$= \frac{1}{h(D)} \sum_{(a,b,c)} \frac{\pi\sqrt{|D|}}{a} + O(1),$$

$ax^2 + bxy + cy^2$
reduzida,
$b^2 - 4ac = D$

$$\tau_{(a,b,c)} = \frac{-b + \sqrt{D}}{2a}$$

e portanto:

Teorema [G–S]

$$h(D) \overset{abc\text{-U}}{\geq} \left(\frac{\pi}{3} + o(1)\right) \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)} \frac{1}{a}$$

q-expansão!

$$\boxed{j(\tau) = \frac{1}{q} + O(1)}$$

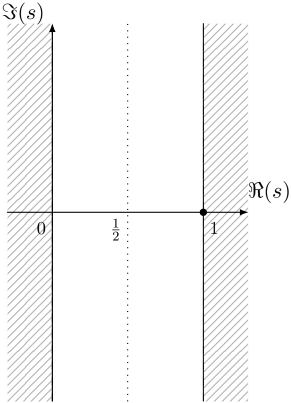
Sumário

A **L-função de Dirichlet** de um caractere primitivo $\chi \pmod q$:

$$L(s,\chi) := \sum_{n\geq 1} \frac{\chi(n)}{n^s}, \qquad (\Re(s) > 1)$$

- Produto de Euler ($\Re(s) > 1$):
  - $L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$
- Equação functional:
  - $L^*(s,\chi) := \gamma_\chi(s)\, L(s,\chi)$ é inteira
  - $L^*(s,\chi) = W(\chi)\, L^*(1-s,\overline{\chi})$

    (onde $|W(\chi)| = 1$)
- TNP de Dirichlet:
  - $L(1+it,\chi) \neq 0$, $\forall t \in \mathbb{R}$.



Regiões livres de zeros clássicas

[Gronwall 1913, Landau 1918, Titchmarsh 1933]

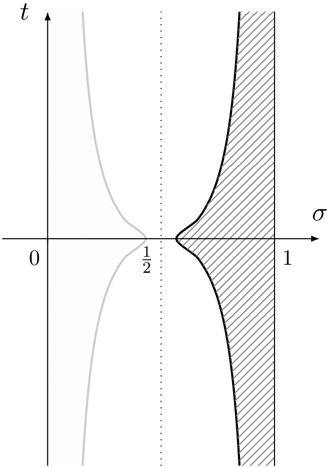
Escreva $s = \sigma + it$ ($\sigma = \Re(s)$, $t = \Im(s)$), e seja $\chi \pmod q$ um caractere primitivo.

Existe $A > 0$ tal que, na região

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{A}{\log q(|t|+2)} \right\},$$

a função $L(s, \chi)$:

- (χ complexo) não possui zeros;
- (χ real) possui no máximo <u>um</u> zero, que se existe é necessariamente *real* e *simples* – o **zero de Siegel**.



Caracteres de Dirichlet reais primitivos

$$\boxed{\left\{\begin{matrix}\text{Caracteres de Dirichlet} \\ \text{reais primitivos}\end{matrix}\right\} \longleftrightarrow \left\{\left(\frac{D}{\cdot}\right), \begin{matrix}D \text{ discriminante} \\ \text{fundamental}\end{matrix}\right\}}$$

	$\mathbb{Q}(\sqrt{D})$	$\chi_D \pmod{\lvert D\rvert}$
Fatoração	$D = D_1 \cdots D_t$	$\chi_D = \chi_{D_1} \cdots \chi_{D_t}$
Ramificação	(p) decomposto	$\chi_D(p) = 1$
	(p) inerte	$\chi_D(p) = -1$
	(p) ramificado	$\chi_D(p) = 0$
∞	Real / Complexo	$\chi_D(-1) = 1$ / $\chi_D(-1) = -1$
L-função	$\zeta_{\mathbb{Q}(\sqrt{D})}(s) = \sum \mathrm{N}(\mathfrak{a})^{-s}$	$L(s, \chi_D) = \sum \chi_D(n) n^{-s}$

<u>Reciprocidade quadrática</u> $\iff$ $\zeta_{\mathbb{Q}(\sqrt{D})}(s) = \zeta(s) L(s, \chi_D)$

Conjectura (“n.e. zeros de Siegel” para $D < 0$)

Existe $\delta > 0$ tal que $L(\beta, \chi_D) \neq 0$ para $1 - \dfrac{\delta}{\log |D|} \leq \beta < 1$.

Equivalentemente:

1. $\dfrac{L'}{L}(1, \chi_D) \ll \log |D|$
2. $\mathrm{ht}(j(\tau_D)) \ll \log |D|$
3. $h(D) \gg \dfrac{\sqrt{|D|}}{\log |D|} \displaystyle\sum_{(a,b,c)} \frac{1}{a}$

Há relações precisas entre as quantidades $\frac{L'}{L}(1, \chi_D)$, $\mathrm{ht}(j(\tau_D))$, e $h(D)$:

$h(D) \longleftrightarrow \mathrm{ht}(j(\tau_D))$	(visto)
$\boxed{\mathrm{ht}(j(\tau_D)) \longleftrightarrow \frac{L'}{L}(1, \chi_D)}$	(Teo 1)
$\frac{L'}{L}(1, \chi_D) \longleftrightarrow$ zero de Siegel	(delicado)

① $\iff$ ② (Teorema 1)

Teorema 1 (T., 2021)

Para discriminantes fundamentais $D < 0$,

$$\frac{L'}{L}(1, \chi_D) = \frac{1}{6}\,\mathrm{ht}(j(\tau_D)) - \frac{1}{2}\log|D| + \underbrace{C + o_{D\to-\infty}(1)}_{\textbf{termo de erro}}$$

onde $C = -1.057770\ldots$.

†**Obs.:** (Colmez, 1993) Seja $E_D/\mathbb{C}$ uma curva elíptica c/ CM por $\mathbb{Z}[\tau_D]$. Então:

$$-2\,\mathrm{ht}_{\mathrm{Fal}}(E_D) = \frac{1}{2}\log|D| + \frac{L'}{L}(0, \chi_D) + \log 2\pi.$$

- Colmez + Teo 1 $\implies$ $|\mathrm{ht}_{\mathrm{Fal}}(E_D) - \frac{1}{12}\mathrm{ht}(j(\tau_D))| = 2.65537\ldots + o_{D\to-\infty}(1)$

	Termo de erro
Colmez (1993)	$O(\log \mathrm{ht}(j(\tau_D)))$
G–S (2000)	$O(\log\log|D|)$

	Termo de erro
Dimitrov (2016)*	$O(1)$
T. (2021)	$\mathbf{-1.05\ldots + o(1)}$

*Não publicado.

Gráfico de $\mathrm{ht}(j(\tau_D))$

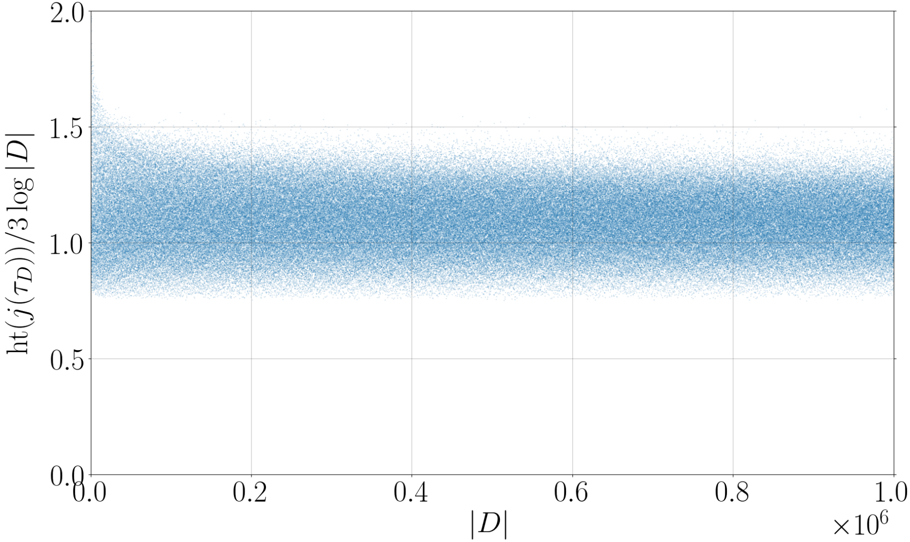
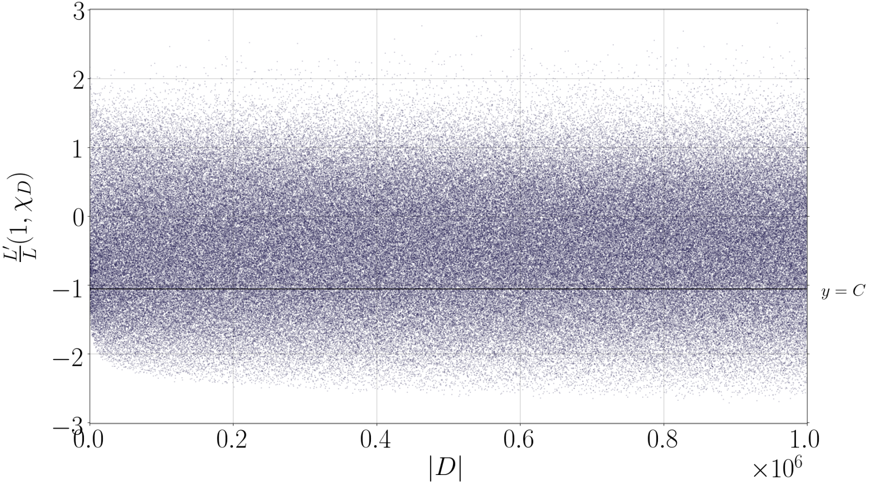


Gráfico de $\frac{L'}{L}(1, \chi_D)$



$$C = -1.057770\ldots$$

A função ζ de Dedekind de $\mathbb{Q}(\sqrt{D})$ é dada por:

$$\zeta_{\mathbb{Q}(\sqrt{D})}(s) := \sum_{\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{D})}} \frac{1}{\mathrm{N}(\mathfrak{a})^s} \qquad \left(\begin{aligned} &= \zeta(s) L(s, \chi_D) \\ &= \sum_{\mathscr{A} \in \mathrm{C}\ell(D)} \zeta(s, \mathscr{A}) \end{aligned} \right)$$

onde $\zeta(s, \mathscr{A}) = \sum_{\substack{\mathfrak{a} \subseteq \mathscr{A} \\ \mathfrak{a} \text{ integral}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$ for $\mathscr{A} \in \mathrm{C}\ell(D)$ — (zeta parcial)

Em geral, conforme $s \to 1$:

- $\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1)$
- $\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + O(s-1)$
- $\zeta_K(s, \mathscr{A}) = \frac{\varkappa_K}{s-1} + \varkappa_K \mathfrak{K}(\mathscr{A}) + O(s-1)$

(Ihara, 2006)
Euler–Kronecker:
$\gamma_K := c_0 / c_{-1}$

Limites de Kronecker:
$\mathfrak{K}(\mathscr{A}),\ \mathscr{A} \in \mathrm{C}\ell_K$

$$\boxed{\gamma_K = \frac{1}{h_K} \sum_{\mathscr{A} \in \mathrm{C}\ell_K} \mathfrak{K}(\mathscr{A})}$$

$$\gamma + \frac{L'}{L}(1, \chi_D) = \frac{1}{h(D)} \sum_{\mathscr{A} \in \mathrm{C}\ell(D)} \mathfrak{K}(\mathscr{A})$$

Correspondência para $D < 0$ (Ideais–Formas–Pontos)

$[f_{\mathfrak{a}}(x,y) \mid \mathfrak{a} \in \mathscr{A}] \longleftarrow\!\!\shortmid \mathscr{A}$

Classes de ideais

$\mathscr{A} \in \mathrm{C}\ell(D)$

$\mathscr{A} \longmapsto [\tau \in \mathfrak{h} \mid f_{\mathfrak{a}}(\tau, 1) = 0,\ \mathfrak{a} \in \mathscr{A}]$

$$f_{\mathfrak{a}}(x,y) := \frac{\mathrm{N}(\alpha x + \beta y)}{\mathrm{N}(\mathfrak{a})}$$

$(\mathfrak{a} = \alpha\mathbb{Z} + \beta\mathbb{Z})$

$[(a,b,c)] \longmapsto \left[\mathbb{Z} + \left(\frac{-b+\sqrt{D}}{2a}\right)\mathbb{Z}\right]$

$[\mathbb{Z} + \tau\mathbb{Z}] \longleftarrow\!\!\shortmid [\tau]$

$[(a,b,c)] \longmapsto \left[\dfrac{-b+\sqrt{D}}{2a}\right]$

Formas quad.s (prim., pos-def)

$[(a,b,c)] \in \mathrm{FormQuad}(D)/\!\sim$

Pontos de Heegner

$[\tau] \in \widetilde{\Lambda}_D/\mathrm{SL}_2(\mathbb{Z})$

$\left[\frac{\sqrt{|D|}}{2\,\Im(\tau)}|x+\tau y|^2\right] \longleftarrow\!\!\shortmid [\tau]$

Zeta parcial		Zeta de Epstein		Série de Eisenstein real-analítica
$\zeta(s, \mathscr{A})$	$\leftrightarrow$	$Z_{[(a,b,c)]}(s)$	$\leftrightarrow$	$E([\tau], s)$
$\displaystyle\sum_{\substack{\mathfrak{a} \subseteq \mathscr{A} \\ \mathfrak{a}\ \text{integral}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$		$\displaystyle\sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\neq(0,0)}} \frac{1}{(ax^2+bxy+cy^2)^s}$		$\displaystyle\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{\Im(\tau)^s}{\|m\tau+n\|^{2s}}$

Limites de Kronecker para $\mathbb{Q}(\sqrt{D})$ $(D < 0)$

$$\zeta(s, \mathscr{A}) = \frac{1}{w_D}\left(\frac{2}{\sqrt{|D|}}\right)^s E(\tau_{\mathscr{A}}, s)$$

$$\boxed{\mathscr{A} \leftrightarrow \begin{matrix}(a,b,c)\\ \text{reduzida}\end{matrix} \leftrightarrow \tau_{\mathscr{A}} = \frac{-b+\sqrt{D}}{2a}}$$

- Para $\tau \in \mathfrak{h}$ fixo, a expansão de Laurent de E em $s = 1$ é:

$$\boxed{E(\tau, s) = \frac{\pi}{s-1} + \pi\left(\frac{\pi}{3}\Im(\tau) - \log \Im(\tau) + \mathcal{U}(\tau)\right) + 2\pi\big(\gamma - \log 2\big) + O(s-1)}$$

onde:

$$\mathcal{U}(\tau) := 4\sum_{n\geq 1}\left(\sum_{d\,|\,n}\frac{1}{d}\right)\frac{\cos(2\pi n\,\Re(\tau))}{e^{2\pi n\,\Im(\tau)}} \qquad \left(= -2\log(|\eta(\tau)|^2) - \frac{\pi}{3}\Im(\tau)\right)$$

(1ª) Fórmula do limite de Kronecker

$$\mathfrak{K}(\mathscr{A}) = \underbrace{\frac{\pi}{3}\Im(\tau_{\mathscr{A}}) - \log \Im(\tau_{\mathscr{A}}) + \mathcal{U}(\tau_{\mathscr{A}})}_{\textit{termo } \mathscr{A}\textit{-dependente}} \underbrace{-\frac{1}{2}\log|D|}_{D\textit{-dependente}} \underbrace{+2\gamma - \log 2}_{\textit{constante}}$$

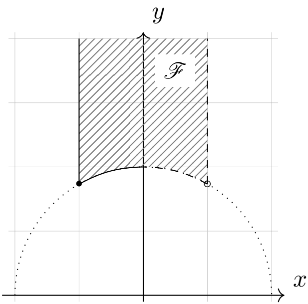
Teorema da equidistribuição de Duke

Teorema (Duke, 1988)

$\Lambda_D = \{\tau_{\mathscr{A}} \mid \mathscr{A} \in \mathrm{C}\ell(D)\}$ *é equidistribuído em* $\mathscr{F}$.

Se $f : \mathscr{F} \to \mathbb{C}$ é Riemann-integrável, então:

$$\boxed{\lim_{D \to -\infty} \frac{1}{h(D)} \sum_{\mathscr{A} \in \mathrm{C}\ell(D)} f(\tau_{\mathscr{A}}) = \int_{\mathscr{F}} f(z)\,\mathrm{d}\mu}$$



$$z = x + iy$$

$$\mathrm{d}\mu = \frac{3}{\pi} \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

(Elemento de área hiperbólico normalizado)

W. Duke

Demonstração do Teorema 1

FLK: $\mathfrak{K}(\mathscr{A}) = \frac{\pi}{3}\Im(\tau_{\mathscr{A}}) - \log \Im(\tau_{\mathscr{A}}) + \mathcal{U}(\tau_{\mathscr{A}}) - \frac{1}{2}\log|D| + (2\gamma - \log 2)$

- $\int_{\mathscr{F}} \mathcal{U}(z)\,\mathrm{d}\mu = 0.000151\ldots$
- $\int_{\mathscr{F}} \log(y)\,\mathrm{d}\mu = 0.952984\ldots$
- $\int_{\mathscr{F}} \left(\log^+|j(z)| - 2\pi y\right)\mathrm{d}\mu = -0.068692\ldots$

$\frac{1}{h(D)} \sum_{\mathscr{A}\in \mathrm{C}\ell(D)} \log \Im(\tau_{\mathscr{A}})$
é **difícil** sem o teorema de Duke!

$\Downarrow$ (pelo Teo. de Duke)

$$\gamma_{\mathbb{Q}(\sqrt{D})} = \frac{1}{6}\left(\frac{1}{h(D)} \sum_{\mathscr{A}\in \mathrm{C}\ell(D)} \log^+|j(\tau_{\mathscr{A}})|\right) - \frac{1}{2}\log|D| + C' + o(1)$$

$\Downarrow$

$$\frac{L'}{L}(1,\chi_D) = \frac{1}{6}\,\mathrm{ht}(j(\tau_D)) - \frac{1}{2}\log|D| + C + o(1) \qquad \square$$

$C = -1.057770\ldots$

Sumário

Teorema principal

Teorema 2 (T., 2021)†

$$\textit{abc uniforme}\ (\text{fraco}) \implies \limsup_{D\to-\infty} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} = 0$$

- **Algébrico:** $\displaystyle \limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{3\log|D|} \overset{abc\text{-U}}{\leq} 1$ †(Granville–Stark, 2000)
- **Analítico:** $\displaystyle \limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{3\log|D|} \geq 1$ (T., 2021) [não-condicional!]

Corolário principal:

abc-U fraco + Prop 1 $\Rightarrow$

$$\max\{\beta\in\mathbb{R} \mid L(\beta,\chi_D)=0\} < 1 - \frac{\sqrt{5}\varphi + o(1)}{\log|D|}$$

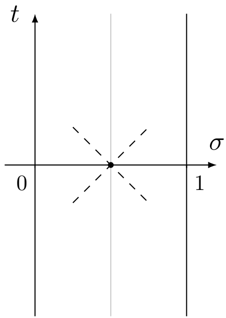
Seja $D < 0$ um discriminante fundamental.

- Fórmula clássica (Eq. functional + produto de Hadamard)

$$\frac{L'}{L}(s,\chi_D) = \left(\sum_{\varrho(\chi_D)} \frac{1}{s-\varrho}\right) - \frac{1}{2}\log\left(\frac{|D|}{\pi}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)$$

- Por reflexão:

$$L(\varrho,\chi) = 0 \implies \begin{cases} \varrho,\ 1-\overline{\varrho} \ \text{ zeros of } \ L(s,\chi) \\ \overline{\varrho},\ 1-\varrho \ \text{ zeros of } \ L(s,\overline{\chi}) \end{cases}$$



- Assim:

$$\boxed{\sum_{\varrho(\chi_D)} \frac{1}{\varrho} = \frac{1}{2}\log|D| + \frac{L'}{L}(1,\chi_D) - \frac{1}{2}\Big(\gamma + \log\pi\Big)}$$

Pareando zeros (1/2)

Em geral, escrevendo ($\varrho \in$ faixa crítica, $\varepsilon > 0$):

$$\Pi_\varepsilon(\varrho) := \frac{1}{\varrho+\varepsilon} + \frac{1}{\overline{\varrho}+\varepsilon} + \frac{1}{1-\varrho+\varepsilon} + \frac{1}{1-\overline{\varrho}+\varepsilon} \quad \underline{\text{(pareamento)}}$$

obtemos:

$$\boxed{\sum_{\varrho(\chi_D)} \frac{\Pi_{\sigma-1}(\varrho)}{4} = \frac{1}{2}\log|D| + \frac{L'}{L}(\sigma,\chi_D) - \frac{1}{2}\left(-\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+1}{2}\right) + \log\pi\right)}$$

- <u>Objetivo</u>: Estimar Π_0 na faixa crítica
- <u>Ideia</u>: Perturbar ε em Π_ε

Lema 1 (lema do pareamento)

$\boxed{\mathbf{i}}$ *Para todo* $s \in \mathfrak{S}$, *temos:*

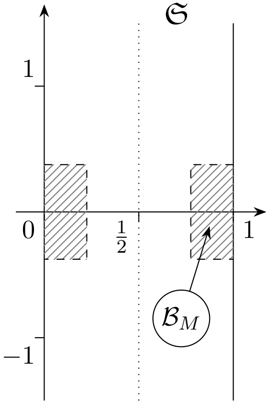
$$\Pi_0(s) > \frac{\Pi_{\varphi-1}(s)}{2\varphi - 1} \qquad \left(c/\ \varphi = \frac{1+\sqrt{5}}{2} \right)$$

$\boxed{\mathbf{ii}}$ *Tome* $0 < \varepsilon < 1$, $M \geq 2$, *e considere*

$$\mathcal{B}_M := \left\{ s \in \mathfrak{S} \;\middle|\; \sigma > 1 - \frac{1}{M},\ |t| < \frac{1}{\sqrt{M}} \right\}$$

Então, em $\mathfrak{S} \setminus \big(\mathcal{B}_M \cup (1 - \mathcal{B}_M)\big)$ *temos:*

$$|\Pi_0(s) - \Pi_\varepsilon(s)| < 5M\varepsilon\, \Pi_\varepsilon(s)$$



Proposição 1

Para todo zero não-trivial ϱ de $L(s, \chi_D)$, temos:

$$\Re\left(\frac{1}{1-\varrho}\right) < \left(1 - \frac{1}{\sqrt{5}}\right)\frac{1}{2}\log|D| + \frac{L'}{L}(1, \chi_D) + (\sqrt{5}\varphi - 1)$$

- Usando que **<u>abc-U fraco</u>** + Teo 1 $\implies \limsup_{D\to-\infty} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} \leq 0$, se $\beta \in [0,1[$ é um zero de $L(s, \chi_D)$, então

$$\frac{1}{1-\beta} < \left(\left(1 - \frac{1}{\sqrt{5}}\right)\frac{1}{2} + o(1)\right)\log|D|.$$

- Como $((1 - \frac{1}{\sqrt{5}})\frac{1}{2})^{-1} = \sqrt{5}\varphi$, rearranjando: $\boxed{\beta < 1 - \frac{\sqrt{5}\varphi + o(1)}{\log|D|}}$. $\square$

Isolando o zero de Siegel (1/2)

Usando o *segundo* lema do pareamento:

Proposição 2 (T., 2021)

Suponha que para $\chi \pmod{q}$ primitivo, $L(s,\chi)$ não possui zeros (exceto um possível zero de Siegel $\beta = \beta_\chi$) na região

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{1}{f(q)},\ |t| \leq \frac{1}{\sqrt{f(q)}} \right\},$$

para alguma $f : \mathbb{Z}_{\geq 2} \to \mathbb{R}$ com $2 \leq f(q) \ll \log q$. Então:

$$\Re\left(\frac{L'}{L}(1,\chi)\right) = \frac{1}{1-\beta} + O\left(\sqrt{f(q)\log q}\right)$$

- RLZ clássica: $f(q) = O(\log q)$.

"n.e. zeros de Siegel" $\iff$ $\dfrac{L'}{L}(1,\chi_D) \ll \log |D|$

Isolando o zero de Siegel (2/2)

Um inteiro q é *k-suave* se todos seus fatores primos são $\leq k$.

Região livre de zeros de Chang (2014)

Para $\chi \pmod{q}$ primitivo, $L(s,\chi)$ não possui zeros (exceto um possível zero de Siegel) em

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{1}{f(q)},\ |t| \leq 1 \right\},$$

onde $f : \mathbb{Z}_{\geq 2} \to \mathbb{R}$ satisfaz:

$f(q) = o(\log q)$ para módulos $q^{o(1)}$-suaves

M.-C. Chang

- RLZ de Chang + Proposição 2:

$$\boxed{\frac{L'}{L}(1,\chi_D) = \frac{1}{1-\beta_D} + O\Big(\sqrt{f(|D|)\,\log|D|}\Big),}$$

onde $\beta_D := \max\{\beta \in \mathbb{R} \mid L(\beta,\chi_D) = 0\}$.

Demonstração do Teorema Principal

$$\boxed{\frac{L'}{L}(1,\chi_D) = \frac{1}{1-\beta_D} + O\Big(\sqrt{f(|D|)\,\log|D|}\Big)}$$

- Como $\dfrac{1}{1-\beta_D} > 0$, e $\sqrt{f(|D|)\,\log|D|} = o(\log|D|)$ para discriminantes fundamentais $|D|^{o(1)}$-suaves, segue que

$$\limsup_{\substack{D\to-\infty \\ |D|^{o(1)}\text{-suave}}} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} \geq 0. \qquad \square$$

Corolário (“n.e. Siegel zeros” forte)

Assuma *abc* uniforme fraco. Para qualquer $A>0$, quando $D\to-\infty$ por discriminantes $|D|^{o(1)}$-suaves, todos salvo um número finito de Ds satisfazem

$$\max\{\beta\in\mathbb{R} \mid L(\beta,\chi_D)\} < 1 - \frac{A}{\log|D|}.$$

Majorantes para $h(D)$

Para discriminantes fundamentais $D < 0$, temos que:

$$\frac{3}{\sqrt{5}} \log|D| + O(1) \;\leq\; \mathrm{ht}(j(\tau_D)) \;\overset{*}{\leq}\; (1+o(1))\, 3\log|D|$$

$$\boxed{(1+o(1))\,\frac{\pi}{3}\frac{\sqrt{|D|}}{\log|D|}\sum_{(a,b,c)}\frac{1}{a} \;\overset{*}{\leq}\; h(D) \;\leq\; \left(\sqrt{5}+O\left(\frac{1}{\log|D|}\right)\right)\frac{\pi}{3}\frac{\sqrt{|D|}}{\log|D|}\sum_{(a,b,c)}\frac{1}{a}}$$

onde:

- desig.s estrela "$\overset{*}{\leq}$" são condicionais em *abc* uniforme fraco.
- se D é $|D|^{o(1)}$-suave, então desig.s estrela tornam-se *igualdades* (assintóticas) estrela.
- HRG $\implies$ igualdades assintóticas estrela *para todo* D, com termo de erro $O(\frac{\log\log|D|}{\log|D|})$. (E.g., $\mathrm{ht}(j(\tau_D)) = 3\log|D| + O(\log\log|D|)$ sob HRG).

¡Muchas gracias!

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